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Picture Fuzzy Lattices, Ideals and Homomorphism

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Abstract. Picture fuzzy set (PFS) is a novel concept for dealing with uncertainty and a generalization of the traditional fuzzy set (FS) and intuitionistic fuzzy set (IFS) and can easily manage the uncertain nature of human thoughts by incorporating the positive, neutral, negative and refusal membership degrees of an object. Many conceptual ideas on PFSs have been developed so far and applied in diversified fields. In this paper, the concept of picture fuzzy sublattices and picture fuzzy ideals are developed and some of their associated properties are established in detail. Moreover, the sum and product of two picture fuzzy ideals are introduced with their properties. Finally, some properties of picture fuzzy ideals under lattice homomorphism are explored.

AMS Subject Classification 2020: 06D72
Keywords and Phrases: Picture fuzzy set, Picture fuzzy lattice, Picture fuzzy ideal, Picture fuzzy homomorphism, Picture fuzzy epimomorphism.

1 Introduction

In this era of globalization, we are to deal with numerous sorts of data for research and innovation in almost all the fields. In real life application, researchers faced many difficulties in conducting many data which are vague than exact. Fuzzy set theory introduced by Zadeh [1] is a generalization of the crisp set theory to handle the uncertain and vague information. A fuzzy set which is expressed by a membership function allows a membership degree for every element of the universal set. The non-membership degree is the direct complement of the membership degree. However, in many researches it is found that, this linguistic negation does not satisfy the logical negation always in the real life applications. Because while selecting the membership degree for an object (element), there may be some kinds of hesitation while defining the membership function, as membership function may be Gaussian, triangular, exponential or any other membership functions. So, due to this hesitation, the non-membership degree is less than or equal to the complement of the membership degree. This is the reason why different results are obtained with different membership functions. To overcome this situation, after about two decades, in 1986, Atanassov K.T. [2] suggested the concept of intuitionistic fuzzy set, where the non-membership degree is not equal to the complement of the membership degree due to the fact that some kinds of hesitations or lack of knowledge is present while defining the membership function. So the intuitionistic fuzzy set theory is an important generalization of fuzzy set theory, where the membership degree and the non-membership degree separately in such a way that, sum of the two degrees

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must not exceed 1. The subtraction of the sum of the membership and non-membership degrees from one is considered the hesitation degree of that element. But in this extension, the degree of neutrality arose a problem in decision making. To overcome this situation, Cuong and Kreinovich [3,4] introduced the notion of PFS where the hesitation degree is divided into two parts such as neutral degree and refusal degree. Hasan et al. developed numerous theoretical concepts in picture fuzzy sets such as minimal and average extension principles, minimal decomposition theorems, compositions in picture fuzzy relations, arithmetic operations in picture fuzzy numbers, several types of operators etc. and discussed many applications of these concepts in real life situations [5–12].

In 1971, Azriel Rosenfeld [13] developed the fundamental theory of fuzzy groups and many researchers discussed about classical and fuzzy algebraic structures [14–22]. Liu W.J. [16] introduced the notion of fuzzy subring and Kuroki [23] discussed some properties of fuzzy semigroups in 1991. The idea of fuzzy sublattices and fuzzy ideals of a lattice were given by Yuan and Wu [24] and applied the concept of fuzzy sets in lattice theory. The theory of fuzzy lattice ordered ideals was studied in [25]. Ajmal N. and Thomas K. V. [26,27] established some structural theorems for fuzzy lattices. They also discussed about some properties and characterizations of a fuzzy sublattice, fuzzy ideal and fuzzy prime ideal including their dual ideals. Moreover, the idea of fuzzy convex sublattice is introduced by them. The IFSs are used to algebra by numerous researchers and developed IF subgroups [28] and IF subring [29]. Swamy U. M. [18] introduced fuzzy ideals on lattices. Tripathy B. K. [30] introduced intuitionistic fuzzy lattices and intuitionistic Boolean algebras. Bharathi P. [31] introduced the idea of picture fuzzy lattices and ideals under picture fuzzy partial order relation.

In this paper, the concept of picture fuzzy sublattices and picture fuzzy ideals are established. Also some properties of picture fuzzy sublattices and picture fuzzy ideals are explored thoroughly. Also, the sum and product of two picture fuzzy ideals are developed with some of their properties. Finally, some properties of picture fuzzy ideals under lattice homomorphism are discussed.

2 Preliminaries

Definition 2.1. [1] A fuzzy set A in a non-empty set U is defined as

$$A = \{(a, u_A(a)) : a \in U\}, \text{ where } u_A : U \to [0, 1].$$

Definition 2.2. [32] Let L be a partially ordered set (poset). Then, the algebraic structure $(L, \wedge, \vee)$ is called a lattice if $\forall a, b \in L$:

$$a \wedge b \in L \text{ and } a \vee b \in L.$$

Here, $\wedge$ and $\vee$ are two binary operations called "Meet" and "Join" respectively. We write

$$a \wedge b = \inf\{a, b\} = \min\{a, b\}, \quad a \vee b = \sup\{a, b\} = \max\{a, b\}.$$

Definition 2.3. [32] A lattice $(L, \wedge, \vee)$ is called a distributive lattice if $\forall a, b, c \in L$:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Definition 2.4. [24] A fuzzy set $A = \{(a, u_A(a)) : a \in L\}$ on a lattice L is called a fuzzy sublattice of L if for all $a, b \in L$:

$$u_A(a \vee b) \geq \min\{u_A(a), u_A(b)\},$$

$$u_A(a \wedge b) \geq \min\{u_A(a), u_A(b)\}.$$

Definition 2.5. [2] An intuitionistic fuzzy set A in a non-empty set U is defined as

$$A = \{(a, u_A(a), v_A(a)) : a \in U\},$$

where the membership and the non-membership degrees are respectively $u_A : U \to [0,1]$ and $v_A : U \to [0,1]$, with $0 \leq u_A(a) + v_A(a) \leq 1, \forall a \in U$.

