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Original Research

Mean-AVaR-Skewness-Kurtosis Optimization Portfolio Selection Model in Uncertain Environments

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ABSTRACT

Several research investigations have indicated that asset returns exhibit notable skewness and kurtosis, which have a substantial impact on the utility function of investors. Additionally, it has been observed that Average Value-at-Risk (AVaR) provides a more accurate estimation of risk compared to variance. This study focuses on the computational challenge associated with portfolio optimization in an uncertain context, employing the Mean-AVaR-skewness-kurtosis paradigm.The uncertainty around the total return is considered and analyzed in the context of the challenge of selecting an optimal portfolio. The concepts of Value-at-Risk (VaR), Average Value-at-Risk (AVaR), skewness, and kurtosis are initially introduced to describe uncertain variables. These concepts are then further explored to identify and analyse relevant aspects within specific distributions. The outcomes of this study will convert the existing models into deterministic forms and uncertain mean-AVaR-skewness-kurtosis optimization models for portfolio selection. These models are designed to cater to the demands of investors and mitigate their apprehensions.

1 Introduction

The primary objective of the optimum portfolio selection theory is to maximize the profits of investors by considering a range of alternative investments based on their individual preferences. Initially, the mean-variance model developed by Markowitz served as the foundation for addressing the portfolio selection problem [1]. Numerous research have subsequently been conducted to explore portfolio optimization within the context of these two moments of the return distribution [2]. Subsequent to this, extensive research has been conducted to explore diverse methodologies that can be employed to model investment risk, aiming to achieve a more accurate estimation of risk. For instance, previous studies have utilised risk functions or Value at Risk (VaR) measures such as [3-13]. However, VaR suffers from certain limitations. Firstly, it fails to provide information regarding the magnitude of losses exceeding the VaR level. Additionally, VaR does not satisfy the coherence criterion. Consequently, the concept of average value-at-risk (AVaR) has emerged as an alternative risk measure.

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The proposal of AVaR is suggested as a potential solution to address the inherent issues associated with VaR. VaR, being a risk measure that lacks coherence in general, necessitates the introduction of AVaR as a more suitable alternative. In numerous articles, alternative terms such as conditional Value-at-Risk (CVaR), Tail Value-at-Risk (TVaR), or Expected Shortfall (ES) are employed to refer to the same concept [14]. However, for the purpose of this discussion, we will adopt the term average value-at-risk (AVaR) as it more accurately captures the essence of the variable under consideration. Risk is derived from the presence of uncertainty, which may be categorised into two main types: objective uncertainty and subjective uncertainty. Stochasticity is a fundamental form of objective uncertainty, whereas probability theory serves as a mathematical discipline dedicated to the analysis of the characteristics and dynamics of random events. The conventional risk metric known as Value at Risk (VaR) or Average Value at Risk (AVaR) has typically been introduced within a stochastic framework. The present research introduces the concept of the credibilistic AVaR as a novel risk measure, offering a more advantageous alternative to VaR within the framework of uncertainty theory as proposed by Liu [15].

Numerous manuscripts addressing optimal portfolio selection problems have been published, highlighting the insufficiency of relying solely on average and variance, or alternative risk estimators such as AVaR, for determining the optimal portfolio allocation. Recent research has emphasised the importance of considering additional factors such as skewness and kurtosis, which have proven to be highly effective and influential in this context [16-31]. In light of these findings, scholars have recently exhibited a growing interest in higher-order moments.

In conventional practise, it has been widely accepted that security returns exhibit stochastic behaviour, hence necessitating the application of probability theory as the primary means for achieving optimal portfolio selection. However, it is evident that the effectiveness of security measures is influenced by a range of factors, such as social, political, economic, human cognitive, and notably psychological factors. Research has demonstrated that historical data does not well capture short-term security returns. Empirical data suggests that the probability distribution of underlying asset returns exhibits greater peaks and heavier tails compared to the normal distribution. Furthermore, it is observed that the first two moments alone are inadequate in characterising this distribution. Several studies have employed fuzzy variables as a means to address the aforementioned problems [32-35]. However, the utilisation of fuzzy variables has been found to present certain paradoxes [36, 37]. Consequently, the concept of uncertainty theory has garnered significant attention, leading many researchers to incorporate Liu’s uncertain measurement theory into their portfolio selection models [38-43].

Some studies applied skewness with respect to portfolio optimization in uncertain or hybrid uncertain spaces [44]. But they have not considered the fourth moment in their studies. So in this article, our attempt to fill this gap is to find AVaR, skewness, and kurtosis in an uncertain environment to study the mean-AVaR-skewness-kurtosis portfolio optimization model.

First, we verify the uncertain model in the framework of uncertain theory for portfolio selection by considering uncertain returns. Second, AVaR is considered as risk, and due to the asymmetry and different kurtosis of financial assets, it is considered uncertain skewness and at the same time examines kurtosis in the case of uncertain variables. Third, we replace Average Value-at-Risk instead of variance and add skewness-kurtosis to the mean-variance model in an uncertain environment and create a mean-AVaR-skewness-kurtosis uncertain portfolio optimization model. The uncertain mean-AVaR-skewness-kurtosis model will be formulated to get the basic opinion of accounting return, risk, skewness, and kurtosis simultaneously in the portfolio optimization problem in general.

The present paper is structured in the following manner. In Section 2, a comprehensive understanding of uncertain and uncertain-random variables will be acquired by a thorough examination of relevant knowledge. Following this, Section 3 will delve into the examination and validation of skewness and kurtosis pertaining to two distinct categories of uncertain random returns. In the fourth section, many models have been developed to address portfolio selection within the framework of mean-variance-skewness-kurtosis. In Section 5, two instances are employed to illustrate the efficacy of the suggested approach. In Section 6, a set of concluding remarks are presented. Other findings research, through a regular and logical process based on the judgment method in a survey of 14 experts in the field of capital market investment and a quantitative and multivariate model of fuzzy network analysis, to assess the level of importance, ranking and refining the effective factors. Portfolio optimization was undertaken. Based on the analysis, the variables of profit volatility, return on capital, company value, market risk, stock profitability, financial structure, liquidity and survival index can be introduced as the most important factors affecting the optimization of the stock portfolio [48].

