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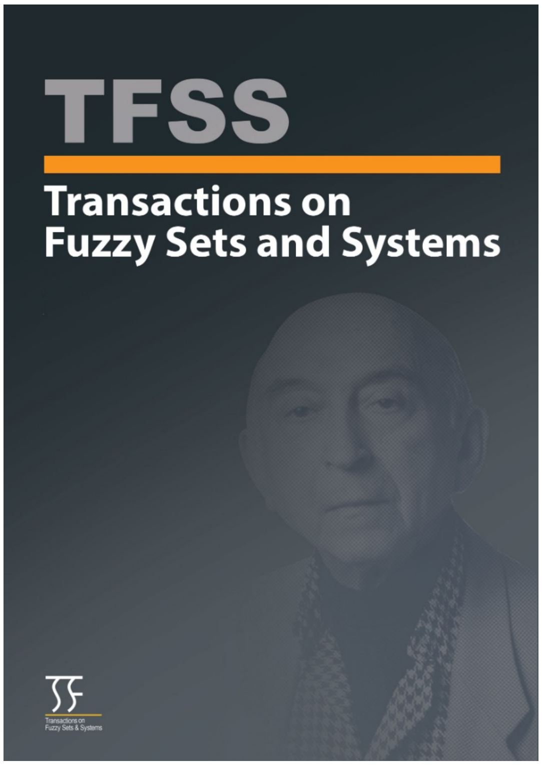

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A Theoretical Development of Linear Diophantine Fuzzy Graph Structures

Saba Ayub*, **Muhammad Shabir**

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. Graph structure (GS) is an advancement of the graph concept which effectively represents intricate situations with various connections, frequently used in computer science and mathematics to illustrate relationships among objects and extensively researched in fuzzy sets (FS), intuitionistic fuzzy set (IFS), pythagorean fuzzy set (PFS) and q-rung orthopair fuzzy set (q-ROFS). Meanwhile, a linear Diophantine fuzzy set (LDFS) is a remarkable extension of the existing notions of a FS, IFS, PFS and q-ROFS by comporting reference parameters that removed all the limitations related to membership degree (MD) and non-membership degree (NMD). According to the best of our knowledge, there is a lack of elegantly proposed GS extension for LDFSs in the current literature. As a result, this research focuses on introducing first linear Diophantine fuzzy graph structure (LDFGS) concept which extends the existing notions of GS in various contexts of FSs. Several key concepts in LDFGSs are presented, such as $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, strength of $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, vertex degree, total $\breve{\rho}_i$-degree of a vertex, and total vertex degree in an LDFGS. In addition, we introduce the $\breve{\rho}_i$-size, size, and order of an LDFGS. Moreover, this article presents the ideas of the maximal product of two LDFGSs, strong LDFGS, degree and $\breve{\rho}_i$-degree of the maximal product, $\breve{\rho}_i$-regular and regular LDFGSs, along with examples for clarification. Certain significant results related to the proposed concepts also demonstrated with explanatory examples such as the maximal product of two strong LDFGSs is also a strong LDFGS, the maximal product of two connected LDFGSs is also a connected LDFGS but the maximal product of two regular LDFGS may not be a regular LDGS. Moreover, many interesting and alternative formulas for calculating $\breve{\rho}_i$-degrees of an LDFGS in various situations are proved with examples. LDFGSs are highly beneficial for solving numerous combinatorial problems involving multiple relations, and they surpass existing concepts of GSs within the FS context due to their flexibility in selecting MD and NMD alongside their reference parameters.

AMS Subject Classification 2020: 03B52; 03E72; 28E10; 18B35
Keywords and Phrases: Linear Diophantine fuzzy sets, Graph structure, Maximal product, Degree of a vertex, Total degree of a vertex.

1 Introduction

Incorporating uncertainties into real-world applications has become essential for addressing a variety of practical issues such as data analysis, computational intelligence, and sustainability. In 1965, Zadeh [1] poineered the concept of FS and fuzzy logic for modelling uncertain situations by assigning the MD to each object rather than absolute membership and absolute non-membership. Since then, FS theory have been studied by

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scholars and scientists in a wide range of practical fields including artificial intelligence, medical science, computational sciences and decision analysis [2, 3, 4]. Since MD is not sufficient to describe many real situations, there is a need for NMD such as educated and uneducated, perfection and imperfection, sick and healthy, etc. In order to deal with such situations, Attanassov [5, 6] proposed the idea of IFS with the addition of NMD such that the sum of MD and NMD is not greater than one. Due to the large space of MD and NMD, IFSs were studied enormously in various fields of applications [7, 8]. However, there are still many real-life problems where the condition of IFS is not satisfied. For instance, a professional is asked to comment on the viability of a strategy to invest in the real estate industry. Imagine that the expert rates this investment plan's degree of feasibility at 0.8 and its degree of impossibility at 0.6. Since $0.8 + 0.6 > 1$, the IFS cannot be utilized to appropriately express this information. As a result, Yager [9] presented the idea of PFS, which meets the requirement that the sum of squares of the MD and the NMD is less than equal to 1 for each element. But if the decision-maker expresses his view as 0.9 for agree and 0.8 for not agree, we can see that $0.9^2 + 0.8^2 > 1$. To deal with the situations, Yager [10] investigated q-ROFS as a more generic version of IFS and PFS. In q-ROFSs, the total of the q-powers for truthfulness and falsehood grades is kept within a unit interval. This indicates that q-ROFSs provide additional data storage to characterize ambiguous or unclear facts. Researchers have given the PFS theory and q-ROFS theory a lot of attention over the past five years, and numerous insightful theoretical and practical findings have been made in a variety of fields. For instance, Yager [11] presented a multi-attribute decision making technique for PFS. Khan et al. [12] developed a new ranking technique for q-ROFSs based on entropy function and hesitancy index with a detailed critical analysis of the previously ranking methods. Liu and Wang [13] proposed some q-ROF aggregation operators and utilized them to solve multi-attribute decision making problems.

Although MD and NMDs are subject to certain restraints under the theories mentioned earlier of IFS, PFS, and q-ROFS. To overcome all these restrictions associated with MD and NMD, Riaz and Hashmi [14] introduced an augmented generalized form of FS known as LDFS with the inclusion of reference parameters. Due to the inclusion of reference or control parameters, LDFSs have a wide space of MD and NMD, in contrast to the commonly used ongoing conceptions, made this theory more advanced, trustworthy and easy to model uncertainties. Due to the advancement of LDFSs and its freedom regarding MD and NMD, various scientists have started to create fresh theories about this emerging and sophisticated concept. For instance, Almagrabi et al. [15] established the concept of q-linear Diophantine fuzzy set (q-LDFS) and its application in emergency decision support system for COVID19. Ayub et al. [16] introduced the notion of linear Diophantine fuzzy relations (LDFRs) and studied their algebraic structures with an application in decision making. Further Ayub et al. [17, 18] studied the roughness of a crisp set by using the level sets of an LDFR and by $\big(\langle s,t\rangle,\langle u,v\rangle\big)$-indiscernibility of an LDFR over dual universes, respectively. A comprehensive details on the study of rough approximations of an LDFS via an LDFR, inituitionistic fuzzy relation (IFR) and fuzzy relation (FR) together with their applications in the field of decision making, respectively, have presented in [19, 20, 21]. Gül and Aydoğdu [22] proposed linear Diophantine fuzzy TOPSIS (LDF-TOPSIS) based on some novel distance and entropy definitions for LDFSs. Iampan et al. [23] presented linear Diophantine fuzzy Einstien aggregation operators for multi-criteria decision-making problems. Inan et al. [24] established a multiple attribute decision model to compare the firms occupational health and safety management perspectives. Riaz et al. [25] introduced linear Diophantine fuzzy soft rough sets with a practical application to select the sustainable material handling equipment. Kamaci [26, 27] studied linear Diophantine fuzzy algebraic structures and introduced the concept of complex linear Diophantine fuzzy sets with their applications using cosine similarity measures, respectively. Further Riaz et al. [28] proposed the concept of spherical linear Diophantine fuzzy sets and presented their applications in modeling uncertainties in MCDM.

The concept of graph theory (GT) started with finding a walk linking seven bridges in Konigsberg. Subsequently, it has developed enormously in all the domains of sciences and humanities with wide applications in the field of operations research, economics and system analysis. A graph is used to represent mathematical networks that define the association between vertices and edges. A vertex can be used to symbolize a workstation, while the edges denote the association between stations. However, graphs often do not reflect many physical processes appropriately due to the obvious complexity of various properties of the structures. Many real-world phenomena have been emphasized to define the concept of fuzzy graphs (FGs). In 1973, Kauffman [29] introduced the concept of the fuzzy graph (FG) based on Zadehs fuzzy relations (FR) [30]. Mordeson [4] have further studied FGs and fuzzy hypergraphs. Fuzzy graph theory (FGT) has many applications in various areas, including, data mining, networking, image segmentation, clustering, communication, planning, image capturing, and scheduling. A detailed study on FGs has presented in [4, 31]. Karunambigai and Parvathi [32] utilized IFS to describe an intuitionistic fuzzy graph (IFG). Shannon and Atanassov [33], and Parvathi et al. [34] utilized IFS to describe intuitionistic fuzzy graphs (IFGs) and their basic operations via intuitionistic fuzzy relation (IFR) [35]. Verma et al. [36] established the concept of pythagorean fuzzy graph (PFG) by first coining the idea of pythagorean fuzzy relation (PFR). Akram et al. [37, 38] studied certain PFS-graphs and q-ROF graphs (q-ROFGs) under Hamacher operators. Hanif et al. [39] presented the concept of an LDF graph (LDFG) by using the idea of an LDF relation (LDFR) which was introduced by Ayub et al. [16].

Since a graph is a pair of set of vertices $\mathscr{V}$ and one relation $\mathscr{E}$ on $\mathscr{V}$, which is capable of describing abundant real-life phenomenons. However, in many real life situations that concern more than one type of relations, GT cannot work efficiently. In order to deal such situations, Sampathkumar [40] generalized the notion of graphs and introduced the concept of graph structures (GSs). GS has n mutually disjoint, symmetric and irreflexive relations. Ramakrishnan and Dinesh [41, 42, 43] introduced fuzzy graph structures (FGSs) and investigated some related properties. Later on, Akram and Sitara [44, 45] and Akram et al. [46] investigated degree, total degree and few properties of semi-strong min product, maximal product and residue product of FGSs. Sharma and Bansal [47, 48] introduced the concept of IF-graph structure (IFGS). Further, Sharma et al. [49] presented the notion of regular IFGSs with a detailed study of their important consequences and useful examples for illustration. Sitara et al. [50] studied the concept of q-rung picture fuzzy graph structure (q-RPFGS).

1.1 Research Gaps and Motivations

The following subsection will summarize the main objectives and areas of knowledge lacking in the theories discussed earlier.

1. GSs are commonly employed in analyzing various structures, such as graphs, signed graphs, semigraphs, edge-colored graphs, and edge-labled graphs. GSs play a crucial role in researching various areas within computer science and computational intelligence. FGSs are more beneficial compared to GS due to their ability to address the uncertainty and ambiguity commonly found in various real-world phenomena.

2. The latest extension of FS theory, called LDFS introduced by Riaz and Hashmi [14], eliminates constraints related to MD and NMD found in previous concepts like FS, IFS, PyFS, and q-ROFS by adding reference parameters. It allows the decision maker greater freedom in their judgment when facing any decision-making issue. Indeed, reference parameters play a significant role in determining the optimal solution in decision analysis.

3. Recently, Hanif et al. [39] proposed the concept of LDF-graph (LDFG) with some fundamental operations and properties. LDFGs are more beneficial than FG, IFG, PFGS, and q-ROFG because they have a broader range of MD and NMD.

4. Since GSs are more valuable than graphs due to their ability to handle multiple relationship issues effectively. By viewing existing literature, it appears that there is a lack of investigation on LDF graph structures (LDFGS).

5. To address this research gap, we explore GS within LDFSs and introduce the concept of LDFGS, which eliminates specific restrictions on MD and NMD found in current FGSs.

6. Several key concepts of LDFGSs are introduced with demonstrative examples. Certain significant and fascinating results are proved using different scenarios along with concrete examples. LDFGSs are certainly better than the current concepts of FGSs, IFGSs, and q-RPFGS because of the expanded scope of MD and NMD. LDFGSs are a valuable resource in addressing issues involving numerous connections within the context of LDFSs.

1.2 Aim of the Proposed Study

The main purposes of this research paper are:

- To establish a detailed study on GS in the context of LDFSs and hence introduce the concept of LDFGS.
- To define key notions such as $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, strength of $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, degree of a vertex, total $\breve{\rho}_i$-degree of a vertex, and total degree of a vertex in an LDFGS, $\breve{\rho}_i$-size of an LDFGS, size, and the order of an LDFGS.
- To introduce the notion of the maximal product of two LDFGSs, strong LDFGS, degree and $\breve{\rho}_i$-degree of the maximal product.
- To present the concept of $\breve{\rho}_i$-regular and regular LDFGS.
- To develop their important consequences with illustrative examples.

1.3 Organization of the Paper

Our remaining part of this paper is organized in the following manners:

Certain basic notions related to FS, IFS, PFS, q-ROFS, LDFS, FR, IFR, LDFR, GS, FGS, and IFGS are presented in Section 2. In Section 3, the concept of LDFGS is introduced with an explanatory example. Furthermore, some fundamental concepts in LDFGS such as $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, strength of $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, degree of a vertex, total $\breve{\rho}_i$-degree of a vertex, and total degree of a vertex in an LDFGS, $\breve{\rho}_i$-size of an LDFGS, size of an LDFGS, and the order of an LDFGS are introduced with constructive examples. In Section 4, the notion of the maximal product of two LDFGSs, strong LDFGS, degree and $\breve{\rho}_i$-degree of the maximal product are introduced. Some important results related to these concepts are also proved with illustrative examples. Section 5 presents the concept of $\breve{\rho}_i$-regular and regular LDFGS with some related consequences and examples. Finally, section 6 consists of some concluding remarks of this research article and some future research directions related to the novel born ideas in this research article.

2 Preliminaries

In this section, some fundamental notions of FS, IFS, PFS, q-ROFS, LDFS, FR, IFR, LDFR, GS, FGS and IFGS are given which are indispensable to understanding the contributions of this paper. For more details, we refer the reader to study [16, 41, 42, 40, 14]. Throughout this research manuscript, $\mathscr{V}, \mathscr{V}_1,$ and $\mathscr{V}_2$ are denoted as universal sets, unless otherwise stated.

Definition 2.1. [1] A FS on $\mathscr{V}$ is defined by $\mathscr{F} = \left\{\left\langle \mathbf{x}, \varkappa^m_{\mathscr{F}}(\mathbf{x})\right\rangle : \mathbf{x} \in \mathscr{V}\right\}$, where $\varkappa^m_{\mathscr{F}} : \mathscr{V} \longrightarrow [0,1]$ is a membership function (MF) which assigns the MD to each object $\mathbf{x} \in \mathscr{V}$.

Definition 2.2. [5] An IFS $\mathscr{I}$ on $\mathscr{V}$ is a set of triplets of the form:

$$\mathscr{I} = \Big\{ \Big(\mathbf{x}, \big\langle \varkappa^m_{\mathscr{I}}(\mathbf{x}), \varkappa^n_{\mathscr{I}}(\mathbf{x}) \big\rangle \Big) : \mathbf{x} \in \mathscr{V} \Big\}, \tag{1}$$

where $\varkappa^m_{\mathscr{I}}, \varkappa^n_{\mathscr{I}} : \mathscr{V} \longrightarrow [0,1]$ specify the MD and NMD, respectively, which satisfy $0 \leq \varkappa^m_{\mathscr{I}}(\mathbf{x}) + \varkappa^n_{\mathscr{I}}(\mathbf{x}) \leq 1$, for all $\mathbf{x} \in \mathscr{V}$.

Definition 2.3. [9] A PFS $\mathscr{P}$ on $\mathscr{V}$ is an object of the form:

$$\mathscr{P} = \Big\{ \Big(\mathbf{x}, \big\langle \varkappa^m_{\mathscr{P}}(\mathbf{x}), \varkappa^n_{\mathscr{P}}(\mathbf{x}) \big\rangle \Big) : \mathbf{x} \in \mathscr{V} \Big\}, \tag{2}$$

where $\varkappa^m_{\mathscr{P}}, \varkappa^n_{\mathscr{P}} : \mathscr{V} \longrightarrow [0,1]$ specify for the MD and NMD, respectively, fulfilling $0 \leq \big(\varkappa^m_{\mathscr{P}}(\mathbf{x})\big)^2 + \big(\varkappa^n_{\mathscr{P}}(\mathbf{x})\big)^2 \leq 1$, for all $\mathbf{x} \in \mathscr{V}$.

Definition 2.4. [11, 10] A q-ROFS $\mathscr{Q}$ on $\mathscr{V}$ is an object of the form:

$$\mathscr{Q} = \Big\{ \Big(\mathbf{x}, \big\langle \varkappa^m_{\mathscr{Q}}(\mathbf{x}), \varkappa^n_{\mathscr{Q}}(\mathbf{x}) \big\rangle \Big) : \mathbf{x} \in \mathscr{V} \Big\}, \tag{3}$$

where $\varkappa^m_{\mathscr{Q}}, \varkappa^n_{\mathscr{Q}} : \mathscr{V} \longrightarrow [0,1]$ are used for the MD and NMD, respectively such that $0 \leq \big(\varkappa^m_{\mathscr{Q}}(\mathbf{x})\big)^q + \big(\varkappa^n_{\mathscr{Q}}(\mathbf{x})\big)^q \leq 1$, for all $\mathbf{x} \in \mathscr{V}$, where $q \in [1, \infty)$.

Definition 2.5. [14] An LDFS $\mathfrak{L}$ over $\mathscr{V}$ is an expression of the following form :

$$\mathfrak{L} = \Big\{ \Big(\mathbf{x}, \big\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathfrak{L}}(\mathbf{x}) \big\rangle, \big\langle \alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x}) \big\rangle \Big) : \mathbf{x} \in \mathscr{V} \Big\}, \tag{4}$$

where $\varkappa^m_{\mathfrak{L}}, \varkappa^n_{\mathfrak{L}} : \mathscr{V} \longrightarrow [0,1]$ are MD and NMD, and $\alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x}) \in [0,1]$ are corresponding reference parameters, respectively, with $0 \leq \alpha_{\mathfrak{L}}(\mathbf{x}) + \beta_{\mathfrak{L}}(x) \leq 1$ and $0 \leq \alpha_{\mathfrak{L}}(\mathbf{x})\varkappa^m_{\mathfrak{L}}(\mathbf{x}) + \beta_{\mathfrak{L}}(\mathbf{x})\varkappa^n_{\mathfrak{L}}(\mathbf{x}) \leq 1$, for all $\mathbf{x} \in \mathscr{V}$. The degree of hesitation of any $\mathbf{x} \in \mathscr{V}$ is denoted and defined as $\boxtimes_{\mathfrak{L}}(\mathbf{x}) = 1 - \big(\alpha_{\mathfrak{L}}(\mathbf{x})\varkappa^m_{\mathfrak{L}}(\mathbf{x}) + \beta_{\mathfrak{L}}(\mathbf{x})\varkappa^n_{\mathfrak{L}}(\mathbf{x})\big)$, for all $\mathbf{x} \in \mathscr{V}$.

From now onward, we will use $\mathbf{LDFS}(\mathscr{V})$ for the set of all LDFSs over $\mathscr{V}$. For simplicity, we will use $\mathfrak{L} = \big(\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathfrak{L}}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x})\rangle\big)$ for an LDFS over $\mathscr{V}$.