Definition 2.6. [25] Let $A = \{(a, u_A(a), v_A(a)) : a \in X\}$ be an IFS of U. Then

$$[A] = \{(a, u_A(a), u_A^c(a)) : a \in U\}, \quad \text{where } u_A^c(a) = 1 - u_A(a)$$

$$\langle A \rangle = \{(a, v_A(a), v_A^c(a)) : a \in U\}, \quad \text{where } v_A^c(a) = 1 - v_A(a)$$

Definition 2.7. [25] An IFS $A = \{(a, u_A(a), v_A(a)) : a \in L\}$ on a lattice L is called an intuitionistic fuzzy sublattice of L if $\forall a, b \in L$:

$$u_A(a \vee b) \geq \min\{u_A(a), u_A(b)\}$$

$$u_A(a \wedge b) \geq \min\{u_A(a), u_A(b)\}$$

$$v_A(a \vee b) \leq \max\{v_A(a), v_A(b)\}$$

$$v_A(a \wedge b) \leq \max\{v_A(a), v_A(b)\}$$

Definition 2.8. [25] An IFS $A = \{(a, u_A(a), v_A(a)) : a \in L\}$ of L is called an intuitionistic fuzzy ideal of L if $\forall a, b \in L$:

$$u_A(a \vee b) \geq \min\{u_A(a), u_A(b)\}$$

$$u_A(a \wedge b) \geq \max\{u_A(a), u_A(b)\}$$

$$v_A(a \vee b) \leq \max\{v_A(a), v_A(b)\}$$

$$v_A(a \wedge b) \leq \min\{v_A(a), v_A(b)\}$$

Definition 2.9. [3,4] A picture fuzzy set A in U is defined as

$$A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in U\},$$

where the positive, neutral, and negative membership degrees are respectively $u_A : U \to [0,1]$, $w_A : U \to [0,1]$, and $v_A : U \to [0,1]$ with $0 \leq u_A(a) + w_A(a) + v_A(a) \leq 1; \forall a \in U$.

Here, $1 - (u_A(a) + w_A(a) + v_A(a)), \forall a \in U$ is the refusal membership degree of a in A.

Definition 2.10. [3,4] For two PFSs A and B on U:

- $A \subseteq B$ if and only if $\forall a \in X, u_A(a) \leq u_B(a), w_A(a) \leq w_B(a)$, and $v_A(a) \geq v_B(a)$;
- $A = B$ if and only if $\forall a \in X, u_A(a) = u_B(a), w_A(a) = w_B(a)$, and $v_A(a) = v_B(a)$;
- $A \cup B = \{(a, \max\{u_A(a), u_B(a)\}, \min\{w_A(a), w_B(a)\}, \min\{v_A(a), v_B(a)\}) : a \in U\}$;
- $A \cap B = \{(a, \min\{u_A(a), u_B(a)\}, \min\{w_A(a), w_B(a)\}, \max\{v_A(a), v_B(a)\}) : a \in U\}$;
- $A^c = \{(a, v_A(a), w_A(a), u_A(a)) : a \in U\}$.

Definition 2.11. [33] For $U \neq \phi$ and $V \neq \phi$ and a mapping $f : U \to V$, a pair of mappings are defined as follows:

$$f : \text{PFS}(U) \to \text{PFS}(V) \quad \text{and} \quad f^{-1} : \text{PFS}(V) \to \text{PFS}(U)$$

defined as:

$$f(A)(b) = (u_{f(A)}(b), w_{f(A)}(b), v_{f(A)}(b)), \quad \text{where } A \in \text{PFS}(U)$$

$$u_{f(A)}(b) = \begin{cases} \vee\{u_A(a) : a \in f^{-1}(b)\} & ; f^{-1}(b) \neq \emptyset \\ 0 & ; \text{Otherwise} \end{cases}$$

$$w_{f(A)}(b) = \begin{cases} \wedge\{w_A(a) : a \in f^{-1}(b)\} & ; f^{-1}(b) \neq \emptyset \\ 0 & ; \text{Otherwise} \end{cases}$$

$$v_{f(A)}(b) = \begin{cases} \wedge\{v_A(a) : a \in f^{-1}(b)\} & ; f^{-1}(b) \neq \emptyset \\ 0 & ; \text{Otherwise} \end{cases}$$

and

$$f^{-1}(B)(a) = (u_{f^{-1}(B)}(a), w_{f^{-1}(B)}(a), v_{f^{-1}(B)}(a)), \quad \text{where } B \in \text{ PFS}(V) \text{ and}$$

$$u_{f^{-1}(B)}(a) = u_B(f(a)), \quad w_{f^{-1}(B)}(a) = w_B(f(a)), \quad v_{f^{-1}(B)}(a) = v_B(f(a))$$

3 Picture Fuzzy Lattices and Ideals

Definition 3.1. *A PFS* $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ *on a lattice* L *is said to be a picture fuzzy sublattice (PFL) of* L *if* $\forall a, b \in L$*:*

(i) $u_A(a \vee b) \geq \min\{u_A(a), u_A(b)\}$

(ii) $u_A(a \wedge b) \geq \min\{u_A(a), u_A(b)\}$

(iii) $w_A(a \vee b) \geq \min\{w_A(a), w_A(b)\}$

(iv) $w_A(a \wedge b) \geq \min\{w_A(a), w_A(b)\}$

(v) $v_A(a \vee b) \leq \max\{v_A(a), v_A(b)\}$

(vi) $v_A(a \wedge b) \leq \max\{v_A(a), v_A(b)\}$

Example 1. Consider the lattice $L = \{1, 2, 3, 6\}$ of "divisors of 6," which is represented in the Hasse diagram.

Let $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ be given by

$$A = \{(1, 0.6, 0.2, 0.1), (2, 0.3, 0.1, 0.5), (3, 0.5, 0.2, 0.3), (6, 0.7, 0.0, 0.3)\}$$

Then, A is a PFL of L.