2 Preliminaries

Consider Γ be a non-empty set, and define the σ -algebra L be a collection of all the events $\Theta \in L$ over Γ. It could be defined as a function that for each event Θ return $\mathcal{M}\{\Theta\}$ which indicates the belief degree which means that we believe Θ will occur. Liu [15] offered the following five axioms, in order to define uncertain measure in an axiomatic form, to ensure that the number $\mathcal{M}\{\Theta\}$ is not arbitrary and has special mathematical properties;

1: (Normality axiom) $\mathcal{M}(\Gamma) = 1$;

2: (Monotonicity axiom) $\mathcal{M}(\Theta_1) \le \mathcal{M}(\Theta_2)$ every where $\Theta_1 \subseteq \Theta_2$;

3: (Duality axiom) $\mathcal{M}(\Theta) + \mathcal{M}(\Theta^c) = 1$ for every event Θ;

4: (Subadditivity axiom) For each sequence of events $\{\Theta_j\}$ that can be counted, we have

$\mathcal{M}(\bigcup_{j=1}^{\infty} \Theta_j) \le \sum_{j=1}^{\infty} \mathcal{M}(\Theta_j)$

Definition 1. *[45]. The set function $\mathcal{M}$ which satisfies the above axioms, is called an uncertain measure.*

Definition 2. *[45]. Consider Γ be a non-empty set, the σ-algebra L be a collection of all the events over Γ, and $\mathcal{M}$ be an uncertain measure according to the above definition, the triple $(\Gamma, L, \mathcal{M})$ is named an uncertain space.*

5: (Product Measure Axiom) [45]. Let the triple $(\Gamma_k, L_k, \mathcal{M}_k)$ for $k = 1, 2, \ldots, n$, where $\Gamma = \Gamma_1 \times \Gamma_2 \times \ldots$ and $L = L_1 \times L_2 \times \ldots$ be uncertainty spaces, then it satisfyed in

$\mathcal{M}(\prod_{k=1}^{\infty} \Theta_k) \le \bigwedge_{k=1}^{\infty} \mathcal{M}_k(\Theta_k)$

Where Θ_k, are arbitrary events and chosen from L_k for $k = 1, 2, \ldots, n$, respectively.

Definition 3. *[45]. The uncertainty distribution for an uncertain variable such as η is defined by function $\Phi: \mathbb{R} \to [0, 1]$ that $\Phi(x) = \mathcal{M}\{\eta \le x\}$.*

Theorem 1. *[46] Let $\Phi_1, \Phi_2, \ldots, \Phi_n$ be uncertainty distributions of independent uncertain variables $\eta_1, \eta_2, \ldots, \eta_n$, respectively. If $f(t_1, t_2, \ldots, t_n)$ be increasing strictly. Then*

$$\eta = f(\eta_1, \eta_2, \ldots, \eta_n), \tag{1}$$

is an uncertain variable with uncertainty distribution

$$\Psi(t) = \sup_{f(t_1, t_2, \ldots, t_n) = t} \left(\min_{1 \le i \le n} \Phi_i(t_i) \right), \quad t \in \mathbb{R}, \tag{2}$$

and following inverse function

$$\Psi^{-1}(\alpha) = f[\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)], \tag{3}$$

Where $\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)$ are unique for each $\alpha \in (0, 1)$.

Definition 4. *[15]. The expected value of an uncertain variable η is defined by*

$$E[\eta] = \int_0^{\infty} M\{\eta \geq r\}dr - \int_{-\infty}^{0} M\{\eta \leq r\}dr, \tag{4}$$

while at least one of the above integrals be finite.

Theorem 2. *[46]. Let a_1 and a_2 be real numbers and η_1 and η_2 be independent uncertain variables where them expected values are finite, then we have*

$$E[a_1\eta_1 + a_2\eta_2] = a_1E[\eta_1] + a_2E[\eta_2]. \tag{5}$$

Theorem 3. *Let η be an uncertain variable where them expected values are finite, with regular uncertainty distribution Φ, and let k be a positive integer. Then the k-th moment of η is*

$$E[\eta^k] = \int_{-\infty}^{+\infty} \alpha^k d\Phi(\alpha) = \int_0^1 (\Phi^{-1}(\lambda))^k d\lambda \tag{6}$$

Definition 5. *Assume that E indicated the operator of expected value and η be an uncertain variable and $E[\eta]$ be finite. the Skewness and kurtosis of η is defined as*

$$S[\eta] = E[(\eta - E[\eta])^3] \tag{7}$$

and

$$K[\eta] = E[(\eta - E[\eta])^4] \tag{8}$$

Theorem 4. *Let η be an uncertain variable with finite expected value $E[\eta]$, and uncertainty distribution Φ then*

$$S[\eta] = \int_{-\infty}^{+\infty} (\eta - E[\eta])^3 d\Phi(x), \tag{9}$$

and

$$K[\eta] = \int_{-\infty}^{+\infty} (\eta - E[\eta])^4 d\Phi(x), \tag{10}$$

Proof. You can find the proof of part 1 in [44]. Now for proofing the Kurtosis formula, assume that $E[\eta] = e$ is the finite expected value of η. From definition (5)

$$K[\eta] = E[(\eta - E[\eta])^4]$$
$$= \int_0^{+\infty} M\{(\eta - e)^4 \geq x\}dx - \int_{-\infty}^{0} M\{(\eta - e)^4 \leq x\}dx$$
$$= \int_0^{+\infty} M\{\eta - e \geq \sqrt[4]{x}\}dx - \int_{-\infty}^{0} M\{\eta - e \leq \sqrt[4]{x}\}dx$$
$$= \int_0^{+\infty} M\{\eta \geq \sqrt[4]{x} + e\}dx - \int_{-\infty}^{0} M\{\eta \leq \sqrt[4]{x} + e\}dx$$

Now let $\sqrt[4]{x} + e = z$, so $x = (z - e)^4$ and we have

$$\int_0^{+\infty} M\{\eta \geq \sqrt[4]{x} + e\}dx - \int_{-\infty}^{0} M\{\eta \leq \sqrt[4]{x} + e\}dx = \int_e^{+\infty} M\{\eta \geq z\}d(z-e)^4 - \int_{-\infty}^{e} M\{\eta \leq z\}d(z-e)^4$$
$$= \int_e^{+\infty} (1 - \Phi(z))d(z-e)^4 - \int_{-\infty}^{e} \Phi(z)d(z-e)^4$$
$$= \int_e^{+\infty} (z-e)^4 d\Phi(z) + \int_{-\infty}^{e} (z-e)^4 d\Phi(z)$$
$$= \int_{-\infty}^{+\infty} (z-e)^4 d\Phi(z).$$

