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Research Paper

Thermoelastic Interaction in A Spherically Symmetric Hollow Sphere via Three-Phase Lag and Memory Effect

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ABSTRACT

This model considers the impact of microscopic structure on a nonsimple thermoelastic hollow sphere with a spherically symmetry exposed to thermal shock-type heat supply on the inner and outer curved surface. The influence of temperature discrepancy factor and phase-lag in the context of a memory-dependent derivative are examined. The study utilizes the Fourier series Laplace inversion method to obtain numerical solutions across different physical domains, and graphically illustrates transient responses such as temperature, displacement, and stress. The study indicates that the memory-dependent derivative accurately represents the memory effect, which is the rate of change influenced by the previous state, potentially aiding in better heat management of nonsimple media. The study compares temperature distributions in non-Fourier and classical Fourier models, revealing wave-like phenomena in fractional heat transfer that are not observed in classical heat conduction. Based on the proposed model, we can derive specific previous thermoelasticity models as special cases. The article's physical viewpoints could be helpful in the development of novel materials for modeling nanoscale heat transport problems in various devices. This study aims to provide future researchers with a comprehensive understanding of nonsimple three-phase lags thermoelasticity, including a detailed analysis of the memory effect.

Keywords: Thermoelasticity; Fractional-order; Three-phase-lags; Nonsimple media; Hollow sphere; Spherically symmetric system.

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1 INTRODUCTION

THERMOELASTICITY theory examines the relationship between stress and strain in solids during heating, providing a crucial tool for understanding and predicting physical behaviour. It is widely used in engineering and is continually advancing through new research. Biot [1] instituted the coupled thermoelasticity theory to address the initial limitations of the classical uncoupled theory, which had two contradicting outcomes. The heat conduction equation in this theory lacks elastic variables and is parabolic, predicting infinite heat wave spread. The classical theory's initial problem is resolved by integrating the governing equations in Biot theory. However, both theories have a parabolic heat equation that exhibits the same second defect.

Thermoelasticity theories, which postulate that heat signals transmit at a limited speed, have gained significant interest over the past forty years. Lord and Shulman [2] were the first to elaborate on this concept, replacing the Fourier law with a novel heat conduction equation. This theory has a single relaxation time, applies to the temporal derivative and heat flow vector, and includes a new constant as a temporal delay. The wave-type heat equation guarantees that heat and elastic waves move at finite speeds, similar to linked and uncoupled theories. This idea was expanded upon by Dhaliwal and Sherief [3] to include anisotropic media with heat sources. Green and Laws' [4] proposal highlighted the disparity in constitutive equations, while Green and Lindsay [5] developed an alternate method. Green and Naghdi [6] presented a more comprehensive theory, thermoelasticity without energy loss, which includes the thermal displacement gradient as one of its independent variables, excluding thermal energy dissipation. The dual-phase-lag (DPL) model, first proposed by Tzou [7, 8], is a macroscopic formulation of the thermoelastic model, which explains large-scale delayed responses due to retarding sources like electron-phonon interactions. Experimental results have confirmed this model as an alternative to the classical thermoelastic model. Tzou [9] suggested the DPL model that incorporates two distinct time translations, a phase-lag in the temperature gradient and a phase-lag in the heat flow, presenting an altered version of the Fourier law. Roy Choudhuri [10] developed the three-phase-lag (TPL) equation, based on the DPL model, to analyze heat conduction. This method assessed the thermal characteristics of heat flow components, temperature gradient, and thermal displacement. The TPL model demonstrated its effectiveness in handling complex phenomena like nuclear boiling, phonon-electron disruptions, exothermic catalytic processes, and phonon scattering.

Fractional calculus, during the past few decades, has been applied in various fields, including signal and image processing, control theory, biology, physics, continuum mechanics, viscoelasticity, and biophysics. This application has significantly changed the physical process models, as the fractional-order differential operator is non-local, requiring the system's updated state to consider both current and historical states. Recent research using fractional calculus has led to revisions to the basic Fourier law in heat conduction, with academic works documenting these revisions. Various researchers [11-18] have adjusted the fractional derivative in the integrand of the Caputo-type formula.

