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Domination numbers and diameters in certain graphs

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Abstract. Regarding the problem mentioned by Brigham et al. "Is it correct that each connected bicritical graph possesses a minimum dominating set having every two appointed vertices of graphs?", we first give a class of graphs that disprove it and second obtain domination numbers and diameters of the graphs of this class. This class of graphs has the property: $\omega(\mathcal{H}) - diam(\mathcal{H}) \to \infty$ when $|\mathcal{V}(\mathcal{H})| = n \to \infty$. Also, for the bicritical graphs of this class, $i(\mathcal{H}) = \omega(\mathcal{H})$.

Keywords: Domination number, bicritical graph, diameter.

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1. Introduction

Presume $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is a graph. Regarding the basic concept mentioned in [2, 6], we first will review some preliminary definitions. $\mathcal{T} \subset \mathcal{V}$ is named a dominating set whenever all vertexes are in $\mathcal{T}$ or are adjacent to a vertex in $\mathcal{T}$, i.e. $\mathcal{V} = \bigcup_{s\in\mathcal{T}} N[s]$. What's more, domination number $\omega(\mathcal{H})$ will be the minimum cardinality of a dominating set of $\mathcal{H}$ and a dominating set of minimum cardinality will be named a $\omega(\mathcal{H}) - set$. A dominating set $\mathcal{T}$ of $\mathcal{H}$ is independent when there exists no two vertices of $\mathcal{T}$ which are adjacent. The minimum cardinality between independent dominating sets of $\mathcal{H}$ is independent domination number $i(\mathcal{H})$. We indicate distance between two vertices $p, q \in \mathcal{V}(\mathcal{H})$ by $d_{\mathcal{H}}(p, q)$. Notice that deleting a vertex can enhance domination number by more than one, but can reduce it by at most one. Also, connectivity of $\mathcal{H}$, considered by $\kappa(\mathcal{H})$, will be the minimum size of $\mathcal{T}$ provided that $\mathcal{H} - \mathcal{T}$ is disconnected or possesses just a vertex. $\mathcal{H}$ will be k-connected if its connectivity is at least k, and it's k-edge-connected when

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each disconnecting set possesses at least k edges. The edge-connectivity of $\mathcal{H}$, given by $\lambda(\mathcal{H})$, will be the minimum size of a disconnecting set. The circulant graph $\mathcal{C}_{n+1}\langle 1,4\rangle$ will be a graph with $\mathcal{V}=\{x_0,\ldots,x_n\}$ and $\mathcal{E}=\{x_kx_{k+l\,(mod\,n+1)}|\ k\in\{0,\ldots,n\}, l\in\{1,4\}\}$. For example, see Figure 1. Also, more details can be found in [2, 3, 5, 9] and references therein.

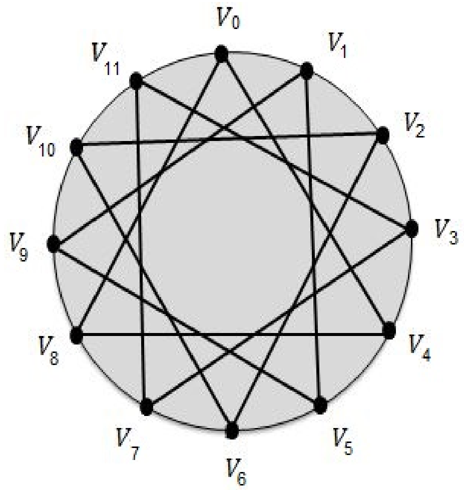

Figure 1. $\mathcal{C}_{12}\langle 1,4\rangle$

Proposition 1.1 [2] For a bicritical graph $\mathcal{H}$ and $p,q\in\mathcal{V}(\mathcal{H})$,

$$\omega(\mathcal{H})-2\leqslant\omega(\mathcal{H}-\{p,q\})\leqslant\omega(\mathcal{H})-1.$$

Since 2005, many researcher have discussed open problems of [2] (for example, see [1, 4, 7, 8, 10]). Also, in 2013, Mojdeh et al. [6] answered these questions by considering their hypothesis. In this paper, we answer this question by disapproving the problem for a class of graphs that seems more simple than previous works.