Definition 2.6. [14] Let $\mathfrak{L}_1 = \Big(\big\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}_1}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x})\big\rangle\Big)$ and $\mathfrak{L}_2 = \Big(\big\langle \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_2}(\mathbf{x})\big\rangle\Big)$ be two LDFSs on $\mathscr{V}$. Then, for all $x \in \mathscr{V}$,

(1) $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ if and only if $\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}) \leq \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}) \geq \varkappa^n_{\mathfrak{L}_2}(\mathbf{x}),$ and $\alpha_{\mathfrak{L}_1}(\mathbf{x}) \leq \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x}) \geq \beta_{\mathfrak{L}_2}(\mathbf{x})$;

(2) $\mathfrak{L}_1 \cup \mathfrak{L}_2 = \Big(\big\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}_1}(\mathbf{x}) \vee \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x}) \wedge \beta_{\mathfrak{L}_2}(\mathbf{x})\big\rangle\Big)$;

(3) $\mathfrak{L}_1 \cap \mathfrak{L}_2 = \Big(\big\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}) \wedge \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}) \vee \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}_1}(\mathbf{x}) \wedge \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x}) \vee \beta_{\mathfrak{L}_2}(\mathbf{x})\big\rangle\Big)$;

(4) $\mathfrak{L}_1^c = \Big(\big\langle \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^m_{\mathfrak{L}_1}(\mathbf{x})\big\rangle, \big\langle \beta_{\mathfrak{L}_1}(\mathbf{x}), \alpha_{\mathfrak{L}_1}(\mathbf{x})\big\rangle\Big)$;

A subset $\mathscr{E}$ of the cartesian product $\mathscr{V}_1 \times \mathscr{V}_2$ is a binary relation from $\mathscr{V}_1$ to $\mathscr{V}_2$ which is basically the set of edges from $\mathscr{V}_1$ to $\mathscr{V}_2$.

Definition 2.7. [30] A FR ρ on $\mathscr{V}_1 \times \mathscr{V}_2$ is defined as:

$$\rho = \Big\{ \big\langle (\mathbf{x}_1, \mathbf{x}_2), \varkappa_\rho^m(\mathbf{x}_1, \mathbf{x}_2) \big\rangle, (\mathbf{x}_1, \mathbf{x}_2) \in \mathscr{V}_1 \times \mathscr{V}_2 \Big\}, \tag{5}$$

where $\varkappa_\rho^m : \mathscr{V}_1 \times \mathscr{V}_2 \longrightarrow [0,1]$ is a MF which specifies the grade of membership to which the objects $\mathbf{x}_1 \in \mathscr{V}_1$ and $\mathbf{x}_2 \in \mathscr{V}_2$ are connected to each other.

Definition 2.8. [35] An IFR $\dot{\rho}$ from $\mathscr{V}_1$ to $\mathscr{V}_2$ is an object of the form:

$$\dot{\rho} = \Big\{ \Big((\mathbf{x}_1, \mathbf{x}_2), \big\langle \varkappa_{\dot{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2), \varkappa_{\dot{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2) \big\rangle \Big) : \mathbf{x}_1 \in \mathscr{V}_1, \mathbf{x}_2 \in \mathscr{V}_2 \Big\}, \tag{6}$$

where $\varkappa_{\dot{\rho}}^m, \varkappa_{\dot{\rho}}^n : \mathscr{V}_1 \times \mathscr{V}_2 \longrightarrow [0,1]$ indicate the MD and NMD from $\mathscr{V}_1$ to $\mathscr{V}_2$, respectively with $0 \leq \varkappa_{\dot{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2) + \varkappa_{\dot{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2) \leq 1$, for all $(\mathbf{x}_1, \mathbf{x}_2) \in \mathscr{V}_1 \times \mathscr{V}_2$.

Definition 2.9. [16] An LDFR $\breve{\rho}$ from $\mathscr{V}_1$ to $\mathscr{V}_2$ is an expression having the form:

$$\breve{\rho} = \Big\{ \Big((\mathbf{x}_1, \mathbf{x}_2), \big\langle \varkappa_{\breve{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2), \varkappa_{\breve{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2) \big\rangle, \big\langle \alpha_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2), \beta_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2) \big\rangle \Big) : \mathbf{x}_1 \in \mathscr{V}_1, \mathbf{x}_2 \in \mathscr{V}_2 \Big\}, \tag{7}$$

where $\varkappa_{\breve{\rho}}^m, \varkappa_{\breve{\rho}}^n : \mathscr{V}_1 \times \mathscr{V}_2 \longrightarrow [0,1]$ denotes the MD and NMD among the entities of $\mathscr{V}_1$ and $\mathscr{V}_2$, and $\alpha_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2)$, $\beta_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2) \in [0,1]$ are the corresponding reference parameters to $\varkappa_{\breve{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2)$ and $\varkappa_{\breve{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2)$, respectively. These MD and NMD obey the constraint $0 \leq \alpha_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2)\varkappa_{\breve{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2) + \beta_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2)\varkappa_{\breve{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2) \leq 1$ for all $(\mathbf{x}_1, \mathbf{x}_2) \in \mathscr{V}_1 \times \mathscr{V}_2$ with $0 \leq \alpha_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2) + \beta_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2) \leq 1$. The degree of hesitation can be calculated as:

$$\gamma \boxtimes_{\breve{\rho}} (\mathbf{x}_1, \mathbf{x}_2) = 1 - \Big(\alpha_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2)\varkappa_{\breve{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2) + \beta_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2)\varkappa_{\breve{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2) \Big), \tag{8}$$

where γ is the corresponding reference parameter of indeterminacy. For simplicity, we shall use $\breve{\rho} = \Big(\big\langle \varkappa_{\breve{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2), \varkappa_{\breve{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2) \big\rangle, \big\langle \alpha_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2), \beta_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2) \big\rangle \Big)$ for an LDFR from $\mathscr{V}_1$ to $\mathscr{V}_2$. The collection of all LDFRs from $\mathscr{V}_1$ to $\mathscr{V}_2$ by $\mathbf{LDFS}(\mathscr{V}_1 \times \mathscr{V}_2)$.

Definition 2.10. [16] Let $\breve{\rho}_1 = \Big(\big\langle \varkappa_{\breve{\rho}_1}^m(\mathbf{x}_1, \mathbf{x}_2), \varkappa_{\breve{\rho}_1}^n(\mathbf{x}_1, \mathbf{x}_2) \big\rangle, \big\langle \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2), \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \big\rangle \Big)$ be an LDFR from $\mathscr{V}_1$ to $\mathscr{V}_2$ and $\breve{\rho}_2 = \Big(\big\langle \varkappa_{\breve{\rho}_2}^m(\mathbf{x}_2, \mathbf{x}_3), \varkappa_{\breve{\rho}_2}^n(\mathbf{x}_2, \mathbf{x}_3) \big\rangle, \big\langle \alpha_{\breve{\rho}_2}(\mathbf{x}_2, \mathbf{x}_3), \beta_{\breve{\rho}_2}(\mathbf{x}_2, \mathbf{x}_3) \big\rangle \Big)$ be an LDFR from $\mathscr{V}_2$ to $\mathscr{V}_3$. Then, their composition is denoted and defined by :

$$\breve{\rho}_1 \circ \breve{\rho}_2 = \Big(\big\langle \big(\varkappa_{\breve{\rho}_1}^m \circ \varkappa_{\breve{\rho}_2}^m\big)(\mathbf{x}_1, \mathbf{x}_3), \big(\varkappa_{\breve{\rho}_1}^n \circ \varkappa_{\breve{\rho}_2}^n\big)(\mathbf{x}_1, \mathbf{x}_3) \big\rangle, \big\langle \big(\alpha_{\breve{\rho}_1} \circ \alpha_{\breve{\rho}_2}\big)(\mathbf{x}_1, \mathbf{x}_3), \big(\beta_{\breve{\rho}_1} \circ \beta_{\breve{\rho}_2}\big)(\mathbf{x}_1, \mathbf{x}_3) \big\rangle \Big) \tag{9}$$

where

$$\big(\varkappa_{\breve{\rho}_1}^m \circ \varkappa_{\breve{\rho}_2}^m\big)(\mathbf{x}_1, \mathbf{x}_3) = \bigvee_{\mathbf{x}_2 \in \mathscr{V}_2} \Big(\varkappa_{\breve{\rho}_1}^m(\mathbf{x}_1, \mathbf{x}_2) \wedge \varkappa_{\breve{\rho}_2}^m(\mathbf{x}_2, \mathbf{x}_3) \Big), \tag{10}$$

$$\big(\varkappa_{\breve{\rho}_1}^n \circ \varkappa_{\breve{\rho}_2}^n\big)(\mathbf{x}_1, \mathbf{x}_3) = \bigwedge_{\mathbf{x}_2 \in \mathscr{V}_2} \Big(\varkappa_{\breve{\rho}_1}^n(\mathbf{x}_1, \mathbf{x}_2) \vee \varkappa_{\breve{\rho}_2}^n(\mathbf{x}_2, \mathbf{x}_3) \Big), \tag{11}$$

$$\big(\alpha_{\breve{\rho}_1} \circ \alpha_{\breve{\rho}_2}\big)(\mathbf{x}_1, \mathbf{x}_3) = \bigvee_{\mathbf{x}_2 \in \mathscr{V}_2} \Big(\alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \wedge \alpha_{\breve{\rho}_2}(\mathbf{x}_2, \mathbf{x}_3) \Big), \tag{12}$$

$$\big(\beta_{\breve{\rho}_1} \circ \beta_{\breve{\rho}_2}\big)(\mathbf{x}_1, \mathbf{x}_3) = \bigwedge_{\mathbf{x}_2 \in \mathscr{V}_2} \Big(\beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \vee \beta_{\breve{\rho}_2}(\mathbf{x}_2, \mathbf{x}_3) \Big), \tag{13}$$

for all $\mathbf{x}_1 \in \mathscr{V}_1, \mathbf{x}_2 \in \mathscr{V}_2, \mathbf{x}_3 \in \mathscr{V}_3$.

Definition 2.11. Let $\breve{\rho} = \Big(\big\langle \varkappa^m_{\breve{\rho}}(\mathbf{x}_1,\mathbf{x}_2), \varkappa^n_{\breve{\rho}}(\mathbf{x}_1,\mathbf{x}_2)\big\rangle, \big\langle \alpha_{\breve{\rho}}(\mathbf{x}_1,\mathbf{x}_2), \beta_{\breve{\rho}}(\mathbf{x}_1,\mathbf{x}_2)\big\rangle\Big)$ be an LDFR from $\mathscr{V}_1$ to $\mathscr{V}_2$. Then, the set

$$Supp(\breve{\rho}) = \big\{(\mathbf{x}_1,\mathbf{x}_2) : \varkappa^m_{\breve{\rho}}(\mathbf{x}_1,\mathbf{x}_2) > 0, \varkappa^n_{\breve{\rho}}(\mathbf{x}_1,\mathbf{x}_2) > 0\rangle, \langle \alpha_{\breve{\rho}}(\mathbf{x}_1,\mathbf{x}_2) > 0, \beta_{\breve{\rho}}(\mathbf{x}_1,\mathbf{x}_2) > 0\big\} \tag{14}$$

is called the support of $\breve{\rho}$.

Definition 2.12. [40] Let $\mathscr{V}$ be any non-empty set known as the vertex set and $\mathscr{E}_1, \mathscr{E}_2, \cdots, \mathscr{E}_k$ be mutually disjoint relations (sets of edges) of $\mathscr{V}$ such that each $\mathscr{E}_i$, $1 \le i \le k$ is symmetric and irreflexive. Then, $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2, \cdots, \mathscr{E}_k\big)$ is called a graph structure (GS).

Definition 2.13. [41, 42] Let $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2, \cdots, \mathscr{E}_k\big)$ be a GS. Then, $\hat{\mathscr{G}} = \big(\mathscr{F}, \rho_1, \rho_2, \cdots, \rho_k\big)$ is called fuzzy graph structure (FGS) of GS $\mathscr{G}$, where $\mathscr{F}$ is a FS on $\mathscr{V}$ and ρ_i are irreflexive, symmetric and mutually exclusive FRs on $\mathscr{V}$, for all $1 \le i \le k$, if $0 \le \varkappa^m_{\rho_i}(\mathbf{x},\mathbf{y}) \le \varkappa^m_{\mathscr{F}}(\mathbf{x}) \wedge \varkappa^m_{\mathscr{F}}(\mathbf{y})$ for all $\mathbf{x},\mathbf{y} \in \mathscr{V}$, $i = 1, 2, \cdots, k$.

Definition 2.14. [47] Let $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2, ..., \mathscr{E}_k\big)$ be a GS, $\mathscr{I} = \big\langle \varkappa^m_{\mathscr{I}}(\mathbf{x}), \varkappa^n_{\mathscr{I}}(\mathbf{x})\big\rangle$ be an IFS on $\mathscr{V}$ and $\dot{\rho}_i = \big\langle \varkappa^m_{\dot{\rho}_i}(\mathbf{x}_1,\mathbf{x}_2), \varkappa^n_{\dot{\rho}_i}(\mathbf{x}_1,\mathbf{x}_2)\big\rangle$ be irreflexive, symmetric and mutually disjoint IFRs on $\mathscr{V}$, $i = 1, 2, ..., n$, where $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in \mathscr{V}$. Then, $\dot{\mathscr{G}} = \big(\mathscr{I}, \dot{\rho}_1, \dot{\rho}_2, ..., \dot{\rho}_n\big)$ is called intuitionistic fuzzy graph structure (IFGS) of $\mathscr{G}$, if

$$\varkappa^m_{\dot{\rho}_i}(\mathbf{x}_1,\mathbf{x}_2) \le \varkappa^m_{\mathscr{I}}(\mathbf{x}) \wedge \varkappa^n_{\mathscr{I}}(\mathbf{x}_2), \text{ and } \varkappa^n_{\dot{\rho}_i}(\mathbf{x}_1,\mathbf{x}_2) \ge \varkappa^n_{\mathscr{I}}(\mathbf{x}_1) \vee \varkappa^n_{\mathscr{I}}(\mathbf{x}_2),$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathscr{V}$, $i = 1, 2, ..., n$.

3 Linear Diophantine Fuzzy Graph Structures (LDFGS)

In this section, we introduce the idea of LDFGS and some basic notions in LDFGSs containing $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, strength of $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, degree of a vertex, total $\breve{\rho}_i$-degree of a vertex, and total degree of a vertex in an LDFGS, $\breve{\rho}_i$-size of an LDFGS, size of an LDFGS, and the order of an LDFGS are introduced with constructive examples. Throughout this section, we will use simply $\mathscr{G}$ for a GS $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2, ..., \mathscr{E}_n\big)$ (see Definition 2.12).

Definition 3.1. Let $\mathfrak{L} = \Big(\big\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathfrak{L}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x})\big\rangle\Big)$ be an LDFS over $\mathscr{V}$, $\mathscr{G}$ be a GS and $\breve{\rho}_i \in$ **LDFS**$(\mathscr{E}_i)$, $i \in \{1, 2, \cdots, k\}$. Then, $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_k\big)$ is called an LDFGS of $\mathscr{G}$, if for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$:

$$\left.\begin{aligned} \varkappa^m_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}) &\le \varkappa^m_{\mathfrak{L}}(\mathbf{x}) \wedge \varkappa^m_{\mathfrak{L}}(\mathbf{y}),\\ \varkappa^n_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}) &\ge \varkappa^n_{\mathfrak{L}}(\mathbf{x}) \vee \varkappa^n_{\mathfrak{L}}(\mathbf{y}),\\ \alpha_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}) &\le \alpha_{\mathfrak{L}}(\mathbf{x}) \wedge \alpha_{\mathfrak{L}}(\mathbf{y}),\\ \beta_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}) &\ge \beta_{\mathfrak{L}}(\mathbf{x}) \vee \beta_{\mathfrak{L}}(\mathbf{y}). \end{aligned}\right\} \tag{15}$$

Example 3.2. Let $\mathscr{V} = \big\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\big\}$, $\mathscr{E}_1 = \big\{(\mathbf{x}_1,\mathbf{x}_2), (\mathbf{x}_1,\mathbf{x}_3), (\mathbf{x}_3,\mathbf{x}_4)\big\}$, and $\mathscr{E}_2 = \big\{(\mathbf{x}_1,\mathbf{x}_4), (\mathbf{x}_2,\mathbf{x}_3), (\mathbf{x}_2,\mathbf{x}_4)\big\}$. Then, $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2\big)$ is the GS. Define an LDFS $\mathfrak{L} \in$ **LDFS**$(\mathscr{V})$ exhibited in TABLE 1.

Consider two LDFRs $\breve{\rho}_1, \breve{\rho}_2$ over $\mathscr{E}_1, \mathscr{E}_2$, respectively which are shown in TABLES 2 and 3, respectively.

By simple calculations, we can easily see that $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2\big)$ is an LDFGS of GS $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2\big)$ shown in Figure 1.

Table 1: Tabular representation of LDFS $\mathfrak{L}$

$\mathscr{V}$	$\left(\left\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathfrak{L}}(\mathbf{x})\right\rangle, \left\langle \alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x})\right\rangle\right)$
$\mathbf{x}_1$	$(\langle 0.4, 0.3\rangle, \langle 0.2, 0.1\rangle)$
$\mathbf{x}_2$	$(\langle 0.6, 0.2\rangle, \langle 0.3, 0.2\rangle)$
$\mathbf{x}_3$	$(\langle 0.4, 0.5\rangle, \langle 0.4, 0.2\rangle)$
$\mathbf{x}_4$	$(\langle 0.7, 0.3\rangle, \langle 0.6, 0.2\rangle)$

Table 2: $\breve{\rho}_1$

$\breve{\rho}_1$	$\left(\left\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}_1}(\mathbf{x},\mathbf{y})\right\rangle, \left\langle \alpha_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}_1}(\mathbf{x},\mathbf{y})\right\rangle\right)$
$(\mathbf{x}_1, \mathbf{x}_2)$	$(\langle 0.4, 0.4\rangle, \langle 0.2, 0.3\rangle)$
$(\mathbf{x}_1, \mathbf{x}_3)$	$(\langle 0.3, 0.6\rangle, \langle 0.2, 0.3\rangle)$
$(\mathbf{x}_3, \mathbf{x}_4)$	$(\langle 0.4, 0.5\rangle, \langle 0.4, 0.2\rangle)$

Table 3: $\breve{\rho}_2$

$\breve{\rho}_2$	$\left(\left\langle \varkappa^m_{\breve{\rho}_2}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}_2}(\mathbf{x},\mathbf{y})\right\rangle, \left\langle \alpha_{\breve{\rho}_2}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}_2}(\mathbf{x},\mathbf{y})\right\rangle\right)$
$(\mathbf{x}_1, \mathbf{x}_4)$	$(\langle 0.4, 0.3\rangle, \langle 0.2, 0.3\rangle)$
$(\mathbf{x}_2, \mathbf{x}_3)$	$(\langle 0.3, 0.5\rangle, \langle 0.2, 0.3\rangle)$
$(\mathbf{x}_2, \mathbf{x}_4)$	$(\langle 0.6, 0.4\rangle, \langle 0.3, 0.4\rangle)$

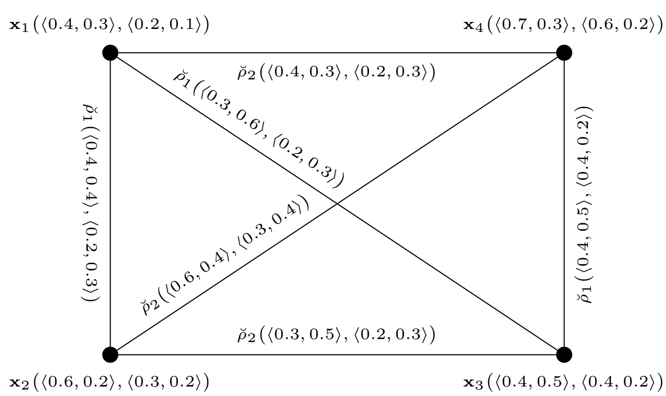

Figure 1: $\breve{\mathscr{G}} = (\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2)$

Definition 3.3. Let $\breve{\mathscr{G}} = (\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_k)$ be an LDFGS with underlying GS $\mathscr{G}$. If $(\mathbf{x},\mathbf{y}) \in Supp(\breve{\rho}_i)$, then $(\mathbf{x},\mathbf{y})$ is called $\breve{\rho}_i$-edge of $\breve{\mathscr{G}}$.