In the result the kurtosis is

$$K[\eta] = \int_{-\infty}^{+\infty} (\eta - E[\eta])^4 d\Phi(x), \tag{11}$$

Theorem 5. *Let a and b be arbitrary real numbers and the expected value of an uncertain variable η be finite, then*

$$S[a\eta + b] = a^4 S[\eta], \tag{12}$$

and

$$K[a\eta + b] = a^4 K[\eta], \tag{13}$$

Proof. We know that $E[ax + b] = aE[x] + b$. It follows from definition (5) that

$$S[a\eta + b] = E[(a\eta + b - (aE[\eta] + b))^3] = a^3 E[(\eta - E(\eta))^3] = a^3 S[\eta], \tag{14}$$

and

$$K[a\eta + b] = E[(a\eta + b - (aE[\eta] + b))^4] = a^4 E[(\eta - E(\eta))^4] = a^4 K[\eta]. \tag{15}$$

The theorem is proved.

Definition 6. *Let $\lambda \in (0,1]$ be a confidence level and η be an uncertain variable, Then the function $VaR: (0,1] \to \mathbb{R}$ denotes Value-at-Risk of η, and defined by*

$$VaR(\lambda) = sup\{x | \mathcal{M}\{\eta \geq x\} \geq \lambda\}.$$

Theorem 6. *For the risk confidence level $\lambda \in (0,1]$,*

$$VaR(\lambda) = \Phi^{-1}(1 - \lambda),$$

where $\Phi^{-1}(1 - \lambda)$ denotes the inverse of uncertainty distribution function $\Phi(\lambda)$.

Definition 7. *Let $\lambda \in (0,1]$ be the confidence level for an uncertain variable η, then the function $AVaR: (0,1] \to \mathbb{R}$ denotes the average Value-at-Risk of η, and defined by*

$$AVaR(\lambda) = \frac{1}{\lambda} \int_0^{\lambda} VaR(\gamma) d\gamma.$$

3 Explanation of the Problem

Lemma 1. *Consider for $\alpha < \beta$, $\eta \sim L(\alpha, \beta)$ be a linear uncertain variable.*

i) The expected value of η is obtained as

$$E[\eta] = \frac{\alpha + \beta}{2} \tag{16}$$

ii) The Value-at-risk of η is obtained as

$$VaR(\lambda) = \lambda\alpha + (1 - \lambda)\beta, \quad 0 \leq \lambda \leq 1. \tag{17}$$

iii) The Average Value-at-risk of η is obtained as

$$AVaR(\lambda) = \frac{\lambda\alpha}{2} + \left(1 - \frac{\lambda}{2}\right)\beta, \quad 0 \leq \lambda \leq 1. \tag{18}$$

iv) The Skewness of η is obtained as

$$S[\eta] = 0 \tag{19}$$

v) The Kurtosis of η is obtained as

$$K[\eta] = \frac{(\alpha - \beta)^4}{80} \tag{20}$$

Proof. i)It is known that the uncertainty distribution of the linear uncertain variable η is [15]

$$\Phi(\tau)=\begin{cases}0, & \tau\le\alpha,\\ \frac{\tau-\alpha}{\beta-\alpha}, & \alpha\le\tau\le\beta,\\ 1, & \tau\ge\beta.\end{cases} \tag{21}$$

So

$$\Phi^{-1}(\tau)=\tau\beta+(1-\tau)\alpha. \tag{22}$$

Using (4),

$$E[\eta]=\int_0^{+\infty}(1-\Phi(\tau))d\tau-\int_{-\infty}^0\Phi(\tau)d\tau. \tag{23}$$

then, if $\alpha\ge 0$,

$$E[\eta]=\left(\int_0^\alpha 1d\tau+\int_\alpha^\beta\left(1-\frac{\tau-\alpha}{\beta-\alpha}\right)d\tau+\int_\beta^{+\infty}0d\tau\right)-\int_{-\infty}^0 0d\tau=\frac{\alpha+\beta}{2} \tag{24}$$

If $\beta\le 0$,

$$E[\eta]=\int_0^{+\infty}0d\tau-\left(\int_{-\infty}^\alpha 0d\tau+\int_\alpha^\beta\frac{\tau-\alpha}{\beta-\alpha}d\tau+\int_\beta^0 1d\tau\right)=\frac{\alpha+\beta}{2} \tag{25}$$

If $\alpha\le 0\le\beta$,

$$E[\eta]=\int_0^\beta\left(1-\frac{\tau-\alpha}{\beta-\alpha}\right)d\tau+\int_\alpha^0\frac{\tau-\alpha}{\beta-\alpha}d\tau=\frac{\alpha+\beta}{2} \tag{26}$$

Thus

$$E[\eta]=\frac{\alpha+\beta}{2} \tag{27}$$

ii)Using (22),

$$\Phi^{-1}(1-\tau)=(1-\tau)\beta+\alpha\tau.\qquad 0\le\tau\le 1. \tag{28}$$

Then using theorem (6), the proof is obvious.

iii)Using (7) and (28), for $0\le\lambda\le 1$,

$$AVaR(\lambda)=\frac{1}{\lambda}\int_0^\lambda VaR(\gamma)d\gamma$$
$$=\frac{1}{\lambda}\int_0^\lambda(\alpha\gamma+\beta(1-\gamma)d\gamma$$
$$=\frac{\lambda\alpha}{2}+\left(1-\frac{\lambda}{2}\right)\beta.$$

iv)Using theorem (3),

$$E[\eta^3]=\int_0^1(\beta\tau+(1-\tau)\alpha)^3d\tau=\frac{\alpha^4-\beta^4}{4(\alpha-\beta)}$$

and

$$Var[\eta]=E[\eta^2]-E^2[\eta]=\frac{(\beta-\alpha)^2}{12}$$

Then using definition (5)

$$S[\eta]=E[\eta^3]-E^3[\eta]-3E[\eta]Var[\eta]$$
$$=\frac{\alpha^4-\beta^4}{4(\alpha-\beta)}-\left(\frac{\alpha+\beta}{2}\right)^3-3\left(\frac{\alpha+\beta}{2}\right)\left(\frac{(\beta-\alpha)^2}{12}\right)=0.$$

v) Using theorem (3),

$$E[\eta^4]=\int_0^1(\beta\tau+(1-\tau)\alpha)^4d\tau=\frac{\alpha^5-\beta^5}{5(\alpha-\beta)}$$