Definition 3.2. *A PFS* $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ *of* L *is called a picture fuzzy ideal (PFI) of* L *if* $\forall a, b \in L$*:*

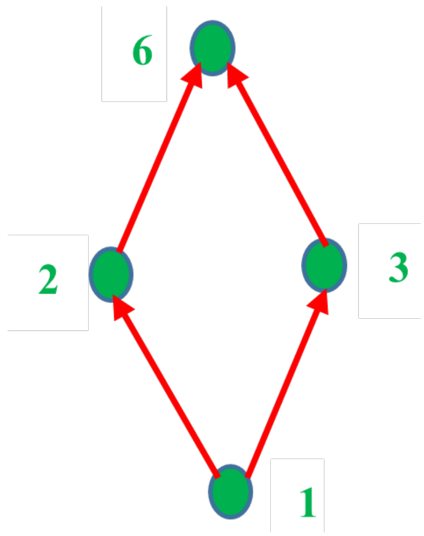

Figure 1

(i) $u_A(a \vee b) \geq \min\{u_A(a), u_A(b)\}$

(ii) $u_A(a \wedge b) \geq \max\{u_A(a), u_A(b)\}$

(iii) $w_A(a \vee b) \geq \min\{w_A(a), w_A(b)\}$

(iv) $w_A(a \wedge b) \geq \max\{w_A(a), w_A(b)\}$

(v) $v_A(a \vee b) \leq \max\{v_A(a), v_A(b)\}$

(vi) $v_A(a \wedge b) \leq \min\{v_A(a), v_A(b)\}$

Example 2. Let $L = \{1, 2, 3, 4, 6, 12\}$ of "factors of 12" which is displayed in the Hasse diagram. Let $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ be given by

$$A = \left\{ \begin{array}{l} (1, 0.5, 0.1, 0.4), \\ (2, 0.4, 0.2, 0.4), \\ (3, 0.7, 0.1, 0.1), \\ (4, 0.3, 0.2, 0.5), \\ (6, 0.6, 0.1, 0.2), \\ (12, 0.4, 0.2, 0.3) \end{array} \right\}$$

It is clear that A is a PFI of L.

Theorem 3.3. *If A and B are two PFLs of L, then $A \cap B$ is also a PFL of L.*

Proof. Let $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ and $B = \{(a, u_B(a), w_B(a), v_B(a)) : a \in L\}$ be two PFLs of L. Then,

$$A \cap B = \{(a, \min\{u_A(a), u_B(a)\}, \min\{w_A(a), w_B(a)\}, \max\{v_A(a), v_B(a)\}) : a \in L\}$$

Now,

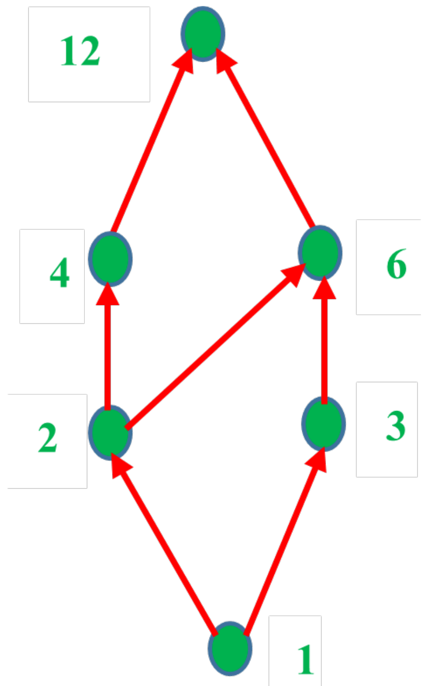

Figure 2

$$\begin{aligned} u_{A\cap B}(a\vee b) &= \min\{u_A(a\vee b), u_B(a\vee b)\} \\ &\geq \min\{\min\{u_A(a), u_A(b)\}, \min\{u_B(a), u_B(b)\}\} \quad (\text{As } A \text{ and } B \text{ are PFLs of } L) \\ &= \min\{\min\{u_A(a), u_B(a)\}, \min\{u_A(b), u_B(b)\}\} \\ &= \min\{u_{A\cap B}(a), u_{A\cap B}(b)\} \end{aligned}$$

$$\therefore u_{A\cap B}(a\vee b) \geq \min\{u_{A\cap B}(a), u_{A\cap B}(b)\}, \quad \forall a, b \in L$$

and

$$\begin{aligned} u_{A\cap B}(a\wedge b) &= \min\{u_A(a\wedge b), u_B(a\wedge b)\} \\ &\geq \min\{\min\{u_A(a), u_A(b)\}, \min\{u_B(a), u_B(b)\}\} \quad (\text{As } A \text{ and } B \text{ are PFLs of } L) \\ &= \min\{\min\{u_A(a), u_B(a)\}, \min\{u_A(b), u_B(b)\}\} \\ &= \min\{u_{A\cap B}(a), u_{A\cap B}(b)\} \end{aligned}$$

$$\therefore u_{A\cap B}(a\wedge b) \geq \min\{u_{A\cap B}(a), u_{A\cap B}(b)\}, \quad \forall a, b \in L$$

Similarly,

$$w_{A\cap B}(a\vee b) \geq \min\{w_{A\cap B}(a), w_{A\cap B}(b)\}$$

and

$$w_{A\cap B}(a\wedge b) \geq \min\{w_{A\cap B}(a), w_{A\cap B}(b)\}, \quad \forall a, b \in L$$

Again,

$$\begin{aligned}
v_{A\cap B}(a\vee b) &= \max\{v_A(a\vee b), v_B(a\vee b)\}\\
&\le \max\{\max\{v_A(a), v_A(b)\}, \max\{v_B(a), v_B(b)\}\} \quad \text{(As } A \text{ and } B \text{ are PFLs of } L\text{)}\\
&= \max\{\max\{v_A(a), v_B(a)\}, \max\{v_A(b), v_B(b)\}\}\\
&= \max\{v_{A\cap B}(a), v_{A\cap B}(b)\}
\end{aligned}$$

$$\therefore v_{A\cap B}(a\vee b) \le \max\{v_{A\cap B}(a), v_{A\cap B}(b)\}, \quad \forall a, b \in L$$

and

$$\begin{aligned}
v_{A\cap B}(a\wedge b) &= \max\{v_A(a\wedge b), v_B(a\wedge b)\}\\
&\le \max\{\max\{v_A(a), v_A(b)\}, \max\{v_B(a), v_B(b)\}\} \quad \text{(As } A \text{ and } B \text{ are PFLs of } L\text{)}\\
&= \max\{\max\{v_A(a), v_B(a)\}, \max\{v_A(b), v_B(b)\}\}\\
&= \max\{v_{A\cap B}(a), v_{A\cap B}(b)\}
\end{aligned}$$

$$\therefore v_{A\cap B}(a\wedge b) \le \max\{v_{A\cap B}(a), v_{A\cap B}(b)\}, \quad \forall a, b \in L$$