Example 3.4. In Example 3.2, $(\mathbf{x}_1,\mathbf{x}_4)$, $(\mathbf{x}_2,\mathbf{x}_3)$, $(\mathbf{x}_2,\mathbf{x}_4)$ are $\breve{\rho}_2$-edges since $Supp(\breve{\rho}_2) = \{(\mathbf{x}_1,\mathbf{x}_4), (\mathbf{x}_2,\mathbf{x}_3), (\mathbf{x}_2,\mathbf{x}_4)\}$ and $(\mathbf{x}_1,\mathbf{x}_2)$, $(\mathbf{x}_1,\mathbf{x}_3)$, $(\mathbf{x}_3,\mathbf{x}_4)$ are $\breve{\rho}_1$-edges since $Supp(\breve{\rho}_1) = \{(\mathbf{x}_1,\mathbf{x}_2), (\mathbf{x}_1,\mathbf{x}_3), (\mathbf{x}_3,\mathbf{x}_4)\}$.

Definition 3.5. Let $\breve{\mathscr{G}} = (\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_k)$ be an LDFGS with underlying GS $\mathscr{G}$. A $\breve{\rho}_i$-path of $\breve{\mathscr{G}}$ is a sequence of vertices $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_l)$ which are distinct except possibly $\mathbf{x}_0 = \mathbf{x}_l$, such that $(\mathbf{x}_{j-1}, \mathbf{x}_j)$ is a $\breve{\rho}_i$-edge for all $j = 1, 2, 3, ..., l$.

Example 3.6. In Example 3.2, $(\mathbf{x}_3, \mathbf{x}_1, \mathbf{x}_2)$, and $(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4)$ are $\breve{\rho}_1$-paths. And, $(\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$, $(\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2)$, and $(\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2, \mathbf{x}_3)$ are $\breve{\rho}_2$-paths.

Definition 3.7. In an LDFGS $\breve{\mathscr{G}} = \left(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_k\right)$ with underlying GS $\mathscr{G}$, two vertices $\mathbf{x}, \mathbf{y}$ of $\breve{\mathscr{G}}$ are said to be $\breve{\rho}_i$-connected, if they are joined by a $\breve{\rho}_i$-path.

Example 3.8. In Example 3.2, all vertices $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ are $\breve{\rho}_1$- and $\breve{\rho}_2$-connected according to the Example 3.6 since they are joined by both $\breve{\rho}_1$- and $\breve{\rho}_2$- paths. Since for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$ they are connected by $\breve{\rho}_i$ for all $i = 1, 2$, so $\breve{\mathscr{G}}$ is connected LDFGS because $\breve{\rho}_1(\mathbf{x}_1, \mathbf{x}_3) > 0$, $\breve{\rho}_1(\mathbf{x}_1, \mathbf{x}_2) > 0$, and $\breve{\rho}_1(\mathbf{x}_3, \mathbf{x}_4) > 0$ so, $\mathbf{x}_1$, $\mathbf{x}_3$ are $\breve{\rho}_1$-connected, $\mathbf{x}_1$, $\mathbf{x}_2$ are $\breve{\rho}_1$-connected, and $\mathbf{x}_3$, $\mathbf{x}_4$ are $\breve{\rho}_1$-connected, respectively. Similarly, $\mathbf{x}_2$, $\mathbf{x}_3$ are $\breve{\rho}_2$-connected, $\mathbf{x}_2$, $\mathbf{x}_4$ are $\breve{\rho}_2$-connected, and $\mathbf{x}_1$, $\mathbf{x}_4$ are $\breve{\rho}_2$-connected.

Definition 3.9. Let $\mathfrak{P} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_l)$ be a $\breve{\rho}_i$-path of an LDFGS $\hat{\mathscr{G}} = \left(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_k\right)$ with underlying GS $\mathscr{G}$. Then, the strength of the $\breve{\rho}_i$-path $\mathfrak{P}$, is denoted and defined as:

$$St(\mathfrak{P}) = \Big(\big\langle \varkappa^m_{St(\mathfrak{P})}, \varkappa^n_{St(\mathfrak{P})}\big\rangle, \big\langle \alpha_{St(\mathfrak{P})}, \beta_{St(\mathfrak{P})}\big\rangle\Big), \tag{16}$$

where

$$\left.\begin{aligned} \varkappa^m_{St(\mathfrak{P})} &= \bigwedge_{j=1}^{k} \varkappa^m_{\breve{\rho}_i}(\mathbf{x}_{j-1}, \mathbf{x}_j), \varkappa^n_{St(\mathfrak{P})} = \bigvee_{j=1}^{k} \varkappa^n_{\breve{\rho}_i}(\mathbf{x}_{j-1}, \mathbf{x}_j) \\ \alpha_{St(\mathfrak{P})} &= \bigwedge_{j=1}^{k} \alpha_{\breve{\rho}_i}(\mathbf{x}_{j-1}, \mathbf{x}_j), \beta_{St(\mathfrak{P})} = \bigvee_{j=1}^{k} \beta_{\breve{\rho}_i}(\mathbf{x}_{j-1}, \mathbf{x}_j) \end{aligned}\right\} \tag{17}$$

for $i = 1, 2, ..., k$.

Example 3.10. (Continued from Example 3.6) We have seen that $(\mathbf{x}_3, \mathbf{x}_1, \mathbf{x}_2)$, and $(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4)$ are $\breve{\rho}_1$-paths. And, $(\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$, $(\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2)$, and $(\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2, \mathbf{x}_3)$ are $\breve{\rho}_2$-paths. We can calculate their strengths as follows:

Strength of $\breve{\rho}_1$-path $\mathfrak{P}_1 = (\mathbf{x}_3, \mathbf{x}_1, \mathbf{x}_2)$:

$$\begin{aligned} \varkappa^m_{St(\mathfrak{P}_1)} &= \wedge_{j=2}^{3} \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_{j-1}, \mathbf{x}_j) = \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \wedge \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) = 0.4 \wedge 0.3 = 0.3 \\ \varkappa^n_{St(\mathfrak{P}_1)} &= \vee_{j=2}^{3} \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_{j-1}, \mathbf{x}_j) = \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \vee \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) = 0.4 \vee 0.6 = 0.6 \\ \alpha_{St(\mathfrak{P}_1)} &= \wedge_{j=2}^{3} \alpha_{\breve{\rho}_1}(\mathbf{x}_{j-1}, \mathbf{x}_j) = \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \wedge \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) = 0.2 \wedge 0.2 = 0.2 \\ \beta_{St(\mathfrak{P}_1)} &= \vee_{j=2}^{3} \beta_{\breve{\rho}_1}(\mathbf{x}_{j-1}, \mathbf{x}_j) = \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \vee \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) = 0.3 \vee 0.3 = 0.3 \end{aligned}$$

So, $St(\mathfrak{P}_1) = \big(\langle 0.3, 0.6\rangle, \langle 0.2, 0.3\rangle\big)$. Similarly, we can calculate strength of $\breve{\rho}_1$-path $\mathfrak{P}_2 = (\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4)$ which is given by $St(\mathfrak{P}_2) = \big(\langle 0.3, 0.6\rangle, \langle 0.2, 0.3\rangle\big)$, strength of $\breve{\rho}_2$-path $\mathfrak{P}_3 = (\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ is $St(\mathfrak{P}_3) = \big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.4\rangle\big)$ and strength of $\breve{\rho}_2$-path $\mathfrak{P}_3 = (\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2, \mathbf{x}_3)$ is $St(\mathfrak{P}_4) = \big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.4\rangle\big)$.

Definition 3.11. Let $\breve{\mathscr{G}} = \left(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_n\right)$ be an LDFGS of GS $\mathscr{G}$. Then, $\breve{\rho}_i$-strength of connectedness of any two vertices $\mathbf{x}_1, \mathbf{x}_2$ is denoted and defined as:

$$\big(\breve{\rho}_i\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2) = \bigg(\Big\langle \big(\varkappa^m_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2), \big(\varkappa^n_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2)\Big\rangle, \Big\langle \big(\alpha_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2), \big(\beta_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2)\Big\rangle\bigg), \tag{18}$$

where

$$\left.\begin{aligned}\left(\varkappa^m_{\breve{\rho}_i}\right)^{\infty}(\mathbf{x}_1,\mathbf{x}_2)=\bigvee_{j=1}^{\infty}\left(\varkappa^m_{\breve{\rho}_i}\right)^{j}(\mathbf{x}_1,\mathbf{x}_2),\ \text{and}\ \left(\varkappa^n_{\breve{\rho}_i}\right)^{\infty}(\mathbf{x}_1,\mathbf{x}_2)=\bigwedge_{j=1}^{\infty}\left(\varkappa^n_{\breve{\rho}_i}\right)^{j}(\mathbf{x}_1,\mathbf{x}_2).\\ \left(\alpha_{\breve{\rho}_i}\right)^{\infty}(\mathbf{x}_1,\mathbf{x}_2)=\bigvee_{j=1}^{\infty}\left(\alpha_{\breve{\rho}_i}\right)^{j}(\mathbf{x}_1,\mathbf{x}_2),\ \text{and}\ \left(\beta_{\breve{\rho}_i}\right)^{\infty}(\mathbf{x}_1,\mathbf{x}_2)=\bigwedge_{j=1}^{\infty}\left(\beta_{\breve{\rho}_i}\right)^{j}(\mathbf{x}_1,\mathbf{x}_2).\end{aligned}\right\}\tag{19}$$

Here, $\left(\breve{\rho}_i\right)^j(\mathbf{x}_1,\mathbf{x}_2)=\left(\left\langle\left(\varkappa^m_{\breve{\rho}_i}\right)^j(\mathbf{x}_1,\mathbf{x}_2),\left(\varkappa^n_{\breve{\rho}_i}\right)^j(\mathbf{x}_1,\mathbf{x}_2)\right\rangle,\left\langle\left(\alpha_{\breve{\rho}_i}\right)^j(\mathbf{x}_1,\mathbf{x}_2),\left(\beta_{\breve{\rho}_i}\right)^j(\mathbf{x}_1,\mathbf{x}_2)\right\rangle\right)=$ $\left((\breve{\rho}_i)^{j-1}\circ\breve{\rho}_i\right)(\mathbf{x}_1,\mathbf{x}_2)$, and the composition $\circ$ among any two LDFRs is provided in Definition 2.10.

Example 3.12. In Example 3.2, we can evaluate the terms as defined in above definition as follows:

$$\left(\varkappa^m_{\breve{\rho}_2}\right)^{\infty}(\mathbf{x}_1,\mathbf{x}_2)=\bigvee_z\left\{\varkappa^m_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{z})\wedge\varkappa^m_{\breve{\rho}_2}(\mathbf{z},\mathbf{x}_2)\right\}=\vee\{\varkappa^m_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{x}_4)\wedge\varkappa^m_{\breve{\rho}_2}(\mathbf{x}_4,\mathbf{x}_2)\}=0.4\wedge 0.6=0.4$$

$$\left(\varkappa^n_{\breve{\rho}_2}\right)^{\infty}(\mathbf{x}_1,\mathbf{x}_2)=\bigwedge_z\{\varkappa^n_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{z})\vee\varkappa^n_{\breve{\rho}_2}(\mathbf{z},\mathbf{x}_2)\}=\wedge\{\varkappa^n_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{x}_4)\vee\varkappa^n_{\breve{\rho}_2}(\mathbf{x}_4,\mathbf{x}_2)\}=0.3\vee 0.4=0.4$$

$$\left(\alpha_{\breve{\rho}_2}\right)^{\infty}(\mathbf{x}_1,\mathbf{x}_2)=\bigvee_z\{\alpha_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{z})\wedge\alpha_{\breve{\rho}_2}(\mathbf{z},\mathbf{x}_2)\}=\vee\{\alpha_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{x}_4)\wedge\alpha_{\breve{\rho}_2}(\mathbf{x}_4,\mathbf{x}_2)\}=0.2\wedge 0.3=0.2$$

$$\left(\beta_{\breve{\rho}_2}\right)^{\infty}(\mathbf{x}_1,\mathbf{x}_2)=\bigwedge_z\{\beta_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{z})\vee\beta_{\breve{\rho}_2}(\mathbf{z},\mathbf{x}_2)\}=\wedge\{\beta_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{x}_4)\vee\beta_{\breve{\rho}_2}(\mathbf{x}_4,\mathbf{x}_2)\}=0.3\vee 0.4=0.4$$

So, $\left(\breve{\rho}_2\right)^{\infty}(\mathbf{x}_1,\mathbf{x}_2)=\left(\langle 0.4,0.4\rangle,\langle 0.2,0.4\rangle\right)$. Similarly, we can find $\left(\breve{\rho}_2\right)^{\infty}(\mathbf{x}_1,\mathbf{x}_3)=\left(\langle 0.3,0.5\rangle,\langle 0.2,0.4\rangle\right)$ and $\left(\breve{\rho}_1\right)^{\infty}(\mathbf{x}_2,\mathbf{x}_3)=\left(\langle 0.3,0.6\rangle,\langle 0.2,0.3\rangle\right)$.

Definition 3.13. Let $\breve{\mathscr{G}}=\left(\mathfrak{L},\breve{\rho}_1,\breve{\rho}_2,...,\breve{\rho}_n\right)$ be an LDFGS of $\mathscr{G}$. Then, $\breve{\mathscr{G}}$ is called connected LDFGS, if each of its two vertices $\mathbf{x}_1,\mathbf{x}_2$ are $\breve{\rho}_i$-connected, that is, $\left(\breve{\rho}_i\right)^{\infty}(\mathbf{x},\mathbf{y})>0$ for any $\mathbf{x},\mathbf{y}\in\mathscr{V}$, and $i=1,2,\dots,n$.

Example 3.14. In Example 3.12, it can be easily observed that $\left(\breve{\rho}_i\right)^{\infty}(\mathbf{x}_j,\mathbf{x}_k)>0$ for all $i=1,2$, and $j,k=1,2,3,4$. Hence, this LDFGS is connected.

Definition 3.15. Let $\breve{\mathscr{G}}=\left(\mathfrak{L},\breve{\rho}_1,\breve{\rho}_2,...,\breve{\rho}_n\right)$ be an LDFGS with underlying GS $\mathscr{G}$. Then $\breve{\rho}_i$-degree of a vertex $\mathbf{x}\in\mathscr{V}$ is denoted and defined by

$$\mathbb{D}_{\breve{\rho}_i}(\mathbf{x})=\left(\left\langle\varkappa^m_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}),\varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x})\right\rangle,\left\langle\alpha_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}),\beta_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x})\right\rangle\right),\tag{20}$$

where

$$\left.\begin{aligned}\varkappa^m_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x})=\sum_{i=1,\mathbf{x}\neq\mathbf{y},(\mathbf{x},\mathbf{y})\in\mathscr{E}_i}^{k}\varkappa^m_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}),\varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x})=\sum_{i=1,\mathbf{x}\neq\mathbf{y},(\mathbf{x},\mathbf{y})\in\mathscr{E}_i}^{k}\varkappa^n_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}),\\ \alpha_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x})=\sum_{i=1,\mathbf{x}\neq\mathbf{y},(\mathbf{x},\mathbf{y})\in\mathscr{E}_i}^{k}\alpha_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}),\beta_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x})=\sum_{i=1,\mathbf{x}\neq\mathbf{y},(\mathbf{x},\mathbf{y})\in\mathscr{E}_i}^{k}\beta_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}).\end{aligned}\right\}\tag{21}$$

Definition 3.16. Let $\breve{\mathscr{G}}=\left(\mathfrak{L},\breve{\rho}_1,\breve{\rho}_2,...,\breve{\rho}_n\right)$ be an LDFGS with underlying GS $\mathscr{G}$. Then the degree of the vertex $\mathbf{x}\in\mathscr{V}$ is denoted and characterized as:

$$\mathbb{D}(\mathbf{x})=\sum_{i=1}^{k}\mathbb{D}_{\breve{\rho}_i}(\mathbf{x})=\left(\left\langle\varkappa^m_{\mathbb{D}}(\mathbf{x}),\varkappa^n_{\mathbb{D}}(\mathbf{x})\right\rangle,\left\langle\alpha_{\mathbb{D}}(\mathbf{x}),\beta_{\mathbb{D}}(\mathbf{x})\right\rangle\right),\tag{22}$$

where

$$\varkappa^m_{\mathbb{D}}(\mathbf{x})=\sum_{i=1}^{k}\varkappa^m_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}),\varkappa^n_{\mathbb{D}}(\mathbf{x})=\sum_{i=1}^{k}\varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}),\alpha_{\mathbb{D}}(\mathbf{x})=\sum_{i=1}^{k}\alpha_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}),\beta_{\mathbb{D}}(\mathbf{x})=\sum_{i=1}^{k}\beta_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}).\tag{23}$$

Example 3.17. If we revisit Example 3.2, then according to Definition 3.15, $\breve{\rho}_i$-degrees of vertices can be calculate as follows:

$$\varkappa^m_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}_1) = \sum_{\mathbf{x}_1 \neq \mathbf{y}, (\mathbf{x}_1, \mathbf{y}) \in \mathscr{E}_1} \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{y}) = \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) + \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) = 0.3 + 0.4 = 0.7$$

$$\varkappa^n_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}_1) = \sum_{\mathbf{x}_1 \neq \mathbf{y}, (\mathbf{x}_1, \mathbf{y}) \in \mathscr{E}_1} \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{y}) = \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) + \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) = 0.6 + 0.4 = 1$$

$$\alpha_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}_1) = \sum_{\mathbf{x}_1 \neq \mathbf{y}, (\mathbf{x}_1, \mathbf{y}) \in \mathscr{E}_1} \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{y}) = \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) + \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) = 0.2 + 0.2 = 0.4$$

$$\beta_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}_1) = \sum_{\mathbf{x}_1 \neq \mathbf{y}, (\mathbf{x}_1, \mathbf{y}) \in \mathscr{E}_1} \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{y}) = \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) + \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) = 0.3 + 0.3 = 0.6$$

So, $\mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_1) = \big(\langle 0.7, 1\rangle, \langle 0.4, 0.6\rangle\big)$. Similarly, we can evaluate $\breve{\rho}_1$- and $\breve{\rho}_2$-degrees of all $\mathbf{x} \in \mathscr{V}$ which are displayed in TABLES 4 and 5, respectively.