2. Results

First, let us study the domination number and the diameter of $\mathcal{C}_{n+1}\langle 1,4\rangle$, and verify their relation.

Lemma 2.1 $\omega(\mathcal{C}_{n+1}\langle 1,4\rangle)\leqslant\begin{cases}2\left\lfloor\frac{n}{9}\right\rfloor+3 & n\equiv 7\ (mod\ 9)\\ 2\left\lfloor\frac{n}{9}\right\rfloor+1 & n\equiv 1 \text{ or } 0\ (mod\ 9)\\ 3 & n=12\\ 2\left\lfloor\frac{n}{9}\right\rfloor+2 & o.w.\end{cases}$.

Proof. Let $n\geqslant 7$ be an integer and $\mathcal{T}$ be one of the following sets. Then $\mathcal{T}$ is a dominating set for corresponding n.

(a) $n=9k+8$, $\mathcal{T}=\{v_0,v_7,v_9,v_{16},v_{18},\ldots,v_{9k-2},v_{9k},v_{9k+7}\}$,
(b) $n=9k+7$, $\mathcal{T}=\{v_0,v_7,v_9,v_{16},v_{18},\ldots,v_{9k},v_{9k+6},v_{9k+7}\}$,
(c) $n=9k+6$, $\mathcal{T}=\{v_0,v_7,v_9,v_{16},v_{18},\ldots,v_{9k-2},v_{9k},v_{9k+5}\}$,
(d) $n=9k+5$, $\mathcal{T}=\{v_0,v_7,v_9,v_{16},v_{18},\ldots,v_{9k-2},v_{9k},v_{9k+4}\}$,

(e) $n = 9k + 4$, $\mathcal{T} = \{v_0, v_7, v_9, v_{16}, v_{18}, \ldots, v_{9k-2}, v_{9k}, v_{9k+3}\}$,
(f) $n = 9k + 3$, $\mathcal{T} = \{v_0, v_7, v_9, v_{16}, v_{18}, \ldots, v_{9k-2}, v_{9k}, v_{9k+2}\}$,
(g) $n = 9k + 2$, $\mathcal{T} = \{v_0, v_7, v_9, v_{16}, v_{18}, \ldots, v_{9k-2}, v_{9k}, v_{9k+1}\}$,
(h) $n = 9k + 1$, $\mathcal{T} = \{v_0, v_7, v_9, v_{16}, v_{18}, \ldots, v_{9k-9}, v_{9k-2}, v_{9k}\}$,
(i) $n = 9k$, $\mathcal{T} = \{v_0, v_7, v_9, v_{16}, v_{18}, \ldots, v_{9k-9}, v_{9k-2}, v_{9k-1}\}$.

This process shows that $\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) \leqslant \begin{cases} 2\left\lfloor \frac{n}{9} \right\rfloor + 3 & n \equiv 7 \ (mod\ 9) \\ 2\left\lfloor \frac{n}{9} \right\rfloor + 1 & n \equiv 1 \text{ or } 0 \ (mod\ 9) \\ 3 & n = 12 \\ 2\left\lfloor \frac{n}{9} \right\rfloor + 2 & o.w. \end{cases}$. ■

Theorem 2.2 $\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) = \begin{cases} 2\left\lfloor \frac{n}{9} \right\rfloor + 3 & n \equiv 7 \ (mod\ 9) \\ 2\left\lfloor \frac{n}{9} \right\rfloor + 1 & n \equiv 1 \text{ or } 0 \ (mod\ 9) \\ 3 & n = 12 \\ 2\left\lfloor \frac{n}{9} \right\rfloor + 2 & o.w. \end{cases}$.