Hence, $A\cap B$ is a PFL of L. □

Theorem 3.4. *If A and B are two PFIs of L, then $A\cap B$ is also a PFI of L.*

Proof. Proof: Same as Theorem 3.5. □

Definition 3.5. *Let $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ be a PFS of L. Then*

$$[A] = \{(a, u_A(a), w_A(a), u^c_A(a)) : a \in L\}, \quad \textit{where } u^c_A(a) = 1 - u_A(a)$$

and

$$\langle A\rangle = \{(a, v_A(a), w_A(a), v^c_A(a)) : a \in L\}, \quad \textit{where } v^c_A(a) = 1 - v_A(a)$$

Proposition 3.6. *A is a PFL of L if and only if $[A]$ and $\langle A\rangle$ are PFLs of L.*

Proof. First consider A is a PFL of L. We have

$$[A] = \{(a, u_A(a), w_A(a), u^c_A(a)) : a \in L\}, \quad \text{where } u^c_A(a) = 1 - u_A(a)$$

Then $\forall a, b \in L$;

$$u_A(a\vee b) \ge \min\{u_A(a), u_A(b)\} \quad \text{and} \quad u_A(a\wedge b) \ge \min\{u_A(a), u_A(b)\}, \quad \text{as } A \text{ is a PFL of } L$$

Now,

$$\begin{aligned}
u^c_A(a\vee b) &= 1 - u_A(a\vee b)\\
&\le 1 - \min\{u_A(a), u_A(b)\}\\
&= \max\{1 - u_A(a), 1 - u_A(b)\}\\
&= \max\{u^c_A(a), u^c_A(b)\}
\end{aligned}$$

$$\therefore u^c_A(a\vee b) \le \max\{u^c_A(a), u^c_A(b)\}$$

and

$$\begin{aligned} u_A^c(a \wedge b) &= 1 - u_A(a \wedge b) \\ &\leq 1 - \min\{u_A(a), u_A(b)\} \\ &= \max\{1 - u_A(a), 1 - u_A(b)\} \\ &= \max\{u_A^c(a), u_A^c(b)\} \end{aligned}$$

$$\therefore u_A^c(a \wedge b) \leq \max\{u_A^c(a), u_A^c(b)\}$$

Hence, $[A]$ is a PFL of L.

Again,

$$\langle A \rangle = \{(a, v_A(a), w_A(a), v_A^c(a)) : a \in L\}, \quad \text{where } v_A^c(a) = 1 - v_A(a)$$

Then $\forall a, b \in L$;

$$v_A(a \vee b) \leq \max\{v_A(a), v_A(b)\} \quad \text{and} \quad v_A(a \wedge b) \leq \max\{v_A(a), v_A(b)\}$$

Now,

$$\begin{aligned} v_A^c(a \vee b) &= 1 - v_A(a \vee b) \\ &\geq 1 - \max\{v_A(a), v_A(b)\} \\ &= \min\{1 - v_A(a), 1 - v_A(b)\} \\ &= \min\{v_A^c(a), v_A^c(b)\} \end{aligned}$$

$$\therefore v_A^c(a \vee b) \geq \min\{v_A^c(a), v_A^c(b)\}$$

and

$$\begin{aligned} v_A^c(a \wedge b) &= 1 - v_A(a \wedge b) \\ &\geq 1 - \max\{v_A(a), v_A(b)\} \\ &= \min\{1 - v_A(a), 1 - v_A(b)\} \\ &= \min\{v_A^c(a), v_A^c(b)\} \end{aligned}$$

$$\therefore v_A^c(a \wedge b) \geq \min\{v_A^c(a), v_A^c(b)\}$$

Hence, $\langle A \rangle$ is a PFL of L.

Conversely, consider that if $[A]$ and $\langle A \rangle$ are PFLs of L, then A is a PFL of L, which holds easily from the definition. □

Lemma 3.7. *The union of two PFLs need not be a PFL.*

Proof.

Proof: Consider the lattice $L = \{1, 2, 5, 10\}$ of "divisors of 10" which is displayed in the Hasse diagram. Define $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ by

$$A = \{(1, 0.5, 0.2, 0.2), (2, 0.4, 0.3, 0.2), (5, 0.6, 0.1, 0.3), (10, 0.4, 0.0, 0.3)\}$$

and $B = \{(a, u_B(a), w_B(a), v_B(a)) : a \in L\}$ by

$$B = \{(1, 0.6, 0.2, 0.1), (2, 0.5, 0.1, 0.3), (5, 0.4, 0.2, 0.4), (10, 0.3, 0.1, 0.1)\}$$

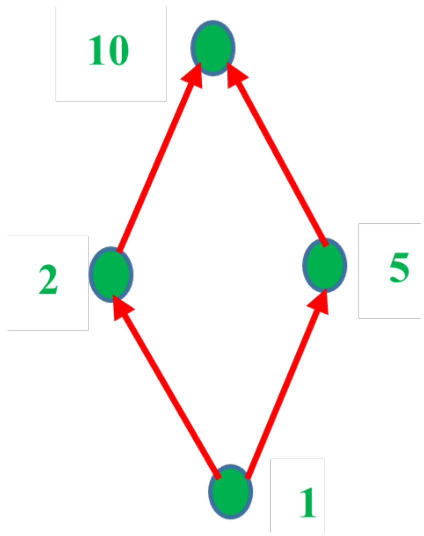

Figure 3

Here note that A and B are two PFLs of L.