Memory effect in the Lord-Shulman extended thermoelasticity theory was seen during the use of memory-dependent derivative (MDD) instead of fractional calculus. This strategy has been used in research by Yu et al. [19], Mondal et al. [20], Ghosh et al. [21], Fahmy [22,23], and the references therein. Ezzat and Bary [24] studied thermoviscoelasticity, focusing on memory-dependent derivatives and two temperatures. They investigated generalized thermo-viscoelasticity using MDD. Sherief and Al-Latief [26] used the fractional order theory of thermoelasticity to study the heat conductivity of a half-space. Ezzat and Bary [27] also studied two temperature models of magnetothermviscoelasticity. Time-fractional-order thermoelasticity is often used to represent MDD behaviours. Hence, the investigation of thermoelasticity using the memory-dependent two-temperature theory is of great significance, particularly for complex media. The author is not aware of any previous analysis for a transient nonsimple heat conduction problem involving a hollow sphere via MDD. It may be because applying the analytical technique is very challenging.

The objective of the given paragraph is to address the gap in knowledge regarding the thermoelastic behavior of nonsimple materials incorporating Memory-Dependent Derivative (MDD) in the heat conduction model. The study aims to develop a comprehensive thermoelasticity model using the Three-Phase Lag (TPL) heat conduction approach for spherically symmetric hollow spheres. The fundamental equations in the Laplace transform domain are subsequently solved using the operational method to determine the thermal field, strain, displacement, and thermoelastic stresses. Fourier series approach is employed for the numerical inversion of the Laplace transform. The study focuses on deriving temperature, displacement, and thermal stress distributions, and it graphically illustrates the effects of parameters such as time delay, kernel function, temperature discrepancy factor, and memory-dependent parameters on the physical fields.

Based on the literature review, there has been no research addressing a nonsimple thermoelastic model incorporating three-phase lags with memory-dependent derivatives under thermal shock conditions. This paper is structured as follows: Section 2 introduces the modified nonsimple heat conduction equation with memory-dependent derivatives. Section 3 details the mathematical formulation of the problem for a spherically symmetric elastic hollow sphere. Solutions to the non-Fourier heat conduction equations in the Laplace transform domain are discussed in Section 4. Section 5 focuses on the numerical inversion of the Laplace transform. Section 6 presents the quantitative results, analysis, and observations. Finally, the conclusions are summarized in Section 7.

2 MATHEMATICAL MODELING

Green and Naghdi [28] proposed a DPL model based on the Fourier law

$$q(t+\tau_q) = -k\nabla\Phi(t+\tau_T) \tag{1}$$

Further, Green and Naghdi [29] modified Eq. (1) as

$$q(t+\tau_q) = -k\nabla\Phi(t+\tau_T) - k^*\nabla u(t+\tau_u) \tag{2}$$

where $\partial u/\partial t = T$ and $k^*(>0)$ is a material constant characteristic of the theory.
Chen and Gurtin [30] proposed two-temperature concepts

$$\Phi = \left(1 + b\frac{\partial}{\partial t}\right)T \tag{3}$$

Using Eqs. (2) and (3), one obtains nonsimple Fourier's law as

$$q(t+\tau_q) = -\left(1+b\frac{\partial}{\partial t}\right)k\nabla T(t+\tau_T) - k^*\vec{\nabla}u(t+\tau_u) \tag{4}$$

Roychoudhuri [31] presented a TPL model as

$$q(t+\tau_q) = -k\nabla T(t+\tau_T) - k^*\nabla u(t+\tau_u) \tag{5}$$

and then Eq. (5) was taken with Taylor's expansion as

$$\left(1+\tau_q\frac{\partial}{\partial t}+\frac{1}{2}\tau_q^2\frac{\partial^2}{\partial t^2}\right)q = -k\left(1+b\frac{\partial}{\partial t}\right)\left(\tau_u^*+k\tau_T\frac{\partial}{\partial t}\right)\nabla T - k^*\nabla u \tag{6}$$

where $\tau_u^* = k + k^*\tau_u$ and $0 \le \tau_u \le \tau_T \le \tau_q$.