Then using definition (5)

$$K[\eta]=E[\eta^4]+E^4[\eta]-4E[\eta^3]E[\eta]+6E[\eta^2]E^2[\eta]-4E^4[\eta]$$
$$=\frac{\alpha^5-\beta^5}{5(\alpha-\beta)}+\left(\frac{\alpha+\beta}{2}\right)^4-4\left(\frac{\alpha^4-\beta^4}{4(\alpha-\beta)}\right)\left(\frac{\beta+\alpha}{2}\right)+6\left(\frac{\alpha^2+\alpha\beta+\beta^2}{3}\right)\left(\frac{\alpha+\beta}{2}\right)^2-4\left(\frac{\alpha+\beta}{2}\right)^4$$
$$=\frac{(\alpha-\beta)^4}{80}.$$

Lemma 2. *Consider for $a < b < c, \eta \sim Z(a،b،c)$ be a Zigzag uncertain variable.*
i) The expected value of η is obtained as

$$E[\eta] = \frac{a+2b+c}{4} \tag{29}$$

ii) The Value-at-risk of η is obtained as

$$VaR(\lambda) = \begin{cases} (2\lambda - 1)a + (1 - \lambda)2b، & 0 < \lambda \leq \frac{1}{2}، \\ 2b\lambda + (1 - 2\lambda)c، & \frac{1}{2} \leq \lambda < 1. \end{cases} \tag{30}$$

iii) AVaR of η is obtained as

$$AVaR(\lambda) = \begin{cases} \left(1 - \frac{\lambda}{2}\right) 2b + a(\lambda - 1)، & 0 < \lambda \leq \frac{1}{2}، \\ \frac{1}{4\lambda}(2b - a) - \lambda c + b\lambda، & \frac{1}{2} \leq \lambda < 1. \end{cases} \tag{31}$$

iv) The Skewness of η is obtained as

$$S[\eta] = \frac{a^3 - a^2c - 2ba^2 - ac^2 + 4abc + c^3 - 2b^2c}{32} \tag{32}$$

v) The Kurtosis of η is obtained as

$$K[\eta] = \frac{1}{1280}(33a^4 + 400b^4 + 33c^4 + 32b^3a + 88a^2b^2 - 72bc^3 + 32b^3c + 88b^2c^2 + 70a^2c^2 - 60a^3c - 60ac^3 - 80ab^2c + 40abc^2 + 40a^2bc) \tag{33}$$

Proof. i)It is known that the uncertainty distribution of the Zigzag uncertain variable η is [15]

$$\Phi(\tau) = \begin{cases} 0، & \tau \leq a، \\ \frac{\tau - a}{2(b-a)}، & a \leq \tau \leq b، \\ \frac{\tau + c - 2b}{2(c-b)}، & b \leq \tau \leq c، \\ 1، & \tau \geq c. \end{cases} \tag{34}$$

So

$$\Phi^{-1}(\tau) = \begin{cases} 2b\tau - a(2\tau - 1)، & 0 \leq \tau \leq \frac{1}{2}، \\ 2b(1 - \tau) + c(2\tau - 1)، & \frac{1}{2} \leq \tau \leq 1. \end{cases} \tag{35}$$

Using (4),

$E[\eta] = \int_0^{+\infty} (1 - \Phi(\tau))d\tau - \int_{-\infty}^0 \Phi(\tau)d\tau.$

then, if $a \geq 0$,

$E[\eta] = (\int_0^a 1d\tau + \int_a^b \frac{2b-\tau-a}{2(b-a)} d\tau + \int_b^c \frac{c-\tau}{2(c-b)} d\tau + \int_c^{+\infty} 0d\tau) - \int_{-\infty}^0 0d\tau = \frac{a+2b+c}{4}$

If $a \leq 0 \leq b$,

$E[\eta] = \int_0^b \frac{2b-a-\tau}{2(b-a)} d\tau + \int_b^c \frac{c-\tau}{2(c-b)} d\tau + \int_c^{+\infty} 0d\tau - \int_{-\infty}^a 0d\tau - \int_a^0 \frac{\tau-a}{2(b-a)} d\tau = \frac{a+2b+c}{4}.$

If $b \leq 0 \leq c$,

$E[\eta] = \int_0^c \frac{c-\tau}{2(c-b)} d\tau + \int_c^{+\infty} 0d\tau - \int_{-\infty}^a 0d\tau - \int_a^b \frac{\tau-a}{2(b-a)} d\tau - \int_b^0 \frac{\tau+c-2b}{2(c-b)} d\tau = \frac{a+2b+c}{4}.$

If $c \leq 0$,

$E[\eta] = \int_0^{+\infty} 0d\tau - \int_{-\infty}^a 0d\tau - \int_a^b \frac{\tau-a}{2(b-a)} d\tau - \int_b^c \frac{\tau+c-2b}{2(c-b)} d\tau - \int_c^0 1d\tau = \frac{a+2b+c}{4}.$

Thus

$E[\eta] = \frac{a+2b+c}{4}$.

ii)Using (35),

$$\Phi^{-1}(1-\tau) = \begin{cases} 2b(1-\tau) - a(2\tau-1), & 0 < \tau \leq \frac{1}{2}, \\ 2b\tau + c(1-2\tau), & \frac{1}{2} \leq \tau < 1. \end{cases}$$

Then using theorem (6), the proof is obvious.