Now,

$$A \cup B = \{(a, \max\{u_A(a), u_B(a)\}, \min\{w_A(a), w_B(a)\}, \min\{v_A(a), v_B(a)\}) : a \in L\}$$

$$= \{(1, 0.6, 0.2, 0.1), (2, 0.5, 0.1, 0.2), (5, 0.6, 0.1, 0.3), (10, 0.4, 0.0, 0.1)\}$$

Here, $u_{A\cup B}(10) = u_{A\cup B}(5 \vee 2) = 0.4$ and

$$\min\{u_{A\cup B}(5), u_{A\cup B}(2)\} = \min\{0.6, 0.5\} = 0.5$$

But, $u_{A\cup B}(10) = u_{A\cup B}(5 \vee 2) = 0.4 \ngeq \min\{u_{A\cup B}(5), u_{A\cup B}(2)\} = 0.5$

So, $A \cup B$ is not a PFL of L.

Lemma 3.10. Every PFI is a PFL. But the converse is not true.

Proof: Let $L = \{1, 2, 5, 10\}$ of "divisors of 10" be a lattice. Let $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ be defined by

$$A = \{(1, 0.8, 0.0, 0.2), (2, 0.4, 0.2, 0.4), (5, 0.5, 0.4, 0.1), (10, 0.5, 0.2, 0.3)\}$$

Here, A is a PFL of L but not a PFI, because

$$u_A(2) = u_A(2 \wedge 10) = 0.4 \ngeq \max\{u_A(2), u_A(10)\} = \max\{0.4, 0.5\} = 0.5$$

Lemma 3.11. The union of two PFIs need not be a PFI.

Proof: Consider the lattice $L = \{1, 2, 3, 4, 6, 12\}$ of "divisors of 12". Define $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ by

$$A = \{(1, 0.5, 0.2, 0.3), (2, 0.6, 0.1, 0.1), (3, 0.7, 0.1, 0.1), (4, 0.3, 0.4, 0.3), (6, 0.4, 0.2, 0.2), (12, 0.8, 0.1, 0.1)\}$$

and $B = \{(a, u_B(a), w_B(a), v_B(a)) : a \in L\}$ by

$B = \{(1, 0.6, 0.3, 0.1), (2, 0.5, 0.4, 0.1), (3, 0.6, 0.2, 0.2), (4, 0.5, 0.1, 0.4), (6, 0.7, 0.1, 0.2), (12, 0.2, 0.3, 0.5)\}$

Here note that A and B are PFIs of L.
Now,

$$A \cup B = \{(a, \max\{u_A(a), u_B(a)\}, \min\{w_A(a), w_B(a)\}, \min\{v_A(a), v_B(a)\}) : a \in L\}$$

$$= \{(1, 0.6, 0.2, 0.1), (2, 0.6, 0.1, 0.1), (3, 0.7, 0.1, 0.1), (4, 0.5, 0.1, 0.3), (6, 0.7, 0.1, 0.2), (12, 0.8, 0.1, 0.1)\}$$

Here, $u_{A\cup B}(12) = u_{A\cup B}(3 \vee 4) = 0.8$ and

$$\max\{u_{A\cup B}(3), u_{A\cup B}(4)\} = \max\{0.7, 0.5\} = 0.7$$

But, $u_{A\cup B}(12) = u_{A\cup B}(3 \vee 4) = 0.8 \not\leq \max\{u_{A\cup B}(3), u_{A\cup B}(4)\} = 0.7$
So, $A \cup B$ is not a PFI of L.
□

Lemma 3.8. *If A is a PFI of L and B is a PFL of L, then $A \cap B$ is a PFL of L but not a PFI of L.*

Proof.
Consider the lattice $L = \{1, 2, 3, 4, 6, 12\}$ of "divisors of 12". Let $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ be defined by

$A = \{(1, 0.6, 0.2, 0.1), (2, 0.5, 0.1, 0.4), (3, 0.4, 0.2, 0.3), (4, 0.7, 0.1, 0.1), (6, 0.5, 0.2, 0.3), (12, 0.3, 0.3, 0.1)\}$

and $B = \{(a, u_B(a), w_B(a), v_B(a)) : a \in L\}$ be defined by

$B = \{(1, 0.3, 0.3, 0.1), (2, 0.4, 0.4, 0.2), (3, 0.5, 0.3, 0.1), (4, 0.4, 0.3, 0.2), (6, 0.2, 0.3, 0.5), (12, 0.5, 0.2, 0.1)\}$

Here, A is a PFI of L and B is a PFL of L.
Now,

$$A \cap B = \{(a, \min\{u_A(a), u_B(a)\}, \min\{w_A(a), w_B(a)\}, \max\{v_A(a), v_B(a)\}) : a \in L\}$$

$$= \{(1, 0.3, 0.2, 0.1), (2, 0.4, 0.1, 0.4), (3, 0.4, 0.2, 0.3), (4, 0.4, 0.1, 0.2), (6, 0.2, 0.2, 0.5), (12, 0.3, 0.2, 0.1)\}$$

Clearly, $A \cap B$ is a PFL of L but not a PFI of L, because

$$u_{A\cap B}(1) = u_{A\cap B}(2 \wedge 3) = 0.3 \not\geq \max\{u_{A\cap B}(2), u_{A\cap B}(3)\} = \max\{0.4, 0.4\} = 0.4$$

□

Proposition 3.9. *A is a PFI of L if and only if $[A]$ and $\langle A \rangle$ are PFIs of L.*

Proof. Same as Theorem 3.8. □

4 Sum and Product of Two Picture Fuzzy Ideals

Definition 4.1. *Let $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ and $B = \{(a, u_B(a), w_B(a), v_B(a)) : a \in L\}$ be two PFSs of L. Then their sum $A + B$ is defined as*

$$A + B = \{(a, u_{A+B}(a), w_{A+B}(a), v_{A+B}(a)) : a \in L\}$$

where,

$$u_{A+B}(a) = \sup_{a=p\vee q} \{\min\{u_A(p), u_B(q)\}\}$$

$$w_{A+B}(a) = \sup_{a=p\vee q} \{\min\{w_A(p), w_B(q)\}\}$$

$$v_{A+B}(a) = \inf_{a=p\vee q} \{\max\{v_A(p), v_B(q)\}\}$$

Theorem 4.2. *The sum of two PFIs in a distributive lattice L is again a PFI of L.*

Proof.