Table 4: $\mathbb{D}_{\breve{\rho}_1}$

$\mathscr{V}$	$\Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}), \varkappa^n_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}), \beta_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x})\big\rangle\Big)$
$\mathbf{x}_1$	$\big(\langle 0.7, 1\rangle, \langle 0.4, 0.6\rangle\big)$
$\mathbf{x}_2$	$\big(\langle 0.4, 0.4\rangle, \langle 0.2, 0.3\rangle\big)$
$\mathbf{x}_3$	$\big(\langle 0.7, 1\rangle, \langle 0.6, 0.5\rangle\big)$
$\mathbf{x}_4$	$\big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.6\rangle\big)$

Table 5: $\mathbb{D}_{\breve{\rho}_2}$

$\mathscr{V}$	$\Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\rho}_2}}(\mathbf{x}), \varkappa^n_{\mathbb{D}_{\breve{\rho}_2}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\rho}_2}}(\mathbf{x}), \beta_{\mathbb{D}_{\breve{\rho}_2}}(\mathbf{x})\big\rangle\Big)$
$\mathbf{x}_1$	$\big(\langle 0.9, 0.9\rangle, \langle 0.5, 0.7\rangle\big)$
$\mathbf{x}_2$	$\big(\langle 0.9, 0.9\rangle, \langle 0.5, 0.7\rangle\big)$
$\mathbf{x}_3$	$\big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.3\rangle\big)$
$\mathbf{x}_4$	$\big(\langle 1, 0.7\rangle, \langle 0.5, 0.7\rangle\big)$

Now, in the light of Definition 3.16, we calculate the degrees $\mathbb{D}(\mathbf{x}) = \sum_{i=1}^{k} \mathbb{D}_{\breve{\rho}_i}(\mathbf{x})$ as follows:

$$\mathbb{D}(\mathbf{x}_1) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_1) + \mathbb{D}_{\breve{\rho}_2}(\mathbf{x}_1) = \big(\langle 0.7, 1\rangle, \langle 0.4, 0.6\rangle\big) + \big(\langle 0.9, 0.9\rangle, \langle 0.5, 0.7\rangle\big) = \big(\langle 1.6, 1.9\rangle, \langle 0.9, 1.3\rangle\big)$$
$$\mathbb{D}(\mathbf{x}_2) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_2) + \mathbb{D}_{\breve{\rho}_2}(\mathbf{x}_2) = \big(\langle 0.4, 0.4\rangle, \langle 0.2, 0.3\rangle\big) + \big(\langle 0.9, 0.9\rangle, \langle 0.5, 0.7\rangle\big) = \big(\langle 1.3, 1.3\rangle, \langle 0.7, 1\rangle\big)$$
$$\mathbb{D}(\mathbf{x}_3) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_3) + \mathbb{D}_{\breve{\rho}_2}(\mathbf{x}_3) = \big(\langle 0.7, 1\rangle, \langle 0.6, 0.5\rangle\big) + \big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.3\rangle\big) = \big(\langle 1, 1.6\rangle, \langle 0.8, 0.8\rangle\big)$$
$$\mathbb{D}(\mathbf{x}_4) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_4) + \mathbb{D}_{\breve{\rho}_2}(\mathbf{x}_4) = \big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.6\rangle\big) + \big(\langle 1, 0.7\rangle, \langle 0.5, 0.7\rangle\big) = \big(\langle 1.4, 1.2\rangle, \langle 0.9, 0.9\rangle\big)$$

which can be also be seen in TABLE 6.

Table 6: $\mathbb{D}(\mathbf{x})$

$\mathscr{V}$	$\big(\langle \varkappa^m_{\mathbb{D}}(\mathbf{x}), \varkappa^n_{\mathbb{D}}(\mathbf{x})\rangle, \langle \alpha_{\mathbb{D}}(\mathbf{x}), \beta_{\mathbb{D}}(\mathbf{x})\rangle\big)$
$\mathbf{x}_1$	$\big(\langle 1.6, 1.9\rangle, \langle 0.9, 1.3\rangle\big)$
$\mathbf{x}_2$	$\big(\langle 1.3, 1.3\rangle, \langle 0.7, 1\rangle\big)$
$\mathbf{x}_3$	$\big(\langle 1, 1.6\rangle, \langle 0.8, 0.8\rangle\big)$
$\mathbf{x}_4$	$\big(\langle 1.4, 1.2\rangle, \langle 0.9, 0.9\rangle\big)$

Definition 3.18. Let $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_n\big)$ be an LDFGS with underlying GS $\mathscr{G}$. Then total $\breve{\rho}_i$-degree of a vertex $\mathbf{x} \in \mathscr{V}$ is denoted and defined as:

$$\mathbb{TD}_{\breve{\rho}_i}(\mathbf{x}) = \mathbb{D}_{\breve{\rho}_i}(\mathbf{x}) + \mathfrak{L}(\mathbf{x}) = \Big(\big\langle \varkappa^m_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \beta_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x})\big\rangle\Big), \tag{24}$$

where

$$\left.\begin{array}{r} \varkappa^m_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}) = \varkappa^m_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) + \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}) = \varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) + \varkappa^n_{\mathfrak{L}}(\mathbf{x}), \\ \alpha_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}) = \alpha_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) + \alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}) = \beta_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) + \beta_{\mathfrak{L}}(\mathbf{x}). \end{array}\right\} \tag{25}$$

Definition 3.19. Let $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_n\big)$ be an LDFGS with underlying GS $\mathscr{G}$. Then the total degree of the vertex $\mathbf{x} \in \mathscr{V}$ is denoted and defined as:

$$\mathbb{TD}(\mathbf{x}) = \sum_{i=1}^{k} \mathbb{TD}_{\breve{\rho}_i}(\mathbf{x}) = \Big(\big\langle \varkappa^m_{\mathbb{TD}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{TD}}(\mathbf{x}), \beta_{\mathbb{TD}}(\mathbf{x})\big\rangle\Big), \tag{26}$$

where

$$\varkappa^m_{\mathbb{TD}}(\mathbf{x}) = \sum_{i=1}^{k} \varkappa^m_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}}(\mathbf{x}) = \sum_{i=1}^{k} \varkappa^n_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \alpha_{\mathbb{TD}}(\mathbf{x}) = \sum_{i=1}^{k} \alpha_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \beta_{\mathbb{TD}}(\mathbf{x}) = \sum_{i=1}^{k} \beta_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}). \tag{27}$$

Example 3.20. (Continued from Examples 3.2 and 3.17) We can calculate the $\breve{\rho}_i$-degrees for each vertex $\mathbf{x} \in \mathscr{V}$ by using Definition 3.18 as follows:

$$\begin{aligned}
\mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_1) &= \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_1) + \mathfrak{L}(\mathbf{x}_1) = \big(\langle 0.7, 1\rangle, \langle 0.4, 0.6\rangle\big) + \big(\langle 0.4, 0.3\rangle, \langle 0.2, 0.1\rangle\big) = \big(\langle 1.1, 1.3\rangle, \langle 0.6, 0.7\rangle\big),\\
\mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_2) &= \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_2) + \mathfrak{L}(\mathbf{x}_2) = \big(\langle 0.4, 0.4\rangle, \langle 0.2, 0.3\rangle\big) + \big(\langle 0.6, 0.2\rangle, \langle 0.3, 0.2\rangle\big) = \big(\langle 1, 0.6\rangle, \langle 0.5, 0.5\rangle\big),\\
\mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_3) &= \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_3) + \mathfrak{L}(\mathbf{x}_3) = \big(\langle 0.7, 1.1\rangle, \langle 0.6, 0.5\rangle\big) + \big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.2\rangle\big) = \big(\langle 1.1, 1.6\rangle, \langle 1, 0.7\rangle\big),\\
\mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_4) &= \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_4) + \mathfrak{L}(\mathbf{x}_4) = \big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.2\rangle\big) + \big(\langle 0.7, 0.3\rangle, \langle 0.6, 0.2\rangle\big) = \big(\langle 1.1, 0.8\rangle, \langle 1, 0.4\rangle\big),
\end{aligned}$$

which is also demonstrated in TABLE 7. Also, $\breve{\rho}_2$-degrees for each vertex $\mathbf{x} \in \mathscr{V}$ are calculated in TABLE 8. Now, according of Definition 3.19, $\mathbb{TD}(\mathbf{x}) = \sum_{i=1}^{k} \mathbb{TD}_{\breve{\rho}_i}(\mathbf{x})$ are calculated as follows:

$$\begin{aligned}
\mathbb{TD}(\mathbf{x}_1) &= \mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_1) + \mathbb{TD}_{\breve{\rho}_2}(\mathbf{x}_1) = \big(\langle 1.1, 1.3\rangle, \langle 0.6, 0.7\rangle\big) + \big(\langle 1.3, 1.2\rangle, \langle 0.7, 0.8\rangle\big) = \big(\langle 2.4, 2.5\rangle, \langle 1.3, 1.5\rangle\big),\\
\mathbb{TD}(\mathbf{x}_2) &= \mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_2) + \mathbb{TD}_{\breve{\rho}_2}(\mathbf{x}_2) = \big(\langle 1, 0.6\rangle, \langle 0.5, 0.5\rangle\big) + \big(\langle 1.5, 1.1\rangle, \langle 0.8, 0.9\rangle\big) = \big(\langle 2.5, 1.7\rangle, \langle 1.3, 1.4\rangle\big),\\
\mathbb{TD}(\mathbf{x}_3) &= \mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_3) + \mathbb{TD}_{\breve{\rho}_2}(\mathbf{x}_3) = \big(\langle 1.1, 1.6\rangle, \langle 1, 0.7\rangle\big) + \big(\langle 0.7, 1\rangle, \langle 0.6, 0.5\rangle\big) = \big(\langle 1.8, 2.6\rangle, \langle 1.6, 1.2\rangle\big),\\
\mathbb{TD}(\mathbf{x}_4) &= \mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_4) + \mathbb{TD}_{\breve{\rho}_2}(\mathbf{x}_4) = \big(\langle 1.1, 1.6\rangle, \langle 1, 0.7\rangle\big) + \big(\langle 1.7, 1\rangle, \langle 1.1, 0.9\rangle\big) = \big(\langle 2.8, 2.6\rangle, \langle 2.1, 1.6\rangle\big),
\end{aligned}$$

which is also shown in TABLE 9.

Table 7: $\mathbb{TD}_{\breve{\rho}_1}$

$\mathscr{V}$	$\left(\left\langle \varkappa^m_{\mathbb{TD}_{\breve{\rho}_1}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}_{\breve{\rho}_1}}(\mathbf{x})\right\rangle, \left\langle \alpha_{\mathbb{TD}_{\breve{\rho}_1}}(\mathbf{x}), \beta_{\mathbb{TD}_{\breve{\rho}_1}}(\mathbf{x})\right\rangle\right)$
$\mathbf{x}_1$	$(\langle 1.1, 1.3\rangle, \langle 0.6, 0.7\rangle)$
$\mathbf{x}_2$	$(\langle 1, 0.6\rangle, \langle 0.5, 0.5\rangle)$
$\mathbf{x}_3$	$(\langle 1.1, 1.6\rangle, \langle 1, 0.7\rangle)$
$\mathbf{x}_4$	$(\langle 1.1, 0.8\rangle, \langle 1, 0.4\rangle)$

Table 8: $\mathbb{TD}_{\breve{\rho}_2}$

$\mathscr{V}$	$\left(\left\langle \varkappa^m_{\mathbb{TD}_{\breve{\rho}_2}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}_{\breve{\rho}_2}}(\mathbf{x})\right\rangle, \left\langle \alpha_{\mathbb{TD}_{\breve{\rho}_2}}(\mathbf{x}), \beta_{\mathbb{TD}_{\breve{\rho}_2}}(\mathbf{x})\right\rangle\right)$
$\mathbf{x}_1$	$(\langle 1.3, 1.2\rangle, \langle 0.7, 0.8\rangle)$
$\mathbf{x}_2$	$(\langle 1.5, 1.1\rangle, \langle 0.8, 0.9\rangle)$
$\mathbf{x}_3$	$(\langle 0.7, 1\rangle, \langle 0.6, 0.5\rangle)$
$\mathbf{x}_4$	$(\langle 1.7, 1\rangle, \langle 1.1, 0.9\rangle)$

Table 9: $\mathbb{TD}$

$\mathscr{V}$	$\left(\left\langle \varkappa^m_{\mathbb{TD}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}}(\mathbf{x})\right\rangle, \left\langle \alpha_{\mathbb{TD}}(\mathbf{x}), \beta_{\mathbb{TD}}(\mathbf{x})\right\rangle\right)$
$\mathbf{x}_1$	$(\langle 2.4, 2.5\rangle, \langle 1.3, 1.5\rangle)$
$\mathbf{x}_2$	$(\langle 2.5, 1.7\rangle, \langle 1.3, 1.4\rangle)$
$\mathbf{x}_3$	$(\langle 1.8, 2.6\rangle, \langle 1.6, 1.2\rangle)$
$\mathbf{x}_4$	$(\langle 2.8, 2.6\rangle, \langle 2.1, 1.6\rangle)$

Definition 3.21. Let $\breve{\mathscr{G}} = (\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_n)$ be an LDFGS with underlying GS $\mathscr{G}$. Then order of $\breve{\mathscr{G}}$ is denoted and described as follows:

$$\mathbb{O}(\breve{\mathscr{G}}) = \left(\left\langle \sum_{\mathbf{x}\in\mathscr{V}} \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \sum_{\mathbf{x}\in\mathscr{V}} \varkappa^n_{\mathfrak{L}}(\mathbf{x})\right\rangle, \left\langle \sum_{\mathbf{x}\in\mathscr{V}} \alpha_{\mathfrak{L}}(\mathbf{x}), \sum_{\mathbf{x}\in\mathscr{V}} \beta_{\mathfrak{L}}(\mathbf{x})\right\rangle\right). \tag{28}$$

Example 3.22. If we consider the Example 3.2, then we can find $\mathbb{O}(\breve{\mathscr{G}})$ as follows:

$$\sum_{\mathbf{x}\in\mathscr{V}} \varkappa^m_{\mathfrak{L}}(\mathbf{x}) = 0.4 + 0.6 + 0.4 + 0.7 = 2,$$
$$\sum_{\mathbf{x}\in\mathscr{V}} \varkappa^n_{\mathfrak{L}}(\mathbf{x}) = 0.3 + 0.2 + 0.5 + 0.3 = 1.3,$$
$$\sum_{\mathbf{x}\in\mathscr{V}} \alpha_{\mathfrak{L}}(\mathbf{x}) = 0.2 + 0.3 + 0.4 + 0.6 = 1.5,$$
$$\sum_{\mathbf{x}\in\mathscr{V}} \beta_{\mathfrak{L}}(\mathbf{x}) = 0.1 + 0.2 + 0.2 + 0.2 = 0.7.$$

Hence, $\mathbb{O}(\breve{\mathscr{G}}) = \big(\langle 2, 1.3\rangle, \langle 1.5, 0.7\rangle\big)$.

Definition 3.23. Let $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_n\big)$ be an LDFGS with underlying GS $\mathscr{G}$. The $\breve{\rho}_i$-size of $\breve{\mathscr{G}}$ is denoted and postulated as:

$$\mathbb{S}_{\breve{\rho}_i}(\breve{\mathscr{G}}) = \Big(\big\langle \varkappa^m_{\mathbb{S}_{\breve{\rho}_i}}, \varkappa^n_{\mathbb{S}_{\breve{\rho}_i}}\big\rangle, \big\langle \alpha_{\mathbb{S}_{\breve{\rho}_i}}, \beta_{\mathbb{S}_{\breve{\rho}_i}}\big\rangle\Big), \tag{29}$$

where

$$\varkappa^m_{\mathbb{S}_{\breve{\rho}_i}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_i} \varkappa^m_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}), \varkappa^n_{\mathbb{S}_{\breve{\rho}_i}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_i} \varkappa^n_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}), \alpha_{\mathbb{S}_{\breve{\rho}_i}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_i} \alpha_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}), \beta_{\mathbb{S}_{\breve{\rho}_i}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_i} \beta_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}). \tag{30}$$

Moreover, the size of $\breve{\mathscr{G}}$ is denoted and characterized as:

$$\mathbb{S}(\breve{\mathscr{G}}) = \sum_{i=1}^{n} \mathbb{S}_{\breve{\rho}_i}(\breve{\mathscr{G}}). \tag{31}$$

Example 3.24. If we revisit Example 3.2, we have

$$\begin{aligned}
\varkappa^m_{\mathbb{S}_{\breve{\rho}_1}} &= \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_1} \varkappa^m_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}) = 0.4 + 0.3 + 0.4 = 1.1,\\
\varkappa^n_{\mathbb{S}_{\breve{\rho}_1}} &= \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_1} \varkappa^n_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}) = 0.4 + 0.6 + 0.5 = 1.5,\\
\alpha_{\mathbb{S}_{\breve{\rho}_1}} &= \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_1} \alpha_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}) = 0.2 + 0.2 + 0.4 = 0.8,\\
\beta_{\mathbb{S}_{\breve{\rho}_1}} &= \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_1} \beta_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}) = 0.3 + 0.3 + 0.2 = 0.8.
\end{aligned}$$

Thus, $\mathbb{S}_{\breve{\rho}_1}(\breve{\mathscr{G}}) = \big(\langle \varkappa^m_{\mathbb{S}_{\breve{\rho}_1}}, \varkappa^n_{\mathbb{S}_{\breve{\rho}_1}}\rangle, \langle \alpha_{\mathbb{S}_{\breve{\rho}_1}}, \beta_{\mathbb{S}_{\breve{\rho}_1}}\rangle\big) = \big(\langle 1.1, 1.5\rangle, \langle 0.8, 0.8\rangle\big)$. Similarly, $\mathbb{S}_{\breve{\rho}_2}(\breve{\mathscr{G}}) = \big(\langle 1.3, 1.2\rangle, \langle 0.7, 1\rangle\big)$.

Further, the size of $\breve{\mathscr{G}}$ is calculated as:

$$\mathbb{S}(\breve{\mathscr{G}}) = \mathbb{S}_{\breve{\rho}_1}(\breve{\mathscr{G}}) + \mathbb{S}_{\breve{\rho}_2}(\breve{\mathscr{G}}) = \big(\langle 1.1, 1.5\rangle, \langle 0.8, 0.8\rangle\big) + \big(\langle 1.3, 1.2\rangle, \langle 0.7, 1\rangle\big) = \big(\langle 2.4, 2.7\rangle, \langle 1.8, 1.5\rangle\big).$$

4 Maximal Product of Two Linear Diophantine Fuzzy Graph Structures

In this section, we introduce the notions of maximal product of two LDFGSs, strong LDFGS, degree and $\breve{\rho}_i$-degree of a vertex in maximal product. Furthermore, certain consequences related to these concepts are proved with some useful examples.