iii)Using (7) and (28), for $0 < \lambda \leq \frac{1}{2}$,

$$AVaR(\lambda) = \frac{1}{\lambda}\int_0^{\lambda} VaR(\gamma)d\gamma$$
$$= \frac{1}{\lambda}\int_0^{\lambda} 2b(1-\gamma) - a(2\gamma-1)d\gamma$$
$$= 2b(1-\frac{\lambda}{2}) + a(\lambda-1).$$

and for $\frac{1}{2} \leq \lambda < 1$,

$$AVaR(\lambda) = \frac{1}{\lambda}\int_0^{\lambda} VaR(\gamma)d\gamma$$
$$= \frac{1}{\lambda}(\int_0^{\frac{1}{2}} 2b(1-\gamma) - a(2\gamma-1)d\gamma + \int_{\frac{1}{2}}^{\lambda} ((1-2\gamma)c + 2b\gamma)d\gamma)$$
$$= 2b(1-\frac{\lambda}{2}) + a(\lambda-1).$$

iv)Using theorem (3),

$$E[\eta^3] = \int_0^{\frac{1}{2}} (2b\tau - a(2\tau-1))^3 d\tau + \int_{\frac{1}{2}}^{1} (c(2\tau-1) + 2b(1-\tau))^3 d\tau = \frac{a^4-b^4}{8(a-b)} + \frac{b^4-c^4}{8(b-c)}$$

and

$$Var[\eta] = E[\eta^2] - E^2[\eta] = \frac{5a^2+4b^2+5c^2-4ab-4bc-6ac}{48}$$

Then using definition (5)

$$S[\eta] = E[\eta^3] - E^3[\eta] - 3E[\eta]Var[\eta]$$
$$= \frac{a^3-a^2c-2ba^2-ac^2+4abc+c^3-2b^2c}{32}$$

v) Using theorem (3),

$$E[\eta^4] = \int_0^{\frac{1}{2}} (2b\tau - a(2\tau-1))^4 d\tau + \int_{\frac{1}{2}}^{1} (c(2\tau-1) + 2b(1-\tau))^4 d\tau = \frac{a^5-b^5}{10(a-b)} + \frac{b^5-c^5}{10(b-c)}$$

Then using definition (5)

$$K[\eta] = E[\eta^4] + E^4[\eta] - 4E[\eta^3]E[\eta] + 6E[\eta^2]E^2[\eta] - 4E^4[\eta]$$
$$= \frac{1}{1280}(33a^4 + 400b^4 + 33c^4 + 32b^3a + 88a^2b^2 - 72bc^3 + 32b^3c$$
$$+88b^2c^2 + 70a^2c^2 - 60a^3c - 60ac^3 - 80ab^2c + 40abc^2 + 40a^2bc).$$

Lemma 3. *Consider a Normal uncertain variable $\eta \sim N(e,\sigma)$ where $e,\sigma > 0$.*

i) The expected value of η is obtained as

$$E[\eta] = e. \tag{36}$$

ii) The VaR of η is obtained as

$$VaR(\lambda) = e - \frac{\sqrt{3}\sigma}{\pi} ln\frac{\lambda}{1-\lambda} \tag{37}$$

iii) The AVaR of η is obtained as

$$AVaR(\lambda) = e - \frac{\sqrt{3}\sigma}{\pi}[ln\frac{\lambda}{1-\lambda} + \frac{ln(1-\lambda)}{\lambda}] \tag{38}$$

iv) The Skewness of η is obtained as

$$S[\eta]=\frac{-36e\sigma^2}{\pi^2} \tag{39}$$

v) The Kurtosis of η is obtained as

$$K[\eta]=\frac{42}{10}\sigma^4-\frac{432\sigma^4}{\pi^4}+\frac{144e^2\sigma^2}{\pi^4} \tag{40}$$

Proof. i)It is known that the uncertainty distribution of the normal uncertain variable η is [15]

$$\Phi(\tau)=(1+exp\left(\frac{\pi(e-\tau)}{\sqrt{3}\sigma}\right))^{-1}, \qquad \tau\in\mathbb{R}, \tag{41}$$

So

$$\Phi^{-1}(\tau)=e-\frac{\sqrt{3}\sigma}{\pi}ln\frac{1-\tau}{\tau} \tag{42}$$

Using (4),

$$E[\eta]=\int_0^{+\infty}(1-\Phi(\tau))d\tau-\int_{-\infty}^{0}\Phi(\tau)d\tau.$$

Then,

$$E[\eta]=\int_0^{+\infty}1-(1+exp(\frac{\pi(e-\tau)}{\sqrt{3}\sigma}))^{-1}d\tau-\int_{-\infty}^{0}(1+exp(\frac{\pi(e-\tau)}{\sqrt{3}\sigma}))^{-1}d\tau=e.$$

ii)Using (42),

$$\Phi^{-1}(1-\tau)=e-\frac{\sqrt{3}\sigma}{\pi}ln\frac{\tau}{1-\tau} \tag{43}$$

Then using theorem (6), the proof is obvious.

iii)Using (7) and (43),

$$AVaR(\lambda)=\frac{1}{\lambda}\int_0^{\lambda}VaR(\gamma)d\gamma$$
$$=\frac{1}{\lambda}\int_0^{\lambda}(e-\frac{\sqrt{3}\sigma}{\pi}ln\gamma+\frac{\sqrt{3}\sigma}{\pi}ln(1-\gamma))d\gamma$$
$$=e-\frac{\sqrt{3}\sigma}{\pi}[ln\lambda+(\frac{1-\lambda}{\lambda})ln(1-\lambda)]$$
$$=e-\frac{\sqrt{3}\sigma}{\pi}[ln\frac{\lambda}{1-\lambda}+\frac{ln(1-\lambda)}{\lambda}].$$

iv)Using theorem (3),

$$E[\eta^3]=\int_0^1(e-\frac{\sqrt{3}\sigma}{\pi}ln(1-\tau)+\frac{\sqrt{3}\sigma}{\pi}ln\tau)^3d\tau=e^3-\frac{36e\sigma^2}{\pi^2}+3e\sigma^2.$$

and

$$Var[\eta]=E[\eta^2]-E^2[\eta]=e^2+\sigma^2-e^2=\sigma^2$$

Then using definition (5)

$$S[\eta]=E[\eta^3]-E^3[\eta]-3E[\eta]Var[\eta]$$
$$=e^3-\frac{36e\sigma^2}{\pi^2}+3e\sigma^2-e^3-3e\sigma^2$$
$$=\frac{-36e\sigma^2}{\pi^2}$$

v) Using theorem (3),

$$E[\eta^4]=\int_0^1(e-\frac{\sqrt{3}\sigma}{\pi}ln(1-\tau)+\frac{\sqrt{3}\sigma}{\pi}ln\tau)^4d\tau=e^4-\frac{432\sigma^4}{\pi^4}+\frac{42}{10}\sigma^4+6e^2\sigma^2.$$

Then using definition (5)

$$K[\eta]=E[\eta^4]+E^4[\eta]-4E[\eta^3]E[\eta]+6E[\eta^2]E^2[\eta]-4E^4[\eta]$$
$$=\frac{42}{10}\sigma^4-\frac{432\sigma^4}{\pi^4}+\frac{144e^2\sigma^2}{\pi^4}.$$

4 Portfolio Selection Problem

Markowitz models had considered the security returns as random variables. As explained in the introduction, there are situations that returns of securities may be uncertain variables. In these cases, uncertain variables will be used to describe the security returns. Let η_i denotes uncertain return of the ith security, and x_i represents the proportion of investment in the ith security, and the given risk confidence level is denoted by $\lambda \in (0,1]$. The investment return is determined by the expected value and risk by AVaR of a portfolio.