Let $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ and $B = \{(a, u_B(a), w_B(a), v_B(a)) : a \in L\}$ be two PFSs of L. Then

$$A + B = \{(a, \mu_{A+B}(a), w_{A+B}(a), v_{A+B}(a)) : a \in L\}$$

where,

$$u_{A+B}(a) = \sup_{a=p\vee q} \{\min\{u_A(p), u_B(q)\}\}$$

$$w_{A+B}(a) = \sup_{a=p\vee q} \{\min\{w_A(p), w_B(q)\}\}$$

$$v_{A+B}(a) = \inf_{a=p\vee q} \{\max\{v_A(p), v_B(q)\}\}$$

Let $a, b \in L$ and $\min\{u_{A+B}(a), u_{A+B}(b)\} = \omega_1$. Then for any $\epsilon > 0$,

$$\omega_1 - \epsilon < u_{A+B}(a) = \sup_{a=p\vee q} \{\min\{u_A(p), u_B(q)\}\}$$

and

$$\omega_1 - \epsilon < u_{A+B}(b) = \sup_{b=r\vee s} \{\min\{u_A(r), u_B(s)\}\}$$

So there exist representations $a = p \vee q$ and $b = r \vee s$ such that,

$$\omega_1 - \epsilon < \min\{u_A(p), u_B(q)\} \quad \text{and} \quad \omega_1 - \epsilon < \min\{u_A(r), u_B(s)\}$$

Then,

$$\omega_1 - \epsilon < u_A(p), \quad \omega_1 - \epsilon < u_B(q), \quad \omega_1 - \epsilon < u_A(r), \quad \text{and} \quad \omega_1 - \epsilon < u_B(s)$$

Therefore,

$$\omega_1 - \epsilon < \min\{u_A(p), u_A(r)\} \leq u_A(p \vee r), \quad \text{since } A \text{ is a PFI of } L$$

Also,

$$\omega_1 - \epsilon < \min\{u_B(q), u_B(s)\} \leq u_B(q \vee s), \quad \text{since } B \text{ is a PFI of } L$$

Therefore,

$$\omega_1 - \epsilon < \min\{u_A(p \vee r), u_B(q \vee s)\}$$

Note that, $a \vee b = (p \vee q) \vee (r \vee s) = (p \vee r) \vee (q \vee s)$. So

$$u_{A+B}(a \vee b) = \sup_{a \vee b = m \vee n} \{\min\{u_A(m), u_B(n)\}\} \geq \min\{u_A(p \vee r), u_B(q \vee s)\} > \omega_1 - \epsilon$$

Since $\epsilon > 0$ is arbitrary,

$$u_{A+B}(a \vee b) \geq \omega_1 = \min\{u_{A+B}(a), u_{A+B}(b)\}. \tag{1}$$

Now, let $\omega_2 = \max\{u_{A+B}(a), u_{A+B}(b)\} = u_{A+B}(a)$ (say). Then for any $\epsilon > 0$,

$$\omega_2 - \epsilon < u_{A+B}(a) = \sup_{a = p \vee q} \{\min\{u_A(p), u_B(q)\}\}$$

So there exists a representation $a = p \vee q$ such that,

$$\omega_2 - \epsilon < \min\{u_A(p), u_B(q)\} \implies \omega_2 - \epsilon < u_A(p), \quad \omega_2 - \epsilon < u_B(q)$$

So for $b = r \vee s$, we have

$$\omega_2 - \epsilon < \max\{u_A(p), u_A(r \vee s)\} \leq u_A(p \wedge (r \vee s)), \quad \text{since } A \text{ is a PFI of } L$$

Also,

$$\omega_2 - \epsilon < \max\{u_B(q), u_B(r \vee s)\} \leq u_B(q \wedge (r \vee s)), \quad \text{since } B \text{ is a PFI of } L$$

Therefore,

$$\omega_2 - \epsilon < \min\{u_A(p \wedge (r \vee s)), u_B(q \wedge (r \vee s))\}$$

Note that, $a \wedge b = (p \vee q) \wedge (r \vee s) = (p \wedge (r \vee s)) \vee (q \wedge (r \vee s))$.
So,

$$u_{A+B}(a \wedge b) = \sup_{a \wedge b = m \vee n} \{\min\{u_A(m), u_B(n)\}\} \geq \min\{u_A(p \wedge (r \vee s)), u_B(q \wedge (r \vee s))\} > \omega_2 - \epsilon$$

Since $\epsilon > 0$ is arbitrary,

$$\mu_{A+B}(a \wedge b) \geq \omega_2 = \max\{u_{A+B}(a), u_{A+B}(b)\}. \tag{2}$$

Similarly, we get

$$w_{A+B}(a \vee b) \geq \omega_3 = \min\{w_{A+B}(a), w_{A+B}(b)\}. \tag{3}$$

$$w_{A+B}(a \wedge b) \geq \omega_4 = \max\{w_{A+B}(a), w_{A+B}(b)\}. \tag{4}$$

Again,
Let $\omega_5 = \max\{v_{A+B}(a), v_{A+B}(b)\}$. Then for any $\epsilon > 0$,

$$\omega_5 + \epsilon > v_{A+B}(a) = \inf_{a=p\vee q}\{\max\{v_A(p), v_B(q)\}\}$$

and

$$\omega_5 + \epsilon > v_{A+B}(b) = \inf_{b=r\vee s}\{\max\{v_A(r), v_B(s)\}\}$$