Proof. For equality, each vertex v_i for $0 \leqslant i \leqslant n$ dominates $\{v_{i-1}, v_{i+1}, v_{i+4}, v_{i-4}\}$, but the vertices v_{i-2} and v_{i+2} are nearest remaining vertices to v_i, which aren't dominated by v_i. If we choose v_{i-2}, it dominates $\{v_{i-1}, v_{i-3}, v_{i+2}, v_{i-6}\}$, and the vertices v_{i+7} are not dominated. If we choose v_{i+3}, then it dominates at most two new vertices. But if we choose v_{i+7}, then four vertices $\{v_{i+3}, v_{i+11}, v_{i+8}, v_{i+6}\}$ will be dominated. If we choose v_{i+5}, at most two new vertices will be dominated. Therefore, we pick v_{i+9} that dominates $\{v_{i+10}, v_{i+13}, v_{i+5}, v_{i+9}\}$. In this case, by choosing four vertices $\{v_i, v_{i-2}, v_{i+7}, v_{i+9}\}$, twenty vertices dominated and the vertices v_{i+12} and v_{i+16} remain. If we select v_{i+12}, at most two vertices will be dominated. But if we pick v_{i+16}, exactly five vertices with itself will be dominated and the vertices v_{i+14} and v_{i+18} remain. If we choose v_{i+14}, at most two vertices will be dominated. But if we select v_{i+18}, four vertices with itself will be dominated. Up to here, with 6 vertices $\{v_i, v_{i-2}, v_{i+7}, v_{i+9}, v_{i+16}, v_{i+18}\}$, at least 29 vertices be dominated. ■

Theorem 2.3 The graph $\mathcal{C}_{n+1}\langle 1,4\rangle$ is bicritical for $n+1 = 9k+3, 9k+4, 9k+8$ in which $k \geqslant 1$.

Proof.

(a) If $n+1 = 9k+3$, then $\mathcal{T} = \{9k+1, 0, 7, 9, \ldots, 9k-2, 9k\}$ is a dominating set for $(\mathcal{C}_{n+1}\langle 1,4\rangle)$. On the other hand

$$\mathcal{T}_1 = \{3, 6, 8, 15, \ldots, 9k-1\},$$
$$\mathcal{T}_2 = \{2, 5, 7, 14, 16, 23, 25, \ldots, 9k-2\},$$
$$\mathcal{T}_3 = \{3, 7, 9, 16, 18, \ldots, 9k\}$$

are dominating sets for $\mathcal{C}_{n+1}\langle 1,4\rangle - \{v_0, v_i\}$ for some i and one of these can be a dominating sets for $\mathcal{C}_{n-1}\langle 1,4\rangle$.

(b) If $n+1 = 9k+8$, then $\mathcal{T} = \{9k+7, 0, 7, 9, \ldots, 9k+6\}$ is a dominating set for $\mathcal{C}_{n+1}\langle 1,4\rangle$.

On the other hand,

$$\begin{aligned}
\mathcal{T}_1 &= \{3, 5, 12, 14, 21, \ldots, 9k+5\},\\
\mathcal{T}_2 &= \{5, 7, 14, 16, 23, \ldots, 9k+7\},\\
\mathcal{T}_3 &= \{6, 5, 12, 14, 21, \ldots, 9k+5\},\\
\mathcal{T}_4 &= \{2, 8, 9, 15, 21, 28, 30, 37, 39, 46, 48, \ldots, 9k+1, 9k+3\},\\
\mathcal{T}_5 &= \{2, 3, 10, 12, 19, \ldots, 9k+3\}
\end{aligned}$$

are dominating sets for $\mathcal{C}_{n+1}\langle 1,4\rangle - \{v_0, v_i\}$ for some i. One of these can be a dominating set for $\mathcal{C}_{n-1}\langle 1,4\rangle$.