One of the problems in portfolio optimization is minimizing the Average Value at Risk (AVaR) in order to reduce risk at a given expected return level ϑ that investors find acceptable. The admissible skewness level, denoted by ϱ, and the maximum tolerable kurtosis level, denoted by κ, are additional factors to consider. In this case, the portfolio optimization model can be displayed as

$$\begin{aligned} & minimize\ \ AVaR[\textstyle\sum_{i=1}^{n} x_i\eta_i] \\ & subject\ \ to \begin{cases} E[\sum_{i=1}^{n} x_i\eta_i] \geq \vartheta, \\ S[\sum_{i=1}^{n} x_i\eta_i] \geq \varrho, \\ K[\sum_{i=1}^{n} x_i\eta_i] \leq \kappa, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \ i = 1,2,\ldots,n. \end{cases} \end{aligned} \tag{44}$$

Alternatively, another portfolio selection problem can be maximizing expected return on the limitation that the skewness is rather than or equal to the admissible level ϱ and the risk which denotes by AVaR does not overpass a preset risk level ϖ and kurtosis does not exceed a preset level κ in advance. This optimization model becomes as

$$\begin{aligned} & maximize\ \ E[\textstyle\sum_{i=1}^{n} x_i\eta_i] \\ & subject\ \ to \begin{cases} AVaR[\sum_{i=1}^{n} x_i\eta_i] \leq \varpi, \\ S[\sum_{i=1}^{n} x_i\eta_i] \geq \varrho, \\ K[\sum_{i=1}^{n} x_i\eta_i] \leq \kappa, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \ i = 1,2,\ldots,n. \end{cases} \end{aligned} \tag{45}$$

This optimization problem can be formulated in some other different kinds, such as maximizing skewness or minimizing kurtosis or multi-objective nonlinear programming model as

$$\begin{aligned} & maximize\ \ E[\textstyle\sum_{i=1}^{n} x_i\eta_i] \\ & maximize\ \ S[\textstyle\sum_{i=1}^{n} x_i\eta_i] \\ & minimize\ \ AVaR[\textstyle\sum_{i=1}^{n} x_i\eta_i] \\ & minimize\ \ K[\textstyle\sum_{i=1}^{n} x_i\eta_i] \\ & subject\ \ to \begin{cases} \sum_{i=1}^{n} x_i = 1. \\ x_i \geq 0, \ i = 1,2,\ldots,n. \end{cases} \end{aligned} \tag{46}$$

In order to solve this problem, consider $w_i, i = 1,2,3,4$ be positive real numbers which indicate the weights of the four appropriated objectives, and $w_i \in [0,1]$, so this multi-objective model can be transformed into a single-objective optimization model as

$$minimize\ \ w_1 AVaR[\textstyle\sum_{i=1}^{n} x_i\eta_i] - w_2 E[\sum_{i=1}^{n} x_i\eta_i] - w_3 S[\sum_{i=1}^{n} x_i\eta_i] + w_4 K[\sum_{i=1}^{n} x_i\eta_i] \tag{47}$$

$$subject\ \ to \begin{cases} \sum_{i=1}^{n} x_i = 1. \\ x_i \geq 0, \ i = 1,2,\ldots,n. \end{cases}$$

Note that if x^* be an optimal solution of model (47), then x^* will be a pareto optimal solution of multi-objective nonlinear programming model (46).

Theorem 7. *Let $\eta_i \in L(\alpha_i, \beta_i)$ for $i = 1, 2, \ldots, n$ be a linear uncertain variable, and $0 < \lambda \leq 1$. Then model (44) can be changed to the crisp equivalent as following form*

$$minimize \ \ \sum_{i=1}^{n} x_i\left(\frac{\lambda\alpha_i}{2} + \left(1 - \frac{\lambda}{2}\right)\beta_i\right) \tag{48}$$

$$subject \ \ to \begin{cases} \sum_{i=1}^{n} x_i\left(\frac{\alpha_i + \beta_i}{2}\right) \geq \vartheta, \\ \sum_{i=1}^{n} x_i\left(\frac{(\alpha_i - \beta_i)^4}{80}\right) \leq \kappa, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \ i = 1, 2, \ldots, n. \end{cases}$$

and model (45) can be changed to the crisp equivalent as following form

$$maximize \ \ \sum_{i=1}^{n} x_i\left(\frac{\alpha_i + \beta_i}{2}\right) \tag{49}$$

$$subject \ \ to \begin{cases} \sum_{i=1}^{n} x_i\left(\frac{\lambda\alpha_i}{2} + \left(1 - \frac{\lambda}{2}\right)\beta_i\right) \leq \varpi, \\ \sum_{i=1}^{n} x_i\left(\frac{(\alpha_i - \beta_i)^4}{80}\right) \leq \kappa, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \ i = 1, 2, \ldots, n. \end{cases}$$

Proof. Since, all of uncertain variables are linear in this mean that $\eta_i \in L(\alpha_i, \beta_i)$ for $i = 1, 2, \ldots, n$. Moreover, the expected value have obtained as $E[\sum_{i=1}^{n} x_i\eta_i] = \sum_{i=1}^{n} x_i(\frac{\alpha_i + \beta_i}{2})$ and the variance $AVaR[\sum_{i=1}^{n} x_i\eta_i] = \sum_{i=1}^{n} x_i(\frac{\lambda\alpha_i}{2} + (1 - \frac{\lambda}{2})\beta_i)$ and the skewness $S[\sum_{i=1}^{n} x_i\eta_i] = 0$ and the kurtosis $K[\sum_{i=1}^{n} x_i\eta_i] = \sum_{i=1}^{n} x_i(\frac{(\alpha_i - \beta_i)^4}{80})$. Substituting the above formulas into model (44) and (45), the theorem will be proved.