Definition 4.1. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_n\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_n\big)$ be two LDFGSs of the GSs $\mathscr{G}_1 = \big(\mathscr{V}_1, \mathscr{E}'_1, \mathscr{E}'_2, \cdots, \mathscr{E}'_n\big)$ and $\mathscr{G}_2 = \big(\mathscr{V}_2, \mathscr{E}''_1, \mathscr{E}''_2, \cdots, \mathscr{E}''_n\big)$, respectively. Then, $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \cdots, \breve{\rho}_n\big)$ is called maximal LDFGS with underlying crisp GS $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2, \cdots, \mathscr{E}_n\big)$, where $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$ and $\mathscr{E}_i = \big\{\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) : \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1, \mathbf{y}_2) \in \mathscr{E}''_i \text{ or } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1, \mathbf{x}_2) \in \mathscr{E}'_i\big\}$. LDF vertex set $\mathfrak{L}$ and LDFRs $\breve{\rho}_i$ in maximal product $\breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ are defined as :

$$\begin{aligned}
\mathfrak{L} &= \mathfrak{L}_1 * \mathfrak{L}_2 \\
&= \Big(\big\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}_1}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x})\big\rangle\Big) * \Big(\big\langle \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}), \varkappa^n_{\mathfrak{L}_2}(\mathbf{y})\big\rangle, \big\langle \alpha_{\mathfrak{L}_2}(\mathbf{y}), \beta_{\mathfrak{L}_2}(\mathbf{y})\big\rangle\Big) \\
&= \Big(\big\langle \big(\varkappa^m_{\mathfrak{L}_1} * \varkappa^m_{\mathfrak{L}_2}\big)(\mathbf{x},\mathbf{y}), \big(\varkappa^n_{\mathfrak{L}_1} * \varkappa^n_{\mathfrak{L}_2}\big)(\mathbf{x},\mathbf{y})\big\rangle, \big\langle \big(\alpha_{\mathfrak{L}_1} * \alpha_{\mathfrak{L}_2}\big)(\mathbf{x},\mathbf{y}), \big(\beta_{\mathfrak{L}_1} * \beta_{\mathfrak{L}_2}\big)(\mathbf{x},\mathbf{y})\big\rangle\Big) \\
&= \Big(\big\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \varkappa^n_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\big\rangle, \big\langle \alpha_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \beta_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\big\rangle\Big), \qquad (32)
\end{aligned}$$

where

$$\left.\begin{aligned}
\varkappa^m_{\mathfrak{L}}(\mathbf{x},\mathbf{y}) &= \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}), \\
\varkappa^n_{\mathfrak{L}}(\mathbf{x},\mathbf{y}) &= \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}), \\
\alpha_{\mathfrak{L}}(\mathbf{x},\mathbf{y}) &= \alpha_{\mathfrak{L}_1}(\mathbf{x}) \vee \alpha_{\mathfrak{L}_2}(\mathbf{y}), \\
\beta_{\mathfrak{L}}(\mathbf{x},\mathbf{y}) &= \beta_{\mathfrak{L}_1}(\mathbf{x}) \wedge \beta_{\mathfrak{L}_2}(\mathbf{y}),
\end{aligned}\right\} \qquad (33)$$

for all $(\mathbf{x},\mathbf{y}) \in \mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$ and $\breve{\rho}_i = \breve{\rho}'_i * \breve{\rho}''_i$ are defined as :

$$\begin{aligned}
\breve{\rho}_i &= \breve{\rho}'_i * \breve{\rho}''_i \\
&= \Big(\big\langle \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{y}_1), \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{y}_1)\big\rangle, \big\langle \alpha_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{y}_1), \beta_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{y}_1)\big\rangle\Big) * \Big(\big\langle \varkappa^m_{\breve{\rho}''_i}(\mathbf{x}_2,\mathbf{y}_2), \varkappa^n_{\breve{\rho}''_i}(\mathbf{x}_2,\mathbf{y}_2)\big\rangle, \big\langle \alpha_{\breve{\rho}''_i}(\mathbf{x}_2,\mathbf{y}_2), \beta_{\breve{\rho}''_i}(\mathbf{x}_2,\mathbf{y}_2)\big\rangle\Big) \\
&= \Big(\big\langle \big(\varkappa^m_{\breve{\rho}'_i} * \varkappa^m_{\breve{\rho}''_i}\big)(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \big(\varkappa^n_{\breve{\rho}'_i} * \varkappa^n_{\breve{\rho}''_i}\big)(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\big\rangle, \big\langle \big(\alpha_{\breve{\rho}'_i} * \alpha_{\breve{\rho}''_i}\big)(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \big(\beta_{\breve{\rho}'_i} * \beta_{\breve{\rho}''_i}\big)(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\big\rangle\Big) \\
&= \Big(\big\langle \varkappa^m_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\big\rangle, \big\langle \alpha_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\big\rangle\Big), \qquad (34)
\end{aligned}$$

where

$$\varkappa^m_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2)\big) = \begin{cases} \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\breve{\rho}''_i}(\mathbf{y}_1,\mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1,\mathbf{y}_2) \in \mathscr{E}''_i \\ \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) \vee \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1,\mathbf{x}_2) \in \mathscr{E}'_i \end{cases} \qquad (35)$$

$$\varkappa^n_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2)\big) = \begin{cases} \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_1) \wedge \varkappa^n_{\breve{\rho}''_i}(\mathbf{y}_1,\mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1,\mathbf{y}_2) \in \mathscr{E}''_i \\ \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_1) \wedge \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1,\mathbf{x}_2) \in \mathscr{E}'_i \end{cases} \qquad (36)$$

$$\alpha_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2)\big) = \begin{cases} \alpha_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \alpha_{\breve{\rho}''_i}(\mathbf{y}_1,\mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1,\mathbf{y}_2) \in \mathscr{E}''_i \\ \alpha_{\mathfrak{L}_2}(\mathbf{y}_1) \vee \alpha_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1,\mathbf{x}_2) \in \mathscr{E}'_i \end{cases} \qquad (37)$$

$$\beta_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2)\big) = \begin{cases} \beta_{\mathfrak{L}_1}(\mathbf{x}_1) \wedge \beta_{\breve{\rho}''_i}(\mathbf{y}_1,\mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1,\mathbf{y}_2) \in \mathscr{E}''_i \\ \beta_{\mathfrak{L}_2}(\mathbf{y}_1) \wedge \beta_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1,\mathbf{x}_2) \in \mathscr{E}'_i \end{cases} \qquad (38)$$

$i = 1, 2, \cdots, n$.

Example 4.2. Consider two LDFGSs $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \breve{\rho}'_3\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1\big)$, which is depicted in Figure 2 with underlying GSs $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}'_1, \mathscr{E}'_2, \mathscr{E}'_3)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}''_1)$, respectively, where $\mathscr{V}_1 = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\mathscr{V}_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ are two sets of vertices and $\mathscr{E}'_1 = \{(\mathbf{u}_1, \mathbf{u}_3)\}$, $\mathscr{E}'_2 = \{(\mathbf{u}_1, \mathbf{u}_2)\}$, and $\mathscr{E}'_3 = \{(\mathbf{u}_2, \mathbf{u}_3)\}$ are the set of edges on $\mathscr{V}_1$, and $\mathscr{E}''_1 = \{(\mathbf{v}_1, \mathbf{v}_2)\}$ is the edges set on $\mathscr{V}_2$ such that $\mathscr{E}'_i$ and $\mathscr{E}''_i$ are irreflexive and symmetric binary relations on $\mathscr{V}_1$ and $\mathscr{V}_2$, respectively. The LDFSs $\mathfrak{L}_1$ on $\mathscr{V}_1$ and $\mathfrak{L}_2$ on $\mathscr{V}_2$ are given in the TABLES 10 and 11, respectively. The LDFRs $\breve{\rho}'_1, \breve{\rho}'_2, \breve{\rho}'_3$ over the $\mathscr{E}'_1, \mathscr{E}'_2, \mathscr{E}'_3$, and $\breve{\rho}''_1$ over $\mathscr{E}''_1$ given in TABLES 12, 13, 14 and 15 respectively. By using Definition 4.1, we obtain the following LDFS $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$ illustrated in FIGURE 3 and shown in TABLE 16 and LDFRs $\breve{\rho}_i = \breve{\rho}'_i * \breve{\rho}''_i$ for $i = 1, 2, 3$ shown in TABLE 17, 18, 19, respectively.

Table 10: LDFS $\mathfrak{L}_1$

$\mathscr{V}_1$	$\big(\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}_1}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x})\rangle\big)$
$\mathbf{u}_1$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.3\rangle\big)$
$\mathbf{u}_2$	$\big(\langle 0.4, 0.3\rangle, \langle 0.5, 0.4\rangle\big)$
$\mathbf{u}_3$	$\big(\langle 0.8, 0.9\rangle, \langle 0.6, 0.3\rangle\big)$

Table 11: LDFS $\mathfrak{L}_2$

$\mathscr{V}_2$	$\big(\langle \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_2}(\mathbf{x})\rangle\big)$
$\mathbf{v}_1$	$\big(\langle 0.7, 0.4\rangle, \langle 0.3, 0.2\rangle\big)$
$\mathbf{v}_2$	$\big(\langle 0.3, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$

Table 12: $\breve{\rho}'_1$

$\mathscr{E}'_1$	$\big(\langle \varkappa^m_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1, \mathbf{u}_3)$	$\big(\langle 0.6, 0.9\rangle, \langle 0.4, 0.5\rangle\big)$

Table 13: $\breve{\rho}'_2$

$\mathscr{E}'_2$	$\big(\langle \varkappa^m_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1, \mathbf{u}_2)$	$\big(\langle 0.4, 0.5\rangle, \langle 0.3, 0.4\rangle\big)$

Table 14: $\breve{\rho}'_3$

$\mathscr{E}'_3$	$\big(\langle \varkappa^m_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_2, \mathbf{u}_3)$	$\big(\langle 0.4, 0.9\rangle, \langle 0.5, 0.4\rangle\big)$

Table 15: $\breve{\rho}''_1$

$\mathscr{E}''_1$	$\big(\langle \varkappa^m_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{v}_1, \mathbf{v}_2)$	$\big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.3\rangle\big)$

Definition 4.3. An LDFGS $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \cdots, \breve{\rho}_n\big)$ is called $\breve{\rho}_i$-strong, if

$$\left.\begin{aligned} \varkappa^m_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}) &= \varkappa^m_{\mathfrak{L}}(\mathbf{x}) \wedge \varkappa^m_{\mathfrak{L}}(\mathbf{y}),\\ \varkappa^n_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}) &= \varkappa^n_{\mathfrak{L}}(\mathbf{x}) \vee \varkappa^n_{\mathfrak{L}}(\mathbf{y}),\\ \alpha_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}) &= \alpha_{\mathfrak{L}}(\mathbf{x}) \wedge \alpha_{\mathfrak{L}}(\mathbf{y}),\\ \beta_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}) &= \beta_{\mathfrak{L}}(\mathbf{x}) \vee \beta_{\mathfrak{L}}(\mathbf{y}), \end{aligned}\right\} \tag{39}$$

for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$. If $\breve{\mathscr{G}}$ is $\breve{\rho}_i$-strong for all $i = 1, 2, \cdots, n$, then $\breve{\mathscr{G}}$ is called strong LDFGS.

Theorem 4.4. *Maximal product of two strong LDFGSs is also a strong LDFGS.*

Table 16: $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$

$\mathscr{V}$	$\big(\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \varkappa^n_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \beta_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$\big(\langle 0.6, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$\big(\langle 0.7, 0.3\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$\big(\langle 0.4, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$\big(\langle 0.8, 0.4\rangle, \langle 0.6, 0.2\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.6, 0.1\rangle\big)$

Table 17: $\breve{\rho}_1$

$\mathscr{E}_1$	$\big(\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2)$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.3\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.3, 0.2\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.4, 0.3\rangle, \langle 0.5, 0.4\rangle\big)$
$(\mathbf{u}_3\mathbf{v}_1,\mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.3, 0.5\rangle, \langle 0.6, 0.3\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2,\mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.6, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$

Table 18: $\breve{\rho}_2$

$\mathscr{E}_2$	$\big(\langle \varkappa^m_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1,\mathbf{u}_3\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_2,\mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.4, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$

Table 19: $\breve{\rho}_3$

$\mathscr{E}_3$	$\big(\langle \varkappa^m_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_3\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2,\mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.6, 0.2\rangle, \langle 0.4, 0.2\rangle\big)$

Proof. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_n\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_n\big)$ be two strong LDFGSs. Then, according to the Definition 4.1, we have the following cases:

Case i: When $\mathbf{x}_1 = \mathbf{x}_2$ and $(\mathbf{y}_1, \mathbf{y}_2) \in \mathscr{E}''_i$. Then,

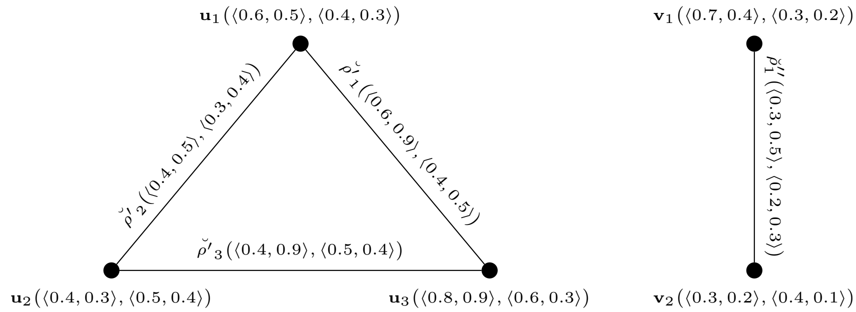

Figure 2: LDFGSs $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \breve{\rho}'_3\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1\right)$

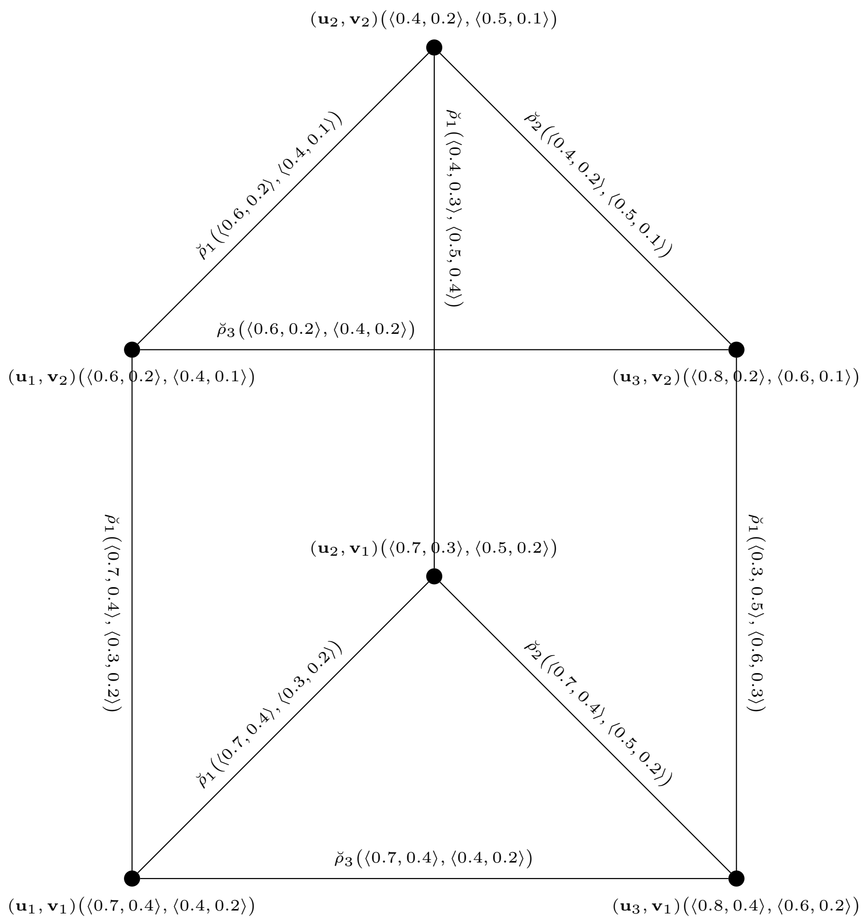

Figure 3: Maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$

$$\begin{aligned}
\varkappa^m_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) &= \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\rho''_i}(\mathbf{y}_1,\mathbf{y}_2)\\
&= \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \left[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) \wedge \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_2)\right]\\
&= \left[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\right] \wedge \left[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_2)\right]\\
&= \left[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\right] \wedge \left[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_2)\right]\\
&= \varkappa^m_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1) \wedge \varkappa^m_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2).
\end{aligned}$$

Similarly we can show that $\varkappa^n_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) = \varkappa^n_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\vee\varkappa^n_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2)$, $\alpha_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) = \alpha_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\wedge\alpha_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2)$, and $\beta_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) = \beta_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\vee\beta_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2)$.

Case ii: When $\mathbf{y}_1=\mathbf{y}_2$ and $(\mathbf{x}_1,\mathbf{x}_2)\in\mathscr{E}'_i$. Then,

$$\begin{aligned}\varkappa^m_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) &= \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\vee\varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{x}_2)\\ &= \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\vee\big[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1)\wedge\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2)\big]\\ &= \big[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1)\big]\wedge\big[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2)\big]\\ &= \big[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1)\big]\wedge\big[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2)\big]\\ &= \varkappa^m_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\wedge\varkappa^m_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2).\end{aligned}$$

In the same way, we can prove that $\varkappa^n_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) = \varkappa^n_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\vee\varkappa^n_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2)$, $\alpha_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) = \alpha_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\wedge\alpha_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2)$, and $\beta_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) = \beta_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\vee\beta_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2)$. Thus, $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is a strong LDFGS. □

Theorem 4.5. *The maximal product of two connected LDFGSs is a connected LDFGS.*

Proof. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1,\breve{\rho}'_1,\breve{\rho}'_2,\cdots,\breve{\rho}'_n\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2,\breve{\rho}''_1,\breve{\rho}''_2,\cdots,\breve{\rho}''_n\big)$ be two connected LDFGSs with underlying GSs $\mathscr{G}_1 = \big(\mathscr{V}_1,\mathscr{E}'_1,\mathscr{E}'_2,\cdots,\mathscr{E}'_n\big)$ and $\mathscr{G}_2 = \big(\mathscr{V}_2,\mathscr{E}''_1,\mathscr{E}''_2,\cdots,\mathscr{E}''_n\big)$, respectively. Let $\mathscr{V}_1=\{\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_m\}$ and $\mathscr{V}_2=\{\mathbf{y}_1,\mathbf{y}_2,\cdots,\mathbf{y}_q\}$. Then, according to the Definition 3.13,

$$\begin{aligned}(\varkappa^m_{\breve{\rho}'_i})^\infty(\mathbf{x}_i,\mathbf{x}_j)>0 \text{ and } (\varkappa^m_{\breve{\rho}''_i})^\infty(\mathbf{y}_i,\mathbf{y}_j)>0;\\ (\varkappa^n_{\breve{\rho}'_i})^\infty(\mathbf{x}_i,\mathbf{x}_j)>0 \text{ and } (\varkappa^n_{\breve{\rho}''_i})^\infty(\mathbf{y}_i,\mathbf{y}_j)>0;\\ (\alpha_{\breve{\rho}'_i})^\infty(\mathbf{x}_i,\mathbf{x}_j)>0 \text{ and } (\alpha_{\breve{\rho}''_i})^\infty(\mathbf{y}_i,\mathbf{y}_j)>0;\\ (\beta_{\breve{\rho}'_i})^\infty(\mathbf{x}_i,\mathbf{x}_j)>0 \text{ and } (\beta_{\breve{\rho}''_i})^\infty(\mathbf{y}_i,\mathbf{y}_j)>0,\end{aligned}$$

for all $\mathbf{x}_i,\mathbf{x}_j\in\mathscr{V}_1$ and $\mathbf{y}_i,\mathbf{y}_j\in\mathscr{V}_2$. Consider m subgraphs of $\mathscr{G}$ with the vertex sets $\big\{(\mathbf{x}_i,\mathbf{y}_1),(\mathbf{x}_i,\mathbf{y}_2),\cdots,(\mathbf{x}_i,\mathbf{y}_q)\big\}$ for $i=1,2,\cdots,m$. Each of these subgraphs of $\mathscr{G}$ is connected since $\mathbf{x}_i$'s are the same and $\mathscr{G}_2$ is connected, each $\mathbf{y}_i$ is adjacent to at least one of the vertices in $\mathscr{V}_2$. Since $\mathscr{G}_1$ is connected, each $\mathbf{x}_i$ is also adjacent to at least one of the vertices in $\mathscr{V}_1$. Therefore, there exists one edge between any pair of the above m subgraphs. Thus, we have

$$\begin{aligned}(\varkappa^m_{\breve{\rho}_i})^\infty\big((\mathbf{x}_i,\mathbf{y}_j),(\mathbf{x}_k,\mathbf{y}_l)\big)>0, (\varkappa^n_{\breve{\rho}_i})^\infty\big((\mathbf{x}_i,\mathbf{y}_j),(\mathbf{x}_k,\mathbf{y}_l)\big)>0, \text{ and}\\ (\alpha_{\breve{\rho}_i})^\infty\big((\mathbf{x}_i,\mathbf{y}_j),(\mathbf{x}_k,\mathbf{y}_l)\big)>0, (\beta_{\breve{\rho}'_i})^\infty\big((\mathbf{x}_i,\mathbf{y}_j),(\mathbf{x}_k,\mathbf{y}_l)\big)>0,\end{aligned}$$

for all $((\mathbf{x}_i,\mathbf{y}_j),(\mathbf{x}_k,\mathbf{y}_l))\in\mathscr{E}_i$. Hence, $\breve{\mathscr{G}}$ is connected LDFGS. □