(c) If $n+1 = 9k+4$, then $\mathcal{T} = \{9k+2, 0, 7, 9, \ldots, 9k\}$ is a dominating set for $\mathcal{C}_{n+1}\langle 1,4\rangle$. On the other hand,

$$\begin{aligned}
\mathcal{T}_1 &= \{2, 6, 8, 15, 17, \ldots, 9k-1\},\\
\mathcal{T}_2 &= \{2, 4, 11, 13, \ldots, 9k-7, 9k-5, 9k+2\},\\
\mathcal{T}_3 &= \{2, 9, 8, 15, 17, \ldots, 9k-1\}
\end{aligned}$$

are dominating sets for $\mathcal{C}_{n+1}\langle 1,4\rangle - \{v_0, v_i\}$ for some i and one of these can be a dominating set for $\mathcal{C}_{n-1}\langle 1,4\rangle$.

In all cases of $\mathcal{T}$, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5$ are a ω-set for $\mathcal{C}_{n+1}\langle 1,4\rangle - \{v_0, v_i\}$ which cardinality is less than $(\omega(\mathcal{C}_{n+1}\langle 1,4\rangle))$. Thus, $\omega(\mathcal{C}_{n+1}\langle 1,4\rangle - \{v_0, v_i\})\omega(\mathcal{C}_{n+1}\langle 1,4\rangle)$ and the proof ends. ∎

Theorem 2.4 $\kappa(\mathcal{C}_{n+1}\langle 1,4\rangle) = 4$, where $n \geqslant 8$.

Proof. Presume $\mathcal{H} = \mathcal{C}_{n+1}\langle 1,4\rangle$ where $n \geqslant 8$. As $\delta(\mathcal{H}) = 4$, it's enough to demonstrate that $\kappa(\mathcal{H}) \geqslant 4$. On the contrary, assume that $\mathcal{T} \subset \mathcal{V}(\mathcal{H})$ with $|\mathcal{T}| < 4$. We show $\mathcal{H} - \mathcal{T}$ is connected. Take $\mathcal{M}, \mathcal{N} \in \mathcal{V}(\mathcal{H}) - \mathcal{T}$. The original circular arrangement contains a clockwise $\mathcal{M}, \mathcal{N}$ path and a counter clockwise $\mathcal{M}, \mathcal{N}$ path a long the circle. Assume that $\mathcal{A}$ and $\mathcal{B}$ are set of internal vertices on these two paths. As $|\mathcal{T}| < 4$, the pigeon hole principle induces that one of $\mathcal{A}, \mathcal{B}, \mathcal{T}$ has fewer than 3 vertices. Note that each vertex in $\mathcal{H}$ contains edges to the next 4 vertices in a specific direction deleting fewer than 3 consecutive vertices cannot travel that direction. Hence, we can find a $\mathcal{M}, \mathcal{N}$ path in $\mathcal{H} - \mathcal{T}$ via $\mathcal{A}$ or $\mathcal{B}$, where $\mathcal{T}$ has fewer than 4 vertices. On the other hand, $\mathcal{C}_{n+1}\langle 1,4\rangle$ is connective by deleting 4 vertices. It is sufficient to delete the adjacent of one vertex. Thus, $\kappa(\mathcal{C}_{n+1}\langle 1,4\rangle) = 4$. ∎

Theorem 2.5 $\kappa'(\mathcal{C}_{n+1}\langle 1,4\rangle) = 4$, where $n \geqslant 8$.

Proof. It is well known that $\kappa \leqslant \kappa' \leqslant \delta$. Therefore, $4 \leqslant \kappa' \leqslant 4$ and $\kappa' = 4$. ∎

Theorem 2.6

$$diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = \begin{cases} \left\lceil \left\lfloor \frac{n+1}{2} \right\rfloor \div 4 \right\rceil + 1 \text{ when } 4 \mid \left\lfloor \frac{n+1}{2} \right\rfloor \\ \left\lceil \left\lfloor \frac{n+1}{2} \right\rfloor \div 4 \right\rceil + 1 \text{ when } 4 \nmid \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \text{ and } 4 \nmid \left\lfloor \frac{n+1}{2} \right\rfloor \\ \left\lceil \left\lfloor \frac{n+1}{2} \right\rfloor \div 4 \right\rceil \text{ when } 4 \mid \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \text{ and } 4 \nmid \left\lfloor \frac{n+1}{2} \right\rfloor \end{cases}.$$