Theorem 8. *Let $\eta_i \in N(e_i, \sigma_i)$ for $i = 1, 2, \ldots, n$ be a Normal uncertain variable, and $0 < \lambda \leq 1$. Then model (44) can be changed to the crisp equivalent as following form*

$$minimize \ \ \sum_{i=1}^{n} x_i\left(e_i\lambda - \frac{\sqrt{3}\sigma_i}{\pi}[\lambda ln\lambda + (1 - \lambda)ln(1 - \lambda)]\right) \tag{50}$$

$$subject \ \ to \begin{cases} \sum_{i=1}^{n} x_i e_i \geq \vartheta, \\ \sum_{i=1}^{n} x_i\left(\frac{-36 e_i \sigma_i^2}{\pi^2}\right) \geq \varrho, \\ \sum_{i=1}^{n} x_i\left(\frac{42}{10}\sigma_i^4 - \frac{432\sigma_i^4}{\pi^4} + \frac{144 e_i^2 \sigma_i^2}{\pi^4}\right) \leq \kappa, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \ i = 1, 2, \ldots, n. \end{cases}$$

and model (45) can be changed to the crisp equivalent as following form

$$maximize \ \ \sum_{i=1}^{n} x_i e_i \tag{51}$$

$$
\text{subject to} \begin{cases} \sum_{i=1}^{n} x_i \left(e_i\lambda - \frac{\sqrt{3}\sigma_i}{\pi}[\lambda ln\lambda + (1-\lambda)ln(1-\lambda)] \right) \leq \varpi, \\ \sum_{i=1}^{n} x_i \left(\frac{-36 e_i \sigma_i^2}{\pi^2} \right) \geq \varrho, \\ \sum_{i=1}^{n} x_i \left(\frac{42}{10}\sigma_i^4 - \frac{432\sigma_i^4}{\pi^4} + \frac{144 e_i^2 \sigma_i^2}{\pi^4} \right) \leq \kappa, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \ i = 1,2,\ldots,n. \end{cases}
$$

Proof. Since, all of uncertain variables are Normal in this mean that $\eta_i \in N(e_i, \sigma_i)$ for $i = 1,2,\ldots,n$. Moreover, the expected value have obtained as $E[\sum_{i=1}^{n} x_i\eta_i] = \sum_{i=1}^{n} x_i e_i$ and the variance $AVaR[\sum_{i=1}^{n} x_i\eta_i] = \sum_{i=1}^{n} x_i(e_i\lambda - \frac{\sqrt{3}\sigma_i}{\pi}[\lambda ln\lambda + (1-\lambda)ln(1-\lambda)])$ and the skewness $S[\sum_{i=1}^{n} x_i\eta_i] = \sum_{i=1}^{n} x_i(\frac{-36 e_i\sigma_i^2}{\pi^2})$ and the kurtosis $K[\sum_{i=1}^{n} x_i\eta_i] = \sum_{i=1}^{n} x_i(\frac{42}{10}\sigma_i^4 - \frac{432\sigma_i^4}{\pi^4} + \frac{144 e_i^2\sigma_i^2}{\pi^4})$. Substituting the above formulas into model (44) and (45), the theorem will be proved.

5 Numerical Example

Example 1. *Suppose that there are 10 stocks which their monthly return rates are estimated by experienced experts and they are Linear uncertain variables. Table 1 represents the simulated expected values of these stocks. An investor would like to create an optimal portfolio, and he wishes to minimize The Average Value at Risk, so solving model (50) to obtain the optimal portfolio is the main concern.*

Table 1: data of securities which are linear uncertain variables for example 1.

stocks	x_1	x_2	x_3	x_4	x_5
	$L(0.4,0.9)$	$L(1.7,2.4)$	$L(0.3,0.7)$	$L(0.1,0.6)$	$L(0.5,1.5)$
stocks	x_6	x_7	x_8	x_9	x_{10}
	$L(2,3.3)$	$L(0.9,1.4)$	$L(2.1,2.3)$	$L(0.8,1)$	$L(1,1.2)$

Consider that in investor's mind, the minimum expected return that can accept is 2.5, and the Avarage Value at Risk not allowed to exceed 0.6, $\lambda = 0.1$ and kurtosis is not allowed to exceed 2. Then the model (50) will be as follows:

$$
\begin{aligned} minimize\ \ & 0.875x_1 + 2.365x_2 + 0.68x_3 + 0.575x_4 + 1.45x_5 \\ & +2.235x_6 + 1.37x_7 + 2.29x_8 + 0.99x_9 + 1.19x_{10} \end{aligned} \quad (52)
$$

$$
\text{subject to} \begin{cases} 0.65x_1 + 2.05x_2 + 0.5x_3 + 0.35x_4 + x_5 + 2.65x_6 + 1.15x_7 + 2.2x_8 + 0.9x_9 \\ +1.1x_{10} \geq 2.5, \\ 0.000781x_1 + 0.003001x_2 + 0.00032x_3 + 0.000871x_4 + 0.0125x_5 + 0.035701x_6 \\ +0.000781x_7 + 0.00002x_8 + 0.00002x_9 + 0.00002x_{10} \leq 2, \\ x_1 + x_2 + \ldots + x_{10} = 1, \\ x_i \geq 0, \quad i = 1,2,\ldots,10. \end{cases}
$$

and the model (51) will be as follows:

$$
\begin{aligned} maximize\ \ & 0.65x_1 + 2.05x_2 + 0.5x_3 + 0.35x_4 + x_5 + 2.65x_6 \\ & +1.15x_7 + 2.2x_8 + 0.9x_9 + 1.1x_{10} \end{aligned} \quad (53)
$$

$$
\text{subject to}\begin{cases}
0.875x_1+2.365x_2+0.68x_3+0.575x_4+1.45x_5+2.235x_6+1.37x_7+2.29x_8\\
+0.99x_9+1.19x_{10}\le 0.6,\\
0.000781x_1+0.003001x_2+0.00032x_3+0.000871x_4+0.0125x_5+0.035701x_6\\
+0.000781x_7+0.00002x_8+0.00002x_9+0.00002x_{10}\le 2,\\
x_1+x_2+\ldots+x_{10}=1,\\
x_i\ge 0, \quad i=1,2,\ldots,10.
\end{cases}
$$

The optimal solution of model (52) is $x_3^*=0.0697$, and $x_6^*=0.9302$, and $x_i^*=0,\ i=1,2,4,5,7,8,9,10$, so the optimal value of objective function is 2.12651. This means that for minimizing the risk with the expected value rather than 2.5 and kurtosis with maximum level 2, the investor must allocate his capital according to x^*. The optimal solution of model (53) is $x_3^*=0.23809$, and $x_4^*=0.76190$, and $x_i^*=0,\ i=1,2,5,6,\ldots,10$, so the optimal value of objective function is 0.3857. This means that for maximizing the expected return with given constraints, the investor must allocate his capital according to x^*.