Definition 4.6. Let $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \big(\mathfrak{L},\breve{\rho}_1,\breve{\rho}_2,\cdots,\breve{\rho}_n\big)$ be the maximal product of LDFGSs $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1,\breve{\rho}'_1,\breve{\rho}'_2,\cdots,\breve{\rho}'_n\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2,\breve{\rho}''_1,\breve{\rho}''_2,\cdots,\breve{\rho}''_n\big)$. Then, the degree of a vertex in $\breve{\mathscr{G}}$ is postulated as follows:

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j) = \Big(\big\langle\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle,\big\langle\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big), \tag{40}$$

where

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \alpha_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \alpha_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\vee\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \beta_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \beta_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\beta_{\mathfrak{L}_1}(\mathbf{x}_i)
\end{aligned}\right\}\tag{41}$$

Also, $\breve{\rho}_i-\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)$ of a vertex $(\mathbf{x}_i,\mathbf{y}_j)$ of maximal product $\breve{\mathscr{G}}$ is defined as follows:

$$\breve{\rho}_i-\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)=\Big(\big\langle\varkappa^m_i-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_i-\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle,\big\langle\alpha_i-\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_i-\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big),\tag{42}$$

where

$$\left.\begin{aligned}
\varkappa^m_i-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_i,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\rho''_i}(\mathbf{y}_j,\mathbf{y}_l)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\varkappa^n_i-\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_i,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_i}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\alpha_i-\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \alpha_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_i,\mathbf{x}_i=\mathbf{x}_k} \alpha_{\breve{\rho}''_i}(\mathbf{y}_j,\mathbf{y}_l)\vee\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\beta_i-\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \beta_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_i,\mathbf{x}_i=\mathbf{x}_k} \beta_{\mathscr{E}''_i}(\mathbf{y}_j,\mathbf{y}_l)\wedge\beta_{\mathfrak{L}_1}(\mathbf{x}_i)
\end{aligned}\right\}\tag{43}$$

Example 4.7. (Continued from Example 4.2) With the same LDFGSs $\breve{\mathscr{G}}_1$, $\breve{\mathscr{G}}_2$ and their maximal product $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2$ with underlying GSs $\mathscr{G}_1$, $\mathscr{G}_2$ and their maximal product $\mathscr{G}=\mathscr{G}_1*\mathscr{G}_2$. According to Definition 4.6, the degrees of vertices in $\breve{\mathscr{G}}$ are calculated as follows:

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^m_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1)\\
&= 0.4\vee0.7+0.6\vee0.7+0.3\vee0.6=2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^m_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1)\\
&= 0.4\vee0.3+0.6\vee0.3+0.3\vee0.6=1.6\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2)\\
&= 0.4\vee0.7+0.4\vee0.7+0.3\vee0.4=1.8\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2)\\
&= 0.4\vee0.3+0.4\vee0.3+0.3\vee0.4=1.2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^m_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3)\\
&= 0.4\vee0.7+0.6\vee0.7+0.3\vee0.8=2.2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^m_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3)\\
&= 0.4\vee0.3+0.6\vee0.3+0.3\vee0.8=1.8
\end{aligned}$$

similarly,

$$\begin{aligned}
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) &= \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1)\\
&= 0.5\wedge 0.4+0.9\wedge 0.4+0.5\wedge 0.5=1.3\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) &= \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1)\\
&= 0.5\wedge 0.2+0.9\wedge 0.2+0.5\wedge 0.5=0.9\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) &= \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2)\\
&= 0.5\wedge 0.4+0.9\wedge 0.4+0.5\wedge 0.3=1.1\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) &= \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2)\\
&= 0.5\wedge 0.2+0.9\wedge 0.2+0.5\wedge 0.3=0.7\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) &= \varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3)\\
&= 0.9\wedge 0.4+0.9\wedge 0.4+0.5\wedge 0.9=1.3\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) &= \varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3)\\
&= 0.9\wedge 0.2+0.9\wedge 0.2+0.5\wedge 0.9=0.9
\end{aligned}$$

In the similar way, $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)$ and $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)$ are calculated for all $\mathbf{x}_i\in\mathscr{V}_1$ and $\mathbf{y}_j\in\mathscr{V}_2$, shown in TABLE 20.

Table 20: $\mathbb{D}_{\breve{\mathscr{G}}}$

$\mathscr{V}$	$\Big(\big\langle\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle,\big\langle\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$(\langle 2,0.9\rangle,\langle 1.1,0.7\rangle)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$(\langle 1.6,1.1\rangle,\langle 1.2,0.5\rangle)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$(\langle 1.8,0.7\rangle,\langle 1.3,0.7\rangle)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$(\langle 1.2,1.3\rangle,\langle 1.4,0.5\rangle)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$(\langle 2.2,0.9\rangle,\langle 1.5,0.7\rangle)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$(\langle 0.8,1.3\rangle,\langle 1.5,0.5\rangle)$

Now, we calculate $\breve{\rho}_i-\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)$ for all $i=1,2,3$ as follows:

$$\begin{aligned}
\varkappa^m_1-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1)=0.4\vee 0.7+0.3\vee 0.6=1.3\\
\varkappa^m_1-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1)=0.4\vee 0.3+0.3\vee 0.6=1\\
\varkappa^m_1-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2)=0.4\vee 0.7+0.3\vee 0.4=1.1\\
\varkappa^m_1-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2)=0.4\vee 0.3+0.3\vee 0.4=0.8\\
\varkappa^m_1-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3)=0.3\vee 0.8=0.8\\
\varkappa^m_1-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3)=0.3\vee 0.8=0.8
\end{aligned}$$

Similarly, $\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)$ can be calculated as:

$$\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1, \mathbf{v}_1) = \varkappa^n_{\breve{\rho}_1'}(\mathbf{u}_1, \mathbf{u}_2) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^n_{\breve{\rho}_1''}(\mathbf{v}_1, \mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1) = 0.5 \wedge 0.4 + 0.5 \wedge 0.5 = 0.9$$
$$\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1, \mathbf{v}_2) = \varkappa^n_{\breve{\rho}_1'}(\mathbf{u}_1, \mathbf{u}_2) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^n_{\breve{\rho}_1''}(\mathbf{v}_1, \mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1) = 0.5 \wedge 0.2 + 0.5 \wedge 0.5 = 0.7$$
$$\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_1) = \varkappa^n_{\breve{\rho}_1'}(\mathbf{u}_2, \mathbf{u}_1) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^n_{\breve{\rho}_1''}(\mathbf{v}_1, \mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2) = 0.5 \wedge 0.4 + 0.5 \wedge 0.3 = 0.7$$
$$\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_2) = \varkappa^n_{\breve{\rho}_1'}(\mathbf{u}_2, \mathbf{u}_1) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^n_{\breve{\rho}_1''}(\mathbf{v}_1, \mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2) = 0.5 \wedge 0.2 + 0.5 \wedge 0.3 = 0.5$$
$$\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_1) = \varkappa^n_{\breve{\rho}_1''}(\mathbf{v}_1, \mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3) = 0.9 \wedge 0.4 = 0.4$$
$$\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_2) = \varkappa^n_{\breve{\rho}_1''}(\mathbf{v}_1, \mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3) = 0.9 \wedge 0.5 = 0.5$$

Moreover, $\alpha_1 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)$ and $\beta_1 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)$ are evaluated by following the same steps, which are given in TABLE 21.

Table 21: $\breve{\rho}_1 - \mathbb{D}_{\breve{\mathscr{G}}}$

$\mathscr{V}$	$\big(\langle \varkappa_1^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\rangle, \langle \alpha_1 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_1 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\rangle\big)$
$(\mathbf{u}_1, \mathbf{v}_1)$	$(\langle 1.3, 0.9\rangle, \langle 0.7, 0.5\rangle)$
$(\mathbf{u}_1, \mathbf{v}_2)$	$(\langle 1, 0.7\rangle, \langle 0.8, 0.4\rangle)$
$(\mathbf{u}_2, \mathbf{v}_1)$	$(\langle 1.1, 0.7\rangle, \langle 0.8, 0.5\rangle)$
$(\mathbf{u}_2, \mathbf{v}_2)$	$(\langle 0.8, 0.5\rangle, \langle 0.9, 0.4\rangle)$
$(\mathbf{u}_3, \mathbf{v}_1)$	$(\langle 0.8, 0.5\rangle, \langle 0.6, 0.3\rangle)$
$(\mathbf{u}_3, \mathbf{v}_2)$	$(\langle 0.8, 0.5\rangle, \langle 0.6, 0.3\rangle)$

Now, we calculate $\breve{\rho}_2 - \mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j)$ as:

$$\varkappa_2^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_1) = \varkappa^m_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.4 \vee 0.7 = 0.7$$
$$\varkappa_2^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_2) = \varkappa^m_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.4 \vee 0.3 = 0.4$$
$$\varkappa_2^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_1) = \varkappa^m_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.4 \vee 0.7 = 0.7$$
$$\varkappa_2^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_2) = \varkappa^m_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.4 \vee 0.3 = 0.4$$
$$\varkappa_2^n - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_1) = \varkappa^n_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.9 \wedge 0.4 = 0.4$$
$$\varkappa_2^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_2) = \varkappa^n_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.9 \wedge 0.2 = 0.2$$
$$\varkappa_2^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_1) = \varkappa^n_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.9 \wedge 0.4 = 0.4$$
$$\varkappa_2^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_2) = \varkappa^n_{\rho_2'}(\mathbf{u}_2, \mathbf{u}_3) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.9 \wedge 0.2 = 0.2$$

In the similar manners, we have calculated $\alpha_2 - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)$ and $\beta_2 - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)$ for all $\mathbf{x}_i \in \mathscr{V}_1$, and $\mathbf{y}_j \in \mathscr{V}_2$ which presented in TABLE 22. By following the similar methodology as above, $\breve{\rho}_3 - \mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j)$ are evaluated for all $\mathbf{x}_i \in \mathscr{V}_1$, and $\mathbf{y}_j \in \mathscr{V}_2$ which are shown in TABLE 23.

Table 22: $\breve{\rho}_2 - \mathbb{D}_{\breve{\mathscr{G}}}$

$\mathscr{V}$	$\Big(\big\langle \varkappa_2^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa_2^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle, \big\langle \alpha_2 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_2 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$\big(\langle 0.4, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$\big(\langle 0.4, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$

Table 23: $\breve{\rho}_3 - \mathbb{D}_{\breve{\mathscr{G}}}$

$\mathscr{V}$	$\Big(\big\langle \varkappa_3^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa_3^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle, \big\langle \alpha_3 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_3 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$\big(\langle 0.6, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$\big(\langle 0.7, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$

Theorem 4.8. *If* $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_k\big)$ *and* $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_k\big)$ *are two LDFGSs such that* $\mathfrak{L}_1 \subseteq \breve{\rho''}_i$, $i = 1, 2, \cdots, k$, *then the degree of any vertex in maximal product* $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \cdots, \breve{\rho}_k\big)$ *is given by:*

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big), \tag{44}$$

where

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).
\end{aligned}\right\} \tag{45}$$

Proof. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_k\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_k\big)$ be two LDFGSs such that $\mathfrak{L}_1 \subseteq \breve{\rho''}_i$, then $\breve{\rho'}_i \subseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$. Thus,

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\\
&= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j);
\end{aligned}$$

Also,

$$\begin{aligned}
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\\
&= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)
\end{aligned}$$

By adopting the procedure, we can show that

$$\mathbb{D}_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) \text{ and } \mathbb{D}_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).$$

□

Theorem 4.9. *If* $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ *and* $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ *are LDFGSs such that* $\mathfrak{L}_1 \subseteq \breve{\rho}''_i$, $i = 1, 2, \cdots, k$, *and* $\mathfrak{L}_2$ *is constant LDFS of LDF value* $\left(\langle a,b\rangle,\langle c,d\rangle\right)$, *where* $a, b, c, d \in [0, 1]$ *are fixed, then the degree of any vertex in maximal product* $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ *is given as:*

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j) = \left(\left\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\right\rangle, \left\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\right\rangle\right), \tag{46}$$

where

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)a + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)b + \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)c + \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)d + \beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).
\end{aligned}\right\} \tag{47}$$

Proof. Let $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ be two LDFGSs such that $\mathfrak{L}_1 \subseteq \breve{\rho}''_i$, then $\breve{\rho}'_i \subseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$ and $\mathfrak{L}_2$ is a constant LDFS. Therefore,

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\\
&= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)a + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).
\end{aligned}$$

Also,

$$\begin{aligned}
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\\
&= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)b + \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).
\end{aligned}$$

Similarly, we can show that

$$\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)c + \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) \text{ and } \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).$$

□

Theorem 4.10. *If $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ are two LDFGSs such that $\mathfrak{L}_2 \subseteq \breve{\rho}'_i$, $i = 1, 2, \cdots, k$, then the degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is given by:*

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big), \tag{48}$$

where

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i),\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i),\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i),\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i).
\end{aligned}\right\} \tag{49}$$

Proof. Let $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ be two LDFGSs such that $\mathfrak{L}_2 \subseteq \breve{\rho}'_i$, then $\breve{\rho}''_i \subseteq \mathfrak{L}_1$, $i = 1, 2, \cdots, k$. So,

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i).
\end{aligned}$$

Also,

$$\begin{aligned}
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i).
\end{aligned}$$

Similarly, we can show that $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)$ and $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)$ $\beta_{\mathfrak{L}_1}(\mathbf{x}_i)$. □

Theorem 4.11. *If $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ are two LDFGSs such that $\mathfrak{L}_2 \subseteq \breve{\rho}'_i$, $i = 1, 2, \cdots, k$, and $\mathfrak{L}_1$ is constant LDFS of LDF value $\left(\langle a, b\rangle, \langle c, d\rangle\right)$, where $a, b, c, d \in [0, 1]$ are fixed, then the degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is given by:*

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big), \tag{50}$$

where

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)a,\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)b,\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)c,\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)d.
\end{aligned}\right\} \tag{51}$$

Proof. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\big)$ be two LDFGSs such that $\mathfrak{L}_2 \subseteq \breve{\rho}'_i$, $i = 1, 2, \cdots, k$, and $\mathfrak{L}_1$ is constant LDFS of LDF value $\big(\langle a, b\rangle, \langle c, d\rangle\big)$. Therefore,

$$\begin{aligned}\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)a.\end{aligned}$$

Also,

$$\begin{aligned}\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)b.\end{aligned}$$

Similarly, it can be shown that $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)c$ and $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)d$.
□

Theorem 4.12. *If $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\big)$ are two LDFGSs such that $\breve{\rho}''_i \subseteq \mathfrak{L}_1$ and $\breve{\rho}'_i \subseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$, then the degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is characterized as:*

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big), \tag{52}$$

where

$$\left.\begin{aligned}\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i),\\ \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i),\\ \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i),\\ \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i).\end{aligned}\right\} \tag{53}$$

Proof. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\big)$ be two LDFGSs such that such that $\breve{\rho}''_i \subseteq \mathfrak{L}_1$ and $\breve{\rho}'_i \subseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$. Then,

$$\begin{aligned}\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i).\end{aligned}$$

Also,

$$\begin{aligned}\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i).\end{aligned}$$

Similarly, we can show that $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)$ and $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i)$. □

Example 4.13. Consider two LDFGSs $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \breve{\rho}'_3\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1\big)$, which is depicted in Figure 4 with underlying GSs $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}'_1, \mathscr{E}'_2, \mathscr{E}'_3)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}''_1)$, respectively, same as in Example 4.2. The LDFSs $\mathfrak{L}_1$ on $\mathscr{V}_1$ and $\mathfrak{L}_2$ on $\mathscr{V}_2$ are given in the TABLES 24 and 25, respectively. The LDFRs $\breve{\rho}'_1, \breve{\rho}'_2, \breve{\rho}'_3$ over the $\mathscr{E}'_1, \mathscr{E}'_2, \mathscr{E}'_3$, and $\breve{\rho}''_1$ over $\mathscr{E}''_1$ given in TABLES 26, 27, 28 and 29 respectively with $\breve{\rho}'_i \subseteq \mathfrak{L}_2$ and $\breve{\rho}''_i \subseteq \mathfrak{L}_1$, for $i = 1, 2, 3$. By using Definition 4.1, the LDFS $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$ is shown in TABLE 30 and LDFRs $\breve{\rho}_i = \breve{\rho}'_i * \breve{\rho}''_i$ for $i = 1, 2, 3$ shown in TABLE 31, 32, 33, respectively. The resulting LDFGS $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \breve{\rho}_3\big)$ is illustrated in FIGURE 5.

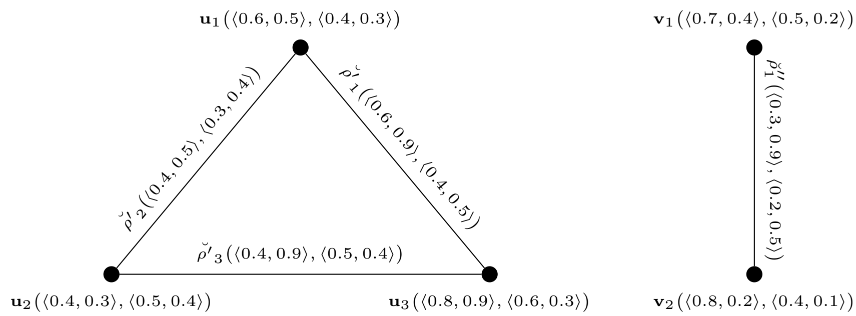

Figure 4: LDFGSs $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \breve{\rho}'_3\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1\big)$

Table 24: LDFS $\mathfrak{L}_1$

$\mathscr{V}_1$	$\big(\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}_1}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x})\rangle\big)$
$\mathbf{u}_1$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.3\rangle\big)$
$\mathbf{u}_2$	$\big(\langle 0.4, 0.3\rangle, \langle 0.5, 0.4\rangle\big)$
$\mathbf{u}_3$	$\big(\langle 0.8, 0.9\rangle, \langle 0.6, 0.3\rangle\big)$

Table 25: LDFS $\mathfrak{L}_2$

$\mathscr{V}_2$	$\big(\langle \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_2}(\mathbf{x})\rangle\big)$
$\mathbf{v}_1$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$\mathbf{v}_2$	$\big(\langle 0.8, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$

Table 26: $\breve{\rho}'_1$

$\mathscr{E}'_1$	$\big(\langle \varkappa^m_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1, \mathbf{u}_3)$	$\big(\langle 0.6, 0.9\rangle, \langle 0.4, 0.5\rangle\big)$

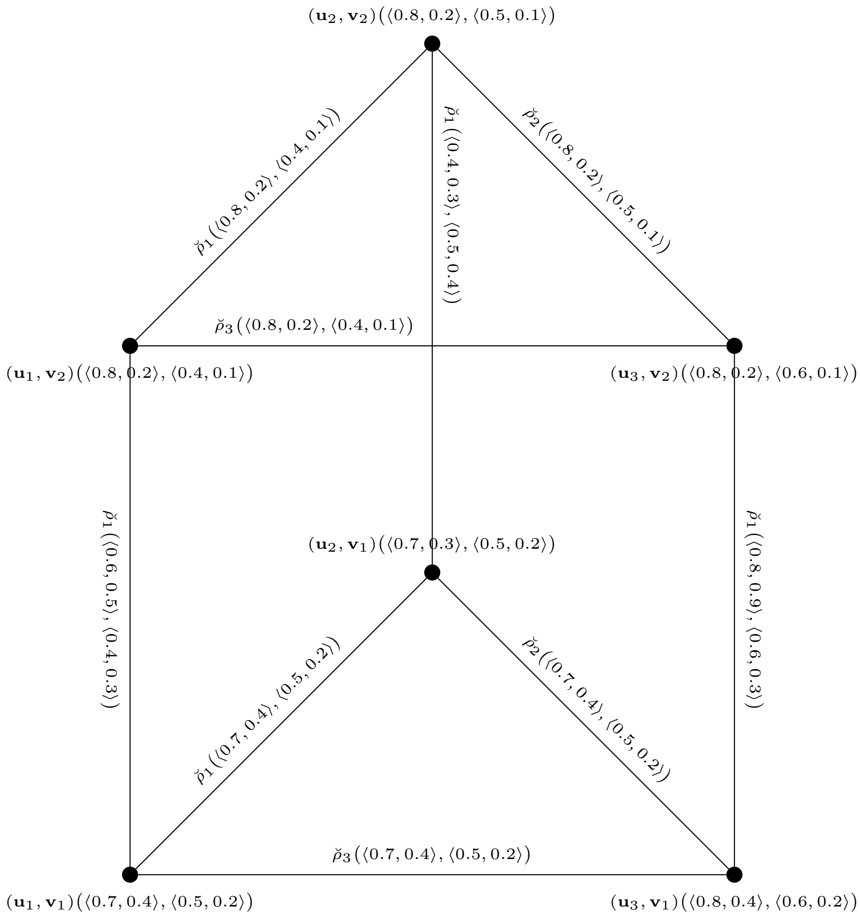

Figure 5: Maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$

Table 27: $\breve{\rho}'_2$

$\mathscr{E}'_2$	$\big(\langle \varkappa^m_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1,\mathbf{u}_2)$	$\big(\langle 0.4,0.5\rangle,\langle 0.3,0.4\rangle\big)$