Proof. Since vertex v_i is adjacent to v_{i+4}, v_{i-4} in $\mathcal{C}_{n+1}\langle 1, 4\rangle$, we have for $k \geqslant 1$ that

- if $\lfloor \frac{n+1}{2} \rfloor = 4k$, then
$$diam(\mathcal{C}_{n+1}\langle 1, 4\rangle) = d(v_0, v_{4k-1}) = d(v_0, v_{4k}) + d(v_{4k}, v_{4k-1}) = k + 1;$$
- if $\lfloor \frac{n+1}{2} \rfloor = 4k + 1$, then
$$diam(\mathcal{C}_{n+1}\langle 1, 4\rangle) = d(v_0, v_{4k-1}) = d(v_0, v_{4k}) + d(v_{4k}, v_{4k-1}) = k + 1;$$
- if $\lfloor \frac{n+1}{2} \rfloor = 4k + 2$, then
$$diam(\mathcal{C}_{n+1}\langle 1, 4\rangle) = d(v_0, v_{4k+2}) = d(v_0, v_{4k}) + d(v_{4k}, v_{4k+2}) = k + 2;$$
- if $\lfloor \frac{n+1}{2} \rfloor = 4k + 3$, then
$$diam(\mathcal{C}_{n+1}\langle 1, 4\rangle) = d(v_0, v_{4k+2}) = d(v_0, v_{4k}) + d(v_{4k}, v_{4k+2}) = k + 2;$$

that these lengths show the diameter in $\mathcal{C}_{n+1}\langle 1, 4\rangle$. ■

Theorem 2.7 In $\mathcal{C}_{n+1}\langle 1, 4\rangle$ for $n \geqslant 7$, we have $diam < \omega$.

Proof. If $n + 1 = 8k$ and $4 \mid \lfloor \frac{n+1}{2} \rfloor$, we have

$$n = 72t + 7, 72t + 15, 72t + 23, 72t + 31, 72t + 39, 72t + 47, 72t + 55, 72t + 63, 72t + 71.$$

If $n = 72t+7$, then $\omega = 16t+3$ and $diam = 9t+2$. Thus, $\omega - diam = 7t+1$. An identical computation demonstrates that

$$\omega - diam = 7t + 2, 7t + 3, 7t + 4, 7t + 5, 7t + 6, 7t + 8.$$

In this case, it is clearly implied

$$\lim_{n\to\infty} (\omega(\mathcal{C}_{n+1}\langle 1, 4\rangle) - diam(\mathcal{C}_{n+1}\langle 1, 4\rangle)) \to \infty.$$

If $n + 1 = 2k$ and $4 \nmid \lfloor \frac{n+1}{2} \rfloor - 1$, we have $n = 8m + 3$ or $n = 8m + 5$. If $n = 8m + 3$, then

$$n = 72t + 11, 72t + 19, 72t + 27, 72t + 35, 72t + 43, 72t + 51, 72t + 59, 72t + 67, 72t + 75.$$

If $n = 72t + 11$, then $\omega = 16t + 4$ and $d = 9t + 3$. Thus,

$$\omega(\mathcal{C}_{n+1}\langle 1, 4\rangle) - diam(\mathcal{C}_{n+1}\langle 1, 4\rangle) = 7t + 1.$$

An identical calculation shows that

$$\omega(\mathcal{C}_{n+1}\langle 1, 4\rangle) - diam(\mathcal{C}_{n+1}\langle 1, 4\rangle) = 71 + 2, 7t + 4, 7t + 5, 7t + 6, 7t + 7.$$

If $n = 8m + 5$,

$$n = 72t + 13, 72t + 21, 72t + 29, 72t + 37, 72t + 45, 72t + 53, 72t + 61, 72t + 69, 72t + 77.$$

Now, if $n = 72t + 13$, then $\omega = 16t + 4$ and $d = 9t + 3$. Thus,

$$\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = 7t + 1.$$

An identical calculation shows that

$$\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = 7t + 2, 7t + 3, 7t + 4, 7t + 6, 7t + 7.$$