Example 2. *Suppose that there are 10 stocks which their monthly return rates are estimated by experienced experts and they are Zigzag uncertain variables. Table 2 represents the simulated expected values of these stocks. An investor would like to create an optimal portfolio, and he wishes to minimize The Average Value at Risk, so solving model (50) to obtain the optimal portfolio is the main concern.*

Table 2: data of securities which are zigzag uncertain variables for example 2.

stocks	x_1	x_2	x_3	x_4	x_5
	$Z(-0.3,2,2.5)$	$Z(-0.3,2.8,3.2)$	$Z(-0.4,2.5,4)$	$Z(-0.2,3,3.5)$	$Z(-0.2,2.5,3)$
stocks	x_6	x_7	x_8	x_9	x_{10}
	$Z(-0.6,3,4)$	$Z(-0.1,2,2.5)$	$Z(-0.4,3,4)$	$Z(-0.1,1.9,3)$	$Z(-0.2,2.1,2.5)$

Consider that in investor's mind, the minimum expected return that can accept is 3.2, and the Avarage Value at Risk not allowed to exceed 1.6, $\lambda=0.2$ and Skewness will be rather than 0.1 and kurtosis is not allowed to exceed 6.5. then the model (44) will be as follows:

$$
\begin{aligned}
\text{minimize}\ & 3.84x_1+5.28x_2+4.82x_3+5.56x_4+4.66x_5 \\
& +5.88x_6+3.68x_7+5.72x_8+3.5x_9+3.94x_{10}
\end{aligned} \tag{54}
$$

$$
\text{subject to}\begin{cases}
1.55x_1+2.125x_2+2.15x_3+2.325x_4+1.95x_5+2.35x_6+1.6x_7+2.4x_8+1.675x_9\\
+1.625x_{10}\ge 3.2,\\
-0.28475x_1-0.80959x_2+0.0905x_3-0.82697x_4-0.46963x_5\\
-0.96925x_6-0.18175x_7-0.702x_8+0.121594x_9-0.30159x_{10}\ge 0.1,\\
6.771004x_1+24.65205x_2+19.68469x_3+32.38537x_4+15.85367x_5+36.27409x_6\\
+6.552548x_7+35.25416x_8+6.156293x_9+7.901944x_{10}\le 6.5,\\
x_1+x_2+\ldots+x_{10}=1,\\
x_i\ge 0, \quad i=1,2,\ldots,10.
\end{cases}
$$

and the model (45) will be as follows:

$$\begin{aligned} maximize\ & 1.55x_1 + 2.125x_2 + 2.15x_3 + 2.325x_4 + 1.95x_5 + 2.35x_6 \\ & +1.6x_7 + 2.4x_8 + 1.675x_9 + 1.625x_{10} \end{aligned} \tag{55}$$

$$subject\ to \begin{cases} 3.84x_1 + 5.28x_2 + 4.82x_3 + 5.56x_4 + 4.66x_5 + 5.88x_6 \\ +3.68x_7 + 5.72x_8 + 3.5x_9 + 3.94x_{10} \le 1.6, \\ -0.28475x_1 - 0.80959x_2 + 0.0905x_3 - 0.82697x_4 - 0.46963x_5 \\ -0.96925x_6 - 0.18175x_7 - 0.702x_8 + 0.121594x_9 - 0.30159x_{10} \ge 0.1, \\ 6.771004x_1 + 24.65205x_2 + 19.68469x_3 + 32.38537x_4 + 15.85367x_5 + 36.27409x_6 \\ +6.552548x_7 + 35.25416x_8 + 6.156293x_9 + 7.901944x_{10} \le 6.5, \\ x_1 + x_2 + \ldots + x_{10} = 1, \\ x_i \ge 0, \quad i = 1,2,\ldots,10. \end{cases}$$

The optimal solution of model (54) is $x_9^* = 1$, and $x_i^* = 0, \ i = 1,\ldots,8,10$, so the optimal value of objective function is 3.5. This means that for minimizing the risk with the expected value rather than 1.6, Skewness rather than 0.1 and kurtosis not allowed to exceed 6.5, the investor must allocate his capital according to x^*. The minimum relevant risk is 3.5.

The optimal solution of model (55) is $x_3^* = 0.025$, and $x_9^* = 0.974$, and $x_i^* = 0, \ i = 1,2,4,5,6,7,8,10$, so the optimal value of objective function is 1.682. This means that for maximizing the expected return with given constraints, the investor must allocate his capital according to x^*. The maximum relevant return is 1.682.

6 Conclusions

This study introduces and calculates the skewness, kurtosis, and Average Value at Risk (AVaR) of uncertain variables by the use of theorems and proofing techniques. Additionally, it describes uncertain models in relation to the mean-AVaR-skewness-kurtosis framework, which is used for optimal portfolio selection. In recent researches mean, variance and skewness in uncertain environment have been used but in some cases such as linear uncertain variables, skeewness is zero and model will be converted to traditional mean-variance model; so using additional parameters such as kurtosis is useful for satisfying mental demands of investors. Also, the advantage of using Average Value at Risk (AVaR) instead of variance is clear in the superiority of the described model.Portfolios are constructed according on investor preferences, taking into account increased moments in an uncertain environment and considering the use of AVaR as a measure of investment risk. The models that have been acquired have been converted into linear programming problems in certain instances involving uncertain variables. The outcomes derived from the developed models for addressing portfolio selection challenges including uncertain returns are expected to hold significant value in the fields of economics and financial mathematics, encompassing both theoretical and practical applications.It is suggested for future research, consider more parameters in the models, such as entropy and higher order moments, in order to gain to best optimal portfolio according to investors mental demands. Also, using of multi-objective models in optimization portfolio models and using of different hibrid algorithms for solving obtained models in cases which investors encountered with too many assets and stocks are suggested to continue this research process.

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