So there exist representations $a = p \vee q$ and $b = r \vee s$ such that,

$$\omega_5 + \epsilon > \max\{v_A(p), v_B(q)\}$$

and

$$\omega_5 + \epsilon > \max\{v_A(r), v_B(s)\}$$

Then,

$$\omega_5 + \epsilon > v_A(p), \quad \omega_5 + \epsilon > v_B(q), \quad \omega_5 + \epsilon > v_A(r), \quad \text{and} \quad \omega_5 + \epsilon > v_B(s)$$

Therefore,

$$\omega_5 + \epsilon > \max\{v_A(p), v_A(r)\} \geq v_A(p \vee r), \quad \text{since } A \text{ is a PFI of } L$$

Also,

$$\omega_5 + \epsilon > \max\{v_B(q), v_B(s)\} \geq v_B(q \vee s), \quad \text{since } B \text{ is a PFI of } L$$

Therefore,

$$\omega_5 + \epsilon > \max\{v_A(p \vee r), v_B(q \vee s)\}$$

Note that, $a \vee b = (p \vee q) \vee (r \vee s) = (p \vee r) \vee (q \vee s)$. So

$$\nu_{A+B}(a \vee b) = \inf_{a\vee b=m\vee n}\{\max\{v_A(m), v_B(n)\}\} \leq \max\{v_A(p \vee r), v_B(q \vee s)\} < \omega_5 + \epsilon$$

Since $\epsilon > 0$ is arbitrary,

$$\nu_{A+B}(a \vee b) \leq \omega_5 = \max\{v_{A+B}(a), v_{A+B}(b)\}. \tag{5}$$

Now, let $\omega_6 = \min\{v_{A+B}(a), v_{A+B}(b)\} = v_{A+B}(a)$ (say). Then for any $\epsilon > 0$,

$$\omega_6 + \epsilon > v_{A+B}(a) = \inf_{a=p\vee q}\{\max\{v_A(p), v_B(q)\}\}$$

So there exists a representation $a = p \vee q$ such that,

$$\omega_6 + \epsilon > \max\{v_A(p), v_B(q)\} \implies \omega_6 + \epsilon > v_A(p), \quad \omega_6 + \epsilon > v_B(q)$$

So for $b = r \vee s$, we have

$$\omega_6 + \epsilon > \min\{v_A(p), v_A(r \vee s)\} \geq v_A(p \wedge (r \vee s)), \quad \text{since } A \text{ is a PFI of } L$$

Also,

$$\omega_6 + \epsilon > \min\{v_B(q), v_B(r \vee s)\} \leq v_B(q \wedge (r \vee s)), \quad \text{since } B \text{ is a PFI of } L$$

Therefore,

$$\omega_6 + \epsilon > \max\{v_A(p \wedge (r \vee s)), v_B(q \wedge (r \vee s))\}$$

Note that, $a \wedge b = (p \vee q) \wedge (r \vee s) = (p \wedge (r \vee s)) \vee (q \wedge (r \vee s))$.
So,

$$\nu_{A+B}(a \wedge b) = \inf_{a \wedge b = m \vee n} \{\max\{\nu_A(m), \nu_B(n)\}\} \leq \max\{v_A(p \wedge (r \vee s)), v_B(q \wedge (r \vee s))\} < \omega_6 + \epsilon$$

Since $\epsilon > 0$ is arbitrary,

$$v_{A+B}(a \wedge b) \leq \omega_6 = \min\{v_{A+B}(a), v_{A+B}(b)\}. \tag{6}$$

From (1), (2), (3), (4), (5), and (6), it is proved that $A + B$ is a PFI of L.
□

Definition 4.3. *Let $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ and $B = \{(a, u_B(a), w_B(a), v_B(a)) : a \in L\}$ be two PFSs of L. Then their product $A \otimes B$ is defined as*

$$A \otimes B = \{(a, u_{A \otimes B}(a), w_{A \otimes B}(a), v_{A \otimes B}(a)) : a \in L\}$$

where,

$$u_{A \otimes B}(a) = \sup_{a = p \wedge q} \{\min\{u_A(p), u_B(q)\}\}$$

$$w_{A \otimes B}(a) = \sup_{a = p \wedge q} \{\min\{w_A(p), w_B(q)\}\}$$

$$v_{A \otimes B}(a) = \inf_{a = p \wedge q} \{\max\{v_A(p), v_B(q)\}\}$$

Theorem 4.4. *The product of two PFIs in a distributive lattice L is again a PFI of L.*

Proof. Same as Theorem 4.2. □

5 Picture Fuzzy Ideals and Homomorphism

Definition 5.1. *Let $f : L \to L'$ be a mapping and $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ be a PFS of L. Then the image $f(A)$ is defined by*

$$f(A) = \{(y, u_{f(A)}(b), w_{f(A)}(b), v_{f(A)}(b)) : b \in L'\},$$

where

$$u_{f(A)}(b) = \begin{cases} \sup\{u_A(a) : a \in f^{-1}(b)\} & ; f^{-1}(b) \neq \emptyset \\ 0 & ; Otherwise \end{cases}$$

$$w_{f(A)}(b) = \begin{cases} \inf\{w_A(a) : a \in f^{-1}(b)\} & ; f^{-1}(b) \neq \emptyset \\ 0 & ; Otherwise \end{cases}$$

$$v_{f(A)}(b) = \begin{cases} \inf\{v_A(a) : a \in f^{-1}(b)\} & ; f^{-1}(b) \neq \emptyset \\ 0 & ; Otherwise \end{cases}$$

Similarly, if $B = \{(b, u_B(b), w_B(b), v_B(b)) : b \in L'\}$ *is a PFS of* L'*, then*

$$f^{-1}(B) = \{(a, u_{f^{-1}(B)}(a), w_{f^{-1}(B)}(a), v_{f^{-1}(B)}(a)) : a \in L\},$$

where

$$u_{f^{-1}(B)}(a) = u_B(f(a)), \quad w_{f^{-1}(B)}(a) = w_B(f(a)), \quad v_{f^{-1}(B)}(a) = v_B(f(a)).$$

Theorem 5.2. *If* $f : L \to L'$ *is a lattice epimorphism (onto homomorphism) and* A *is a PFI of* L*, then* $f(A)$ *is a PFI of* L'*.*

Proof.