If $n + 1 = 2k$ and $4 \nmid \left\lfloor \frac{n+1}{2} \right\rfloor, 4 \left\lfloor \frac{n+1}{2} \right\rfloor - 1$, we have $n = 8m + 1$ and then

$$n = 72t + 9, 72t + 17, 72t + 25, 72t + 33, 72t + 41, 72t + 49, 72t + 57, 72t + 65, 72t + 73.$$

If $n = 72t + 9$, then $\omega = 16t + 3$ and $d = 9t + 2$. Thus,

$$\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = 71t + 1.$$

An identical calculation shows that

$$\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = 7t + 3, 7t + 4, 7t + 5, 7t + 6, 7t + 7.$$

If $n + 1 = 2k + 1$ and $4 \mid \left\lfloor \frac{n+1}{2} \right\rfloor, 4 \left\lfloor \frac{n+1}{2} \right\rfloor$, we have n= $8m$ and then

$$n = 72t + 8, 72t + 16, 72t + 24, 72t + 32, 72t + 40, 72t + 48, 72t + 56, 72t + 64, 72t + 72.$$

If $n = 72t + 8$, then $\omega = 16t + 2$ and $d = 9t + 2$. Thus,

$$\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = 7t.$$

An identical calculation shows that

$$\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = 7t + 2, 7t + 3, 7t + 4, 7t + 5, 7t + 6, 7t + 7.$$

If $n + 1 = 2k + 1$ and $4 \nmid \left\lfloor \frac{n+1}{2} \right\rfloor, 4 \left\lfloor \frac{n+1}{2} \right\rfloor - 1$, we have $n = 8m + 4$ or $n = 8m + 6$. If $n = 8m + 4$, then

$$n = 72t + 12, 72t + 20, 72t + 28, 72t + 36, 72t + 44, 72t + 52, 72t + 60, 72t + 68, 72t + 76.$$

If $n = 72t + 12$, then $\omega = 16t + 4, d = 9t + 3$. Thus,

$$\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = 7t + 1.$$

An identical calculation shows that

$$\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = 7t + 2, 7t + 3, 7t + 5, 7t + 6, 7t + 7.$$

Now, if $n = 8m + 6$,

$$n = 72t + 14, 72t + 22, 72t + 30, 72t + 38, 72t + 46, 72t + 54, 72t + 62, 72t + 7, 72t + 78.$$

If $n = 72t + 14$, then $\omega = 16t + 4$ and $d = 9t + 3$. Thus,

$$\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = 7t + 1.$$

An identical calculation shows that

$$\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = 7t + 2, 7t + 3, 7t + 4, 7t + 5, 7t + 7.$$

If $n + 1 = 2k + 1$, then $4 \left\lfloor \frac{n+1}{2} \right\rfloor - 1, 4 \nmid \left\lfloor \frac{n+1}{2} \right\rfloor$ and $n = 8m + 2$. Then

$$n = 72t + 10, 72t + 18, 72t + 26, 72t + 34, 72t + 42, 72t + 50, 72t + 58, 72t + 66, 72t + 74.$$

If $n = 72t + 10$, then $\omega = 16t + 3$ and $d = 9t + 2$. Thus,

$$\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = 7t + 1.$$

An identical calculation shows that

$$\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle) = 7t + 2, 7t + 4, 7t + 5, 7t + 6, 7t + 7, 7t + 8.$$

All of them show that $(\omega(\mathcal{C}_{n+1}\langle 1,4\rangle) - diam(\mathcal{C}_{n+1}\langle 1,4\rangle)) \to \infty$ when $|\mathcal{V}(\mathcal{H})| = n \to \infty$. ■

Corollary 2.8 Given the problem mentioned above, we found bicritical graphs such as $\mathcal{C}_{n+1}\langle 1,4\rangle$ for $n + 1 = 9k + 3, 9k + 4, 9k + 8$ that have the property: $\omega(\mathcal{H}) = i(\mathcal{H})$ and so we could disprove the validity of the problem mentioned in abstract.

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