Table 28: $\breve{\rho}'_3$

$\mathscr{E}'_3$	$\big(\langle \varkappa^m_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_2,\mathbf{u}_3)$	$\big(\langle 0.4,0.9\rangle,\langle 0.5,0.4\rangle\big)$

Table 29: $\breve{\rho}''_1$

$\mathscr{E}''_1$	$\big(\langle \varkappa^m_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{v}_1,\mathbf{v}_2)$	$\big(\langle 0.3,0.9\rangle,\langle 0.2,0.5\rangle\big)$

Table 30: $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$

$\mathscr{V}$	$\big(\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \varkappa^n_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \beta_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$\big(\langle 0.7, 0.3\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$\big(\langle 0.8, 0.4\rangle, \langle 0.6, 0.2\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.6, 0.1\rangle\big)$

Table 31: $\breve{\rho}_1$

$\mathscr{E}_1$	$\big(\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2)$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.3\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.4, 0.3\rangle, \langle 0.5, 0.4\rangle\big)$
$(\mathbf{u}_3\mathbf{v}_1,\mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.8, 0.9\rangle, \langle 0.6, 0.3\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2,\mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$

Table 32: $\breve{\rho}_2$

$\mathscr{E}_2$	$\big(\langle \varkappa^m_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1,\mathbf{u}_3\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_2,\mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$

Table 33: $\breve{\rho}_3$

$\mathscr{E}_3$	$\big(\langle \varkappa^m_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_3\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2,\mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$

Then, using the formula given in Theorem 4.12, we calculate the degrees of the vertices in the maximal product as follows:

$$\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_1)\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1) = (2)(0.7) + (1)(0.6) = 2$$
$$\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_2)\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1) = (2)(0.8) + (1)(0.6) = 2.2$$
$$\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_1)\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2) = (2)(0.7) + (1)(0.4) = 1.8$$
$$\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_2)\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2) = (2)(0.8) + (1)(0.4) = 2$$
$$\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_3)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_1)\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3) = (2)(0.7) + (1)(0.8) = 2.2$$
$$\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_3)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_2)\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3) = (2)(0.8) + (1)(0.8) = 2.4$$

And,

$$\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_1)\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1) = (2)(0.4) + (1)(0.5) = 1.3$$
$$\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_2)\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1) = (2)(0.2) + (1)(0.5) = 0.9$$
$$\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_1)\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2) = (2)(0.4) + (1)(0.3) = 1.1$$
$$\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_2)\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2) = (2)(0.2) + (1)(0.3) = 0.7$$
$$\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_3)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_1)\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3) = (2)(0.4) + (1)(0.9) = 1.7$$
$$\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_3)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_2)\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3) = (2)(0.6) + (1)(0.9) = 1.3$$

In the similar way, we get $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)$ and $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)$ for all $\mathbf{x}_i \in \mathscr{V}_1$ and $\mathbf{y}_i \in \mathscr{V}_2$, which are shown in TABLE 34.

Table 34: $\mathbb{D}_{\breve{\mathscr{G}}}$

$\mathscr{V}$	$\left(\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\rangle, \langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\rangle\right)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$(\langle 2, 1.3\rangle, \langle 1.4, 0.7\rangle)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$(\langle 2.2, 0.9\rangle, \langle 1.2, 0.5\rangle)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$(\langle 1.8, 1.1\rangle, \langle 1.5, 0.8\rangle)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$(\langle 2, 0.7\rangle, \langle 1.3, 0.6\rangle)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$(\langle 2.2, 1.7\rangle, \langle 1.6, 0.7\rangle)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$(\langle 2.4, 1.3\rangle, \langle 1.4, 0.5\rangle)$

Theorem 4.14. *If $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k)$ are two LDFGSs, such that $\breve{\rho}''_i \supseteq \mathfrak{L}_1$, $i = 1, 2, \cdots, k$, then the total degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is described as:*

$$\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j) = \left(\langle \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\rangle, \langle \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\rangle\right), \quad (54)$$

where

$$\left.\begin{aligned}\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j)+\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\ \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j)+\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{y}_j),\\ \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j)+\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\ \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j)+\beta_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).\end{aligned}\right\} \tag{55}$$

Proof. Let $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ be two LDFGSs such that such that $\breve{\rho}''_i \supseteq \mathfrak{L}_1$, then $\breve{\rho}'_i \supseteq \mathfrak{L}_2$ and $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ $i = 1, 2, \cdots, k$. We have,

$$\begin{aligned}\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l}\varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j)+\sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k}\varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)+\varkappa^m_{\mathfrak{L}}(\mathbf{x}_i,\mathbf{y}_j)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l}\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j)+\sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k}\varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)+\left[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j)\right]\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j)+\left(\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)+\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j)\right)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j)+\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).\end{aligned}$$

Also,

$$\begin{aligned}\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l}\varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j)+\sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k}\varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)+\varkappa^n_{\mathfrak{L}}(\mathbf{x}_i,\mathbf{y}_j)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l}\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j)+\sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k}\varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)+\left[\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j)\right]\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j)+\left(\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)+\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j)\right)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j)+\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).\end{aligned}$$

Similarly, we can show that $\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j)+\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)$ and $\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j)+\beta_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)$. □

Example 4.15. Let $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}'_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}'_2)$ be GSs with $\mathscr{V}_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$, $\mathscr{V}_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$, $\mathscr{E}'_1 = \{(\mathbf{u}_1, \mathbf{u}_2)\}$ and $\mathscr{E}''_1 = \{(\mathbf{v}_1, \mathbf{v}_2)\}$. The LDFGSs $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho}'_1)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho}''_1)$ with underlying GSs $\mathscr{G}_1$ and $\mathscr{G}_2$, respectively are shown in FIGURE 6, where $\mathfrak{L}_1$ on $\mathscr{V}_1$ and $\mathfrak{L}_2$ on $\mathscr{V}_2$ are given in TABLES 35 and 36, respectively, and LDFRs $\breve{\rho}'_1$ and $\breve{\rho}''_1$ presented in TABLES 37 and 38, respectively with the condition $\mathfrak{L}_1 \subseteq \breve{\rho}''_1$. By using the Definition 4.1, we obtain the maximal LDFGS $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \left(\mathfrak{L}, \breve{\rho}\right)$ is portrayed in FIGURE 7, where $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$ given in TABLE 39 on $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2 = \left\{(\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_1, \mathbf{v}_2), (\mathbf{u}_2, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2)\right\}$ and LDFR $\breve{\rho}_1 = \breve{\rho}'_1 * \breve{\rho}''_1$ on $\mathscr{E}_1 = \mathscr{E}'_1 \times \mathscr{E}''_1 = \left\{(\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_1\mathbf{v}_2), (\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_1), (\mathbf{u}_1\mathbf{v}_2, \mathbf{u}_2\mathbf{v}_2), (\mathbf{u}_2\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_2)\right\}$ presented in TABLE 40.

Table 35: $\mathfrak{L}_1$

$\mathscr{V}_1$	$\left(\langle\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\rangle, \langle\alpha_{\mathfrak{L}_1}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x})\rangle\right)$
$\mathbf{u}_1$	$\left(\langle 0.6, 0.5\rangle, \langle 0.4, 0.2\rangle\right)$
$\mathbf{u}_2$	$\left(\langle 0.5, 0.7\rangle, \langle 0.3, 0.5\rangle\right)$

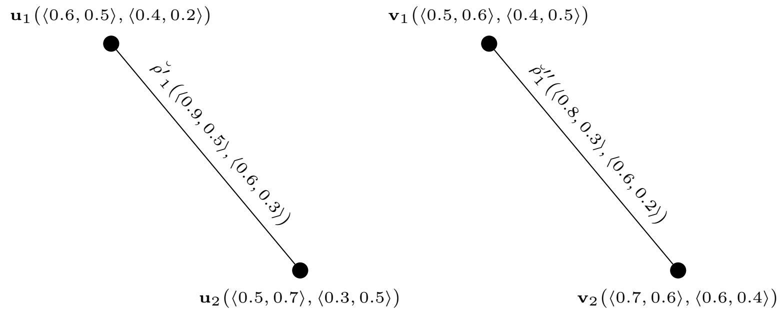

Figure 6: LDFGSs $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1\big)$

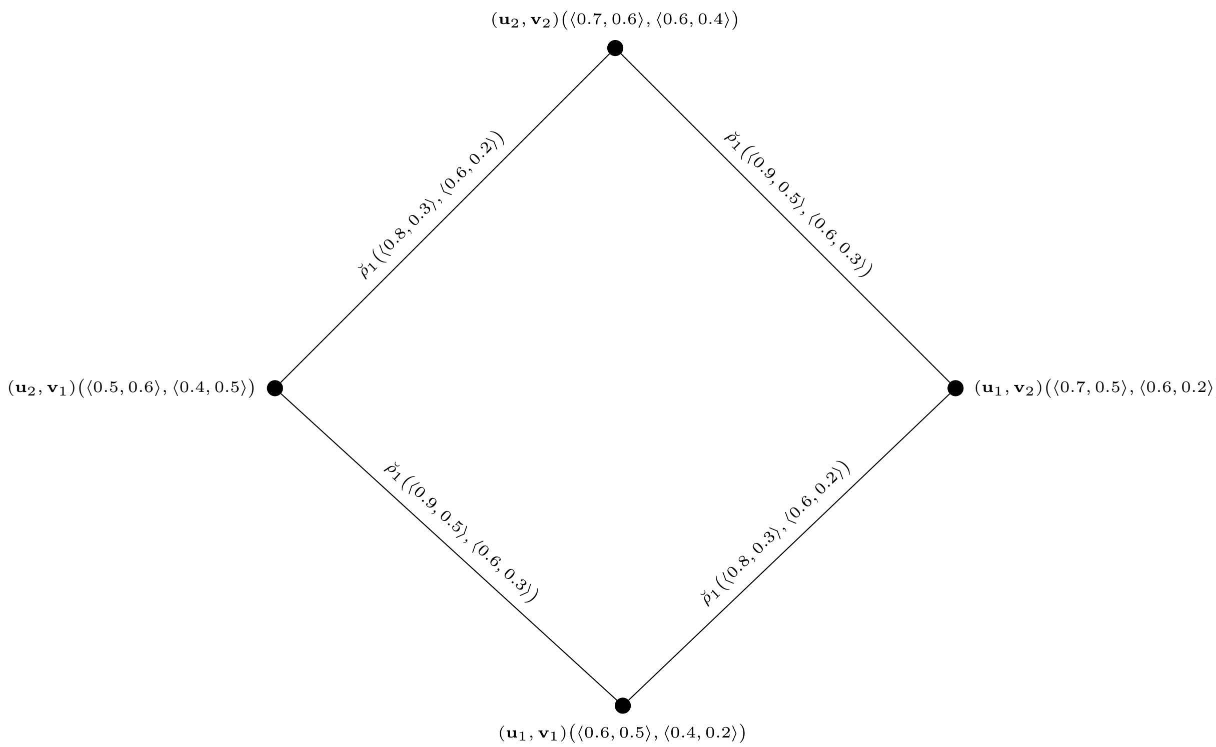

Figure 7: The maximal LDFGS $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$

Table 36: $\mathfrak{L}_2$

$\mathscr{V}_2$	$\big(\langle \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_2}(\mathbf{x})\rangle\big)$
$\mathbf{v}_1$	$\big(\langle 0.5, 0.6\rangle, \langle 0.4, 0.5\rangle\big)$
$\mathbf{v}_2$	$\big(\langle 0.7, 0.6\rangle, \langle 0.6, 0.4\rangle\big)$

Using the Formula given Theorem 4.14, we calculate the total degrees of all the vertices of the maximal product in the sequel:

Table 37: $\breve{\rho'}_1$

$\mathscr{E}'_1$	$\big(\langle \varkappa^m_{\breve{\rho'}_1}(\mathbf{x}), \varkappa^n_{\breve{\rho'}_1}(\mathbf{x})\rangle, \langle \alpha_{\breve{\rho'}_1}(\mathbf{x}), \beta_{\breve{\rho'}_1}(\mathbf{x})\rangle\big)$
$(\mathbf{u}_1, \mathbf{u}_2)$	$\big(\langle 0.9, 0.5\rangle, \langle 0.6, 0.3\rangle\big)$

Table 38: $\breve{\rho}''_1$

$\mathscr{E}''_1$	$\big(\langle \varkappa^m_{\breve{\rho}''_1}(\mathbf{x}), \varkappa^n_{\breve{\rho}''_1}(\mathbf{x})\rangle, \langle \alpha_{\breve{\rho}''_1}(\mathbf{x}), \beta_{\breve{\rho}''_1}(\mathbf{x})\rangle\big)$
$(\mathbf{v}_1, \mathbf{v}_2)$	$\big(\langle 0.8, 0.3\rangle, \langle 0.6, 0.2\rangle\big)$

Table 39: $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$

$\mathscr{V}$	$\big(\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}, \mathbf{y}), \varkappa^n_{\mathfrak{L}}(\mathbf{x}, \mathbf{y})\rangle, \langle \alpha_{\mathfrak{L}}(\mathbf{x}, \mathbf{y}), \beta_{\mathfrak{L}}(\mathbf{x}, \mathbf{y})\rangle\big)$
$(\mathbf{u}_1, \mathbf{v}_1)$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_1, \mathbf{v}_2)$	$\big(\langle 0.7, 0.5\rangle, \langle 0.6, 0.2\rangle\big)$
$(\mathbf{u}_2, \mathbf{v}_1)$	$\big(\langle 0.5, 0.6\rangle, \langle 0.4, 0.5\rangle\big)$
$(\mathbf{u}_2, \mathbf{v}_2)$	$\big(\langle 0.7, 0.6\rangle, \langle 0.6, 0.4\rangle\big)$

Table 40: $\breve{\rho}_1 = \breve{\rho}'_1 \times \breve{\rho}''_1$

$\mathscr{E}_1$	$\big(\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_1\mathbf{v}_2)$	$\big(\langle 0.8, 0.3\rangle, \langle 0.6, 0.2\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_1)$	$\big(\langle 0.9, 0.5\rangle, \langle 0.6, 0.3\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2, \mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.9, 0.5\rangle, \langle 0.6, 0.3\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.8, 0.3\rangle, \langle 0.6, 0.2\rangle\big)$

$$\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_1, \mathbf{v}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_1) = (1)(0.5) + (0.8 + 0.5) = 1.8$$
$$\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_1, \mathbf{v}_2) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_2) = (1)(0.7) + (0.8 + 0.7) = 2.2$$
$$\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_1) = (1)(0.5) + (0.8 + 0.5) = 1.8$$
$$\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_2) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_2) = (1)(0.7) + (0.8 + 0.7) = 2.2$$

Also,

$$\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_1)=(1)(0.6)+(0.3+0.6)=1.5$$
$$\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_2)=(1)(0.6)+(0.3+0.6)=1.5$$
$$\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_1)=(1)(0.6)+(0.3+0.6)=1.5$$
$$\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_2)=(1)(0.6)+(0.3+0.6)=1.5$$

In the similar way, we've calculated $\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j)+\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)$, and $\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j)+\beta_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)$ for all $\mathbf{x}_i\in\mathscr{V}_1$, and $\mathbf{y}_j\in\mathscr{V}_2$, which are listed in the TABLE 41.

Table 41: $\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)$

$\mathscr{V}$	$\big(\langle\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\rangle,\langle\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$(\langle 1.8,1.5\rangle,\langle 1.4,1.2\rangle)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$(\langle 2.2,1.5\rangle,\langle 1.8,1.1\rangle)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$(\langle 1.8,1.5\rangle,\langle 1.4,1.2\rangle)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$(\langle 2.2,1.5\rangle,\langle 1.8,1.1\rangle)$

Theorem 4.16. *If* $\breve{\mathscr{G}}_1=\big(\mathfrak{L}_1,\breve{\rho}'_1,\breve{\rho}'_2,\cdots,\breve{\rho}'_k\big)$ *and* $\breve{\mathscr{G}}_2=\big(\mathfrak{L}_2,\breve{\rho}''_1,\breve{\rho}''_2,\cdots,\breve{\rho}''_k\big)$ *are two LDFGSs, such that* $\breve{\rho}''_i\supseteq\mathfrak{L}_1$, $i=1,2,\cdots,k$, *and* $\mathfrak{L}_2$ *is constant LDFS of LDF value* $\big(\langle a,b\rangle,\langle c,d\rangle\big)$, *where* $a,b,c,d\in[0,1]$ *are fixed, then the total degree of any vertex in maximal product* $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2$ *is characterized as:*

$$\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)=\Big(\big\langle\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle,\big\langle\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big),\tag{56}$$

where

$$\left.\begin{aligned}\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)+\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)a,\\ \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)+\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)b,\\ \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)+\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)c,\\ \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)+\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)d.\end{aligned}\right\}\tag{57}$$

Proof. Analogous to the proof of Theorems 4.9 and 4.14. □

Theorem 4.17. *If* $\breve{\mathscr{G}}_1=\big(\mathfrak{L}_1,\breve{\rho}'_1,\breve{\rho}'_2,\cdots,\breve{\rho}'_k\big)$ *and* $\breve{\mathscr{G}}_2=\big(\mathfrak{L}_2,\breve{\rho}''_1,\breve{\rho}''_2,\cdots,\breve{\rho}''_k\big)$ *are two LDFGSs, such that* $\breve{\rho}'_i\supseteq\mathfrak{L}_2$, $i=1,2,\cdots,k$, *then the total degree of any vertex in maximal product* $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2$ *is postulated as:*

$$\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)=\Big(\big\langle\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle,\big\langle\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big),\tag{58}$$

where

$$\left.\begin{aligned}\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)+\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\ \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)+\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\ \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)+\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\ \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i)+\beta_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i).\end{aligned}\right\}\tag{59}$$

Proof. Identical to the proof of Theorems 4.10 and 4.14. □

Theorem 4.18. *If* $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_k\big)$ *and* $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_k\big)$ *are two LDFGSs, such that* $\breve{\rho'}_i \supseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$, *and* $\mathfrak{L}_1$ *is constant LDFS of LDF value* $\big(\langle a, b\rangle, \langle c, d\rangle\big)$, *where* $a, b, c, d \in [0, 1]$ *are fixed, then the total degree of any vertex in maximal product* $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ *is given by:*

$$\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big), \tag{60}$$

where

$$\left.\begin{aligned} \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)a + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\ \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)b + \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\ \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)c + \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\ \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)d + \beta_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i). \end{aligned}\right\} \tag{61}$$

Proof. Analogous to the proof of Theorems 4.11 and 4.14. □

5 Regular Linear Diophantine Fuzzy Graph Structures

In this section, we have defined the notions of $\breve{\rho}_i$-regular and regular LDFGSs. Some fascinating consequences are also proved with illustrative examples.