Let $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ be a PFI of L. Then

$$f(A) = \{(b, u_{f(A)}(b), w_{f(A)}(b), v_{f(A)}(b)) : b \in L'\}.$$

Let $b, c \in L'$. Then

$$u_{f(A)}(b \vee c) = \sup\{u_A(a) : a \in f^{-1}(b \vee c)\}$$

$$\geq \sup\{u_A(m \vee n) : m \in f^{-1}(b), n \in f^{-1}(c)\}$$

$$\geq \sup\{\min\{u_A(m), u_A(n)\} : m \in f^{-1}(b), n \in f^{-1}(c)\}$$

$$= \min\{\sup u_A(m) : m \in f^{-1}(b), \sup u_A(n) : n \in f^{-1}(c)\}$$

$$= \min\{u_{f(A)}(b), u_{f(A)}(c)\}, \text{ since } A \text{ is a PFI of } L$$

$$\therefore u_{f(A)}(b \vee c) \geq \min\{u_{f(A)}(b), u_{f(A)}(c)\}.$$

Also,

$$u_{f(A)}(b \wedge c) = \sup\{u_A(a) : a \in f^{-1}(b \wedge c)\}$$

$$\geq \sup\{u_A(m \wedge n) : m \in f^{-1}(b), n \in f^{-1}(c)\}$$

$$\geq \sup\{\max\{u_A(m), u_A(n)\} : m \in f^{-1}(b), n \in f^{-1}(c)\}$$

$$= \max\{\sup u_A(m) : m \in f^{-1}(b), \sup u_A(n) : n \in f^{-1}(c)\}$$

$$= \max\{u_{f(A)}(b), u_{f(A)}(c)\}, \text{ since } A \text{ is a PFI of } L$$

$$\therefore u_{f(A)}(b \wedge c) \geq \max\{u_{f(A)}(b), \mu_{f(A)}(c)\}.$$

Again,

$$w_{f(A)}(b \vee c) = \inf\{w_A(a) : a \in f^{-1}(b \vee c)\}$$

$$\leq \inf\{w_A(m \vee n) : m \in f^{-1}(b), n \in f^{-1}(c)\}$$

$$\leq \inf\{\max\{w_A(m), w_A(n)\} : m \in f^{-1}(b), n \in f^{-1}(c)\}$$

$$= \max\{\inf w_A(m) : m \in f^{-1}(b), \inf w_A(n) : n \in f^{-1}(c)\}$$

$$= \max\{w_{f(A)}(b), w_{f(A)}(c)\}, \text{ since } A \text{ is a PFI of } L$$

$$\therefore w_{f(A)}(b \vee c) \leq \max\{w_{f(A)}(b), w_{f(A)}(c)\}.$$

□

Definition 5.3. *Let $f : L \to L'$ be a function and $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ be a PFS of L. Then A is said to be f-invariant if*

$$f(a) = f(b) \Rightarrow u_A(a) = u_A(b), w_A(a) = w_A(b), \text{ and } v_A(a) = v_A(b)$$

Proposition 5.4. *If a PFS A is f-invariant, then $f^{-1}[f(A)] = A$.*

Proof. Proof: Same as Theorem 5.4. □

Theorem 5.5. *Let $f : L \to L'$ be a function and A, B be two PFSs of L and A', B' be two PFSs of L'. Then*

$$A \subseteq B \Rightarrow f(A) \subseteq f(B);$$

$$A' \subseteq B' \Rightarrow f^{-1}(A') \subseteq f^{-1}(B').$$

Proof.

(i) Let $A = \{(a, u_A(a), w_A(a), v_A(a)) : a \in L\}$ and $B = \{(a, u_B(a), w_B(a), v_B(a)) : a \in L\}$ be two PFSs of L. Then

$$A \subseteq B \Rightarrow u_A(a) \leq u_B(a), w_A(a) \leq w_B(a), \text{ and } v_A(a) \geq v_B(a).$$

Then

$$f(A) = \{(b, u_{f(A)}(b), w_{f(A)}(b), v_{f(A)}(b)) : b \in L'\}$$

and

$$f(B) = \{(b, u_{f(B)}(b), w_{f(B)}(b), v_{f(B)}(b)) : b \in L'\}.$$

Now,

$$u_{f(A)}(b) = \sup\{u_A(a) : a \in f^{-1}(b)\}$$

$$\leq \sup\{u_B(a) : a \in f^{-1}(b)\}, \text{ since } A \subseteq B \text{ and } u_A(a) \leq u_B(a)$$

$$= u_{f(B)}(b)$$

$$\therefore u_{f(A)}(b) \leq u_{f(B)}(b)$$

$$w_{f(A)}(b) = \inf\{w_A(a) : a \in f^{-1}(b)\}$$

$$\leq \inf\{w_B(a) : a \in f^{-1}(b)\}, \text{ since } A \subseteq B \text{ and } w_A(a) \leq w_B(a)$$

$$= w_{f(B)}(b)$$

$$\therefore w_{f(A)}(b) \leq w_{f(B)}(b)$$

and

$$v_{f(A)}(b) = \inf\{v_A(a) : a \in f^{-1}(b)\}$$

$$\geq \inf\{v_B(a) : a \in f^{-1}(b)\}, \text{ since } A \subseteq B \text{ and } v_A(a) \geq v_B(a)$$

$$= v_{f(B)}(b)$$

$$\therefore v_{f(A)}(b) \geq v_{f(B)}(b)$$

Hence, $f(A) \subseteq f(B)$.

□

6 Conclusion

Picture fuzzy set is capable enough to handle uncertain situation. Since the innovation of this concept, a host of researchers have involved to develop this concept in several dimensions. In this work, the notion of picture fuzzy sublattices and picture fuzzy ideals are introduced with some of their properties. In addition, the sum and product of two picture fuzzy ideals are defined and some of their properties are described. Finally, some properties of picture fuzzy ideals under lattice homomorphism are established. In future, the outcomes of this paper will open diverse areas to explore more algebraic structures and their properties in boolean algebra in terms of picture fuzzy fields with applications especially in switching circuits.

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