Definition 5.1. An LDFGS $\breve{\mathscr{G}}$ is said to be $\big(\langle \mathbf{a}, \mathbf{b}\rangle, \langle \mathbf{c}, \mathbf{d}\rangle\big)$-$\breve{\rho}_i$ regular, if $\mathbb{D}_{\breve{\rho}_i}(\mathbf{x}) = \big(\langle \mathbf{a}, \mathbf{b}\rangle, \langle \mathbf{c}, \mathbf{d}\rangle\big)$, for all $\mathbf{x} \in \mathscr{V}$. Moreover, $\breve{\mathscr{G}}$ is called $\big(\langle \mathbf{a}, \mathbf{b}\rangle, \langle \mathbf{c}, \mathbf{d}\rangle\big)$-regular, if $\mathbb{D}(\mathbf{x}) = \big(\langle \mathbf{a}, \mathbf{b}\rangle, \langle \mathbf{c}, \mathbf{d}\rangle\big)$, for all $\mathbf{x} \in \mathscr{V}$.

Example 5.2. From Example 3.2, we can easily see that $\breve{\mathscr{G}}$ is neither $\breve{\rho}_1$ nor $\breve{\rho}_2$ regular. Also, not regular LDFGS.

Remark 5.3. The maximal product of two regular LDFGSs may not be regular, which can justified through Example 5.4.

Example 5.4. Let $\mathscr{V}_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$, $\mathscr{V}_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$, $\mathscr{E}'_1 = \{(\mathbf{u}_1, \mathbf{u}_2)\}$ and $\mathscr{E}''_1 = \{(\mathbf{v}_1, \mathbf{v}_2)\}$. Then, $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}'_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}'_2)$ are GSs.

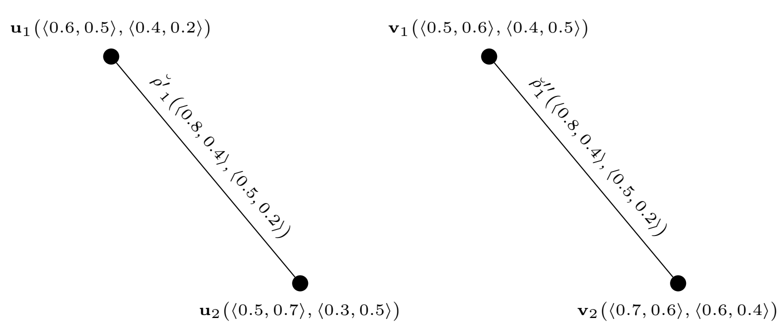

Figure 8: LDFGSs $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1\big)$

Consider LDFSs $\mathfrak{L}_1$ on $\mathscr{V}_1$ and $\mathfrak{L}_2$ on $\mathscr{V}_2$ which are given in TABLES 42 and 43, respectively. LDFRs $\breve{\rho}'_1$ and $\breve{\rho}''_1$ are exhibited in TABLES 44 and 45, respectively.

Table 42: $\mathfrak{L}_1$

$\mathscr{V}_1$	$\big(\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}_1}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x})\rangle\big)$
$\mathbf{u}_1$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.2\rangle\big)$
$\mathbf{u}_2$	$\big(\langle 0.5, 0.7\rangle, \langle 0.3, 0.5\rangle\big)$

Table 43: $\mathfrak{L}_2$

$\mathscr{V}_2$	$\big(\langle \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_2}(\mathbf{x})\rangle\big)$
$\mathbf{v}_1$	$\big(\langle 0.5, 0.6\rangle, \langle 0.4, 0.5\rangle\big)$
$\mathbf{v}_2$	$\big(\langle 0.7, 0.6\rangle, \langle 0.6, 0.4\rangle\big)$

Table 44: $\breve{\rho'}_1$

$\mathscr{E}'_1$	$\big(\langle \varkappa^m_{\breve{\rho'}_1}(\mathbf{x}), \varkappa^n_{\breve{\rho'}_1}(\mathbf{x})\rangle, \langle \alpha_{\breve{\rho'}_1}(\mathbf{x}), \beta_{\breve{\rho'}_1}(\mathbf{x})\rangle\big)$
$(\mathbf{u}_1, \mathbf{u}_2)$	$\big(\langle 0.8, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$

Table 45: $\breve{\rho}''_1$

$\mathscr{E}''_1$	$\big(\langle \varkappa^m_{\breve{\rho}''_1}(\mathbf{x}), \varkappa^n_{\breve{\rho}''_1}(\mathbf{x})\rangle, \langle \alpha_{\breve{\rho}''_1}(\mathbf{x}), \beta_{\breve{\rho}''_1}(\mathbf{x})\rangle\big)$
$(\mathbf{v}_1, \mathbf{v}_2)$	$\big(\langle 0.8, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$

It becomes evident that $\breve{\mathscr{G}}_1 = (\mathscr{V}_1, \breve{\rho'}_1)$ and $\breve{\mathscr{G}}_2 = (\mathscr{V}_2, \breve{\rho''}_1)$ are LDFGSs which are depicted in FIGURE 8 and they are $\big(\langle 0.8, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$-regular.

By employing Definition 4.1, we obtain the following LDFS $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$ given in TABLE 16 on $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2 = \big\{(\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_1, \mathbf{v}_2), (\mathbf{u}_2, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2)\big\}$ and LDFR $\breve{\rho}_1 = \breve{\rho}'_1 * \breve{\rho}''_1$ shown in $\mathfrak{L}$ on $\mathscr{V}$ is calculated in TABLE 46 on $\mathscr{E}_1 = \mathscr{E}'_1 \times \mathscr{E}''_1 = \big\{(\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_1\mathbf{v}_2), (\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_1), (\mathbf{u}_1\mathbf{v}_2, \mathbf{u}_2\mathbf{v}_2), (\mathbf{u}_2\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_2)\big\}$. LDFR $\breve{\rho}_1 = \breve{\rho}'_1 \times \breve{\rho}''_1$ is calculated in Table 47.

Then the maximal LDFGS $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \big(\mathfrak{L}, \breve{\rho}_1\big)$ is portrayed in FIGURE 9.

Table 46: $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$

$\mathscr{V}$	$\big(\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}, \mathbf{y}), \varkappa^n_{\mathfrak{L}}(\mathbf{x}, \mathbf{y})\rangle, \langle \alpha_{\mathfrak{L}}(\mathbf{x}, \mathbf{y}), \beta_{\mathfrak{L}}(\mathbf{x}, \mathbf{y})\rangle\big)$
$(\mathbf{u}_1, \mathbf{v}_1)$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_1, \mathbf{v}_2)$	$\big(\langle 0.7, 0.5\rangle, \langle 0.6, 0.2\rangle\big)$
$(\mathbf{u}_2, \mathbf{v}_1)$	$\big(\langle 0.5, 0.6\rangle, \langle 0.4, 0.5\rangle\big)$
$(\mathbf{u}_2, \mathbf{v}_2)$	$\big(\langle 0.7, 0.6\rangle, \langle 0.6, 0.4\rangle\big)$

From Definition 4.6, we can calculate the $\breve{\rho}_1$-degrees of each vertex of $\mathfrak{L}$ as follows:

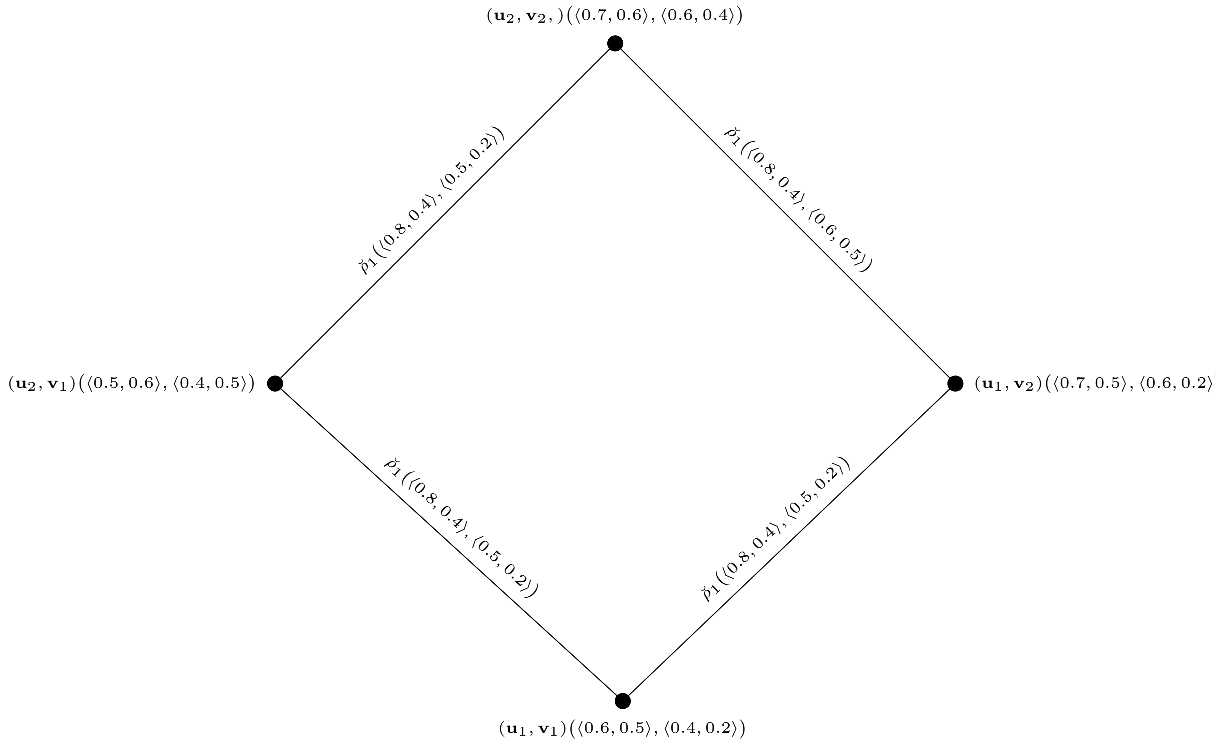

Figure 9: The maximal LDFGS $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$

Table 47: $\breve{\rho}_1 = \breve{\rho}_1' \times \breve{\rho}_1''$

$\mathscr{E}_1$	$\big(\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2)$	$\big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1)$	$\big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2,\mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.8,0.4\rangle,\langle 0.6,0.5\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)$

$$\begin{aligned}\mathbb{D}_{\breve{\rho}_1}(\mathbf{u}_1,\mathbf{v}_1) &= \big(\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2)+\varkappa^m_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1), \varkappa^n_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2)+\varkappa^n_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1)\rangle,\\ &\quad \langle \alpha_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2)+\alpha_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1), \beta_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2)+\beta_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1)\rangle\big)\\ &= \big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)+\big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)\\ &= \big(\langle 1.6,0.8\rangle,\langle 1,0.4\rangle\big)\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{D}_{\breve{\rho}_1}(\mathbf{u}_1,\mathbf{v}_2) &= \big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)+\big(\langle 0.8,0.4\rangle,\langle 0.6,0.2\rangle\big)=\big(\langle 1.6,0.8\rangle,\langle 1.1,0.4\rangle\big)\\ \mathbb{D}_{\breve{\rho}_1}(\mathbf{u}_2,\mathbf{v}_1) &= \big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)+\big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)=\big(\langle 1.6,0.8\rangle,\langle 1,0.4\rangle\big)\\ \mathbb{D}_{\breve{\rho}_1}(\mathbf{u}_2,\mathbf{v}_2) &= \big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)+\big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)=\big(\langle 1.6,0.8\rangle,\langle 1,0.4\rangle\big)\end{aligned}$$

Clearly, $\breve{\mathscr{G}}$ is not regular since $\mathbb{D}_{\breve{\rho}_1}(\mathbf{u}_1, \mathbf{v}_1) = \big(\langle 1.6, 0.8\rangle, \langle 1, 0.4\rangle\big) \neq \big(\langle 1.6, 0.8\rangle, \langle 1.1, 0.4\rangle\big) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{u}_1, \mathbf{v}_2)$.

Theorem 5.5. *If $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, ..., \breve{\rho}'_k\big)$ is $\big(\langle \mathbf{r}, \mathbf{s}\rangle, \langle \mathbf{s}, \mathbf{t}\rangle\big)$-regular LDFGS and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, ..., \breve{\rho}''_k\big)$ is an LDFGS, such that $\breve{\rho}''_i \supseteq \mathfrak{L}_1$, $i = 1, 2, ..., k$, and $\mathfrak{L}_2$ is constant LDFS of LDF value $\big(\langle a, b\rangle, \langle c, d\rangle\big)$, where $a, b, c, d \in [0, 1]$ are fixed, then maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular if and only if $\breve{\mathscr{G}}_2$ is regular.*

Proof. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, ..., \breve{\rho}'_k\big)$ be partially regular LDFGS and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, ..., \breve{\rho}''_k\big)$ be an LDFGS, such that $\breve{\rho}''_i \supseteq \mathfrak{L}_1$, $i = 1, 2, ..., k$, and $\mathfrak{L}_2 = \big(\langle a, b\rangle, \langle c, d\rangle\big)$ be a constant LDFGS. Then,

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big)$$

where

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)a + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j);\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)b + \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j);\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)c + \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j);\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)d + \beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).
\end{aligned}$$

This holds for all vertices of $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$. Hence, maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular.
Conversely, suppose that maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular. Then, for any two vertices of $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$,

$$\begin{aligned}
&\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)\\
&\Rightarrow \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_1)a + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_2)a + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_2)\\
&\Rightarrow \mathbf{r}a + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_1) = \mathbf{r}a + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_2)\\
&\Rightarrow \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_1) = \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_2)
\end{aligned}$$

Similarly, $\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies that $\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_1) = \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_2)$; $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies that $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_1) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_2)$; $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies that $\beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_1) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_2)$. This holds for all vertices of $\breve{\mathscr{G}}_2$. Hence, $\breve{\mathscr{G}}_2$ is regular LDFGS. □

Theorem 5.6. *If $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, ..., \breve{\rho}'_k\big)$ is partially regular LDFGS and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, ..., \breve{\rho}''_k\big)$ is an LDFGS, such that $\breve{\rho}'_i \supseteq \mathfrak{L}_2$, $i = 1, 2, ..., k$, and $\mathfrak{L}_2$ is constant LDFS of LDF value $\big(\langle a, b\rangle, \langle c, d\rangle\big)$, where $a, b, c, d \in [0, 1]$ are fixed, then maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular if and only if $\breve{\mathscr{G}}_1$ is regular.*

Proof. Suppose with the given assumptions, we have from Theorem 4.11,

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big),$$

where

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)a;\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)b;\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)c;\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)d.
\end{aligned}$$

which holds for all vertices of $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$. Hence, maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular.
Conversely, assume that maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular. Then for any two vertices of $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$, we have:

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2) \\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_1) + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{y}_1)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_2) + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{y}_2)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2) \\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_1) + \mathbf{r}_2\mathbf{a} &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_2) + \mathbf{r}_2\mathbf{a} \\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_1) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_2)
\end{aligned}$$

Similarly, $\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies that $\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_1) = \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_2)$; $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies that $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_1) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_2)$; $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies that $\beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_1) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_2)$. This proves that $\breve{\mathscr{G}}_1$ regular LDFGS. □

Theorem 5.7. *If $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, ..., \breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, ..., \breve{\rho}''_k\right)$ are two $\left(\langle r_1, s_1\rangle, \langle t_1, u_1\rangle\right)$-regular and $\left(\langle r_2, s_2\rangle, \langle t_2, u_2\rangle\right)$-regular LDFGSs, respectively, such that $\breve{\rho}''_i \subseteq \mathfrak{L}_1$ and $\breve{\rho}''_i \subseteq \mathfrak{L}_2$, $i = 1, 2, ..., k$ and $\mathfrak{L}_2$ is a constant LDFS of LDF value $\left(\langle a, b\rangle, \langle c, d\rangle\right)$, where $a, b, c, d \in [0, 1]$ are fixed, then maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular if and only if $\mathfrak{L}_1$ is a constant LDFS of LDF value $\left(\langle a', b'\rangle, \langle c', d'\rangle\right)$, where $a', b', c', d' \in [0, 1]$ are fixed.*

Proof. With the given assumptions, we have from Theorem 4.12,

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \left(\left\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\right\rangle, \left\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\right\rangle\right),$$

where

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i) = r_1 a + r_2 a'; \\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i) = s_1 b + s_2 b'; \\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i) = t_1 c + t_2 c'; \\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i) = u_1 d + u_2 d';
\end{aligned}$$

which holds for all vertices of $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$. Hence, $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular.
Conversely, assume that $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular. For any two vertices of $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$, we have:

$$\begin{aligned}
&\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2) \\
&\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_1)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_1)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_2)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_2) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_2)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2) \\
&r_1\varkappa^m_{\mathfrak{L}_2}(\mathbf{x}_1) + r_2\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) = r_1\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) + r_2\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \\
&\varkappa^m_{\mathfrak{L}_2}(\mathbf{x}_1) = \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)
\end{aligned}$$

Similarly, $\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies $\varkappa^n_{\mathfrak{L}_2}(\mathbf{x}_1) = \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_1)$; $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies $\alpha_{\mathfrak{L}_2}(\mathbf{x}_1) = \alpha_{\mathfrak{L}_2}(\mathbf{y}_1)$; $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies $\beta_{\mathfrak{L}_2}(\mathbf{x}_1) = \beta_{\mathfrak{L}_2}(\mathbf{y}_1)$, which holds for all vertices of $\mathscr{G}_1$. Hence, $\mathfrak{L}_1$ is constant LDFS. □

6 Conclusion

Graphs are used in various applications such as social networks, recommendation systems, routing algorithms, and many more. A GS has n mutually disjoint, symmetric and irreflexive relations. Understanding these structures and their properties is key to leveraging graphs effectively in solving real-world problems. However, in certain scenarios, several features of GT might be uncertain. FGSs have many advantages to cope with vagueness and uncertainty. FGSs are more advantageous to circumvent uncertainty. In this research study, we have applied the notion of LDFSs to GSs and introduced a novel concept LDFGS. We have defined $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, strength of $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, vertex degree, total $\breve{\rho}_i$-degree of a vertex, and total vertex degree in an LDFGS. Also, we have introduced the $\breve{\rho}_i$-size, size, and order of an LDFGS. Moreover, the ideas of the maximal product of two LDFGSs, strong LDFGS, degree and $\breve{\rho}_i$-degree of the maximal product, $\breve{\rho}_i$-regular and regular LDFGS are introduced, along with examples for clarification. Certain significant results related to the proposed concepts also demonstrated with explanatory examples such as the maximal product of two strong LDFGSs is also a strong LDFGS, the maximal product of two connected LDFGSs is also a connected LDFGS but the maximal product of two regular LDFGS may not be a regular LDGS. Moreover, many interesting and alternative formulas for calculating $\breve{\rho}_i$-degrees of an LDFGS in various situations are proved with examples. LDFGSs are highly beneficial for solving numerous combinatorial problems involving multiple relations than the existing GSs in the context of FS, IFS, PFS and q-ROFS. LDFGSs as an extension of IFGS and LDFG to GSs deals the graph theoretical aspects in more appropriate way due to their flexibility in selecting MD and NMD alongside their reference parameters.

In the future, we aim to extend our approach to (1) rough linear Diophantine fuzzy graph structures, (2) rough linear Diophantine fuzzy soft graph structures, (3) linear Diophantine fuzzy soft graph structures, and (4) Spherical linear Diophantine fuzzy graph structures.

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