The rise in entropy S leads to the following relations [32]

$$-\vec{\nabla}\cdot\vec{q} + Q = \rho T_0\frac{\partial S}{\partial t} \tag{7}$$

$$\rho T_0 S = \rho C_e T + \gamma T_0\vec{\nabla}\cdot u \tag{8}$$

From Eqs. (8) and (9), one obtains the energy equation as

$$\rho C_e\frac{\partial T}{\partial t} + \gamma T_0\frac{\partial(\vec{\nabla}\cdot u)}{\partial t} = -\vec{\nabla}\cdot q + Q \tag{9}$$

Taking divergence on both sides of Eq. (7), and using Eq. (10), and differentiating both equations with respect to time, one obtains

$$\left(1+\tau_q \frac{\partial}{\partial t}+\frac{1}{2}\tau_q^2 \frac{\partial^2}{\partial t^2}\right)\left(\rho C_e \frac{\partial^2 T}{\partial t^2}+\gamma T_0 \frac{\partial^2 (\nabla \cdot u)}{\partial t^2}-\dot{Q}\right) = k\left(1+b\frac{\partial}{\partial t}\right)\left(\tau_u^* + k\tau_T \frac{\partial}{\partial t}\right)\nabla^2\dot{\theta}+k^*\nabla^2\theta \tag{10}$$

Diethelm [33] introduced a kernel as follows:

$$D_a^{\alpha}Y(t)=\int_a^t K_{\alpha}(t-\xi)Y^{(m)}(\xi)\,d\xi \tag{11}$$

where

$$K_{\alpha}(t-\xi)=\frac{(t-\xi)^{m-\alpha-1}}{\Gamma(n-\alpha)} \tag{12}$$

Wang and Li [34] proposed a first-order common derivative of MDD as

$$D_{\tau}Y(t)=\frac{1}{\tau}\int_{t-\tau}^{t} K_{\alpha}(t-\xi)Y'(\xi)\,d\xi \tag{13}$$

Eq. (11) can be written using Eq. (14) as

$$\left(D_{\tau_q}^0+\tau_q D_{\tau_q}^1+\frac{1}{2}\tau_q^2 D_{\tau_q}^2\right)\left(\rho C_e \frac{\partial^2 T}{\partial t^2}+\gamma T_0 \frac{\partial^2 e}{\partial t^2}-\dot{Q}\right) = k\frac{\partial}{\partial t}\left(1+b\frac{\partial}{\partial t}\right)\left(\tau_u^*+k\tau_T D_{\tau_T}^1\right)\nabla^2 T+k^*\nabla^2 T \tag{14}$$

The basic thermoelastic equation under transient temperature:

$$\sigma_{ij}=2\mu e_{ij}+(\lambda e_{kk}-\gamma T)\delta_{ij} \tag{15}$$

$$e_{ij}=(u_{j,i}+u_{i,j})/2 \tag{16}$$

$$\mu u_{i,jj}+(\lambda+\mu)u_{j,ij}-\gamma T_{,i}+F_i=\rho\ddot{u}_i \tag{17}$$

3 STATEMENT OF PROBLEM

In its undisturbed state, let us consider an elastic hollow sphere with spherical symmetry and a constant temperature T_0 with an inner radius a and outer radius b. This sphere is homogenous, transradially isotropic, and thermally conductive. The origin of the spherical polar coordinate system (r,θ,ϕ) is defined as the centre of the sphere, as depicted in Figure 1. The notation occupying the space $D \subset \mathbb{R}^3$ defined by

$$D=\{(r,\theta,\varphi)\subset \mathbb{R}^3 \mid a<r\le b, 0<\theta\le 2\pi, 0<\varphi\le\pi\}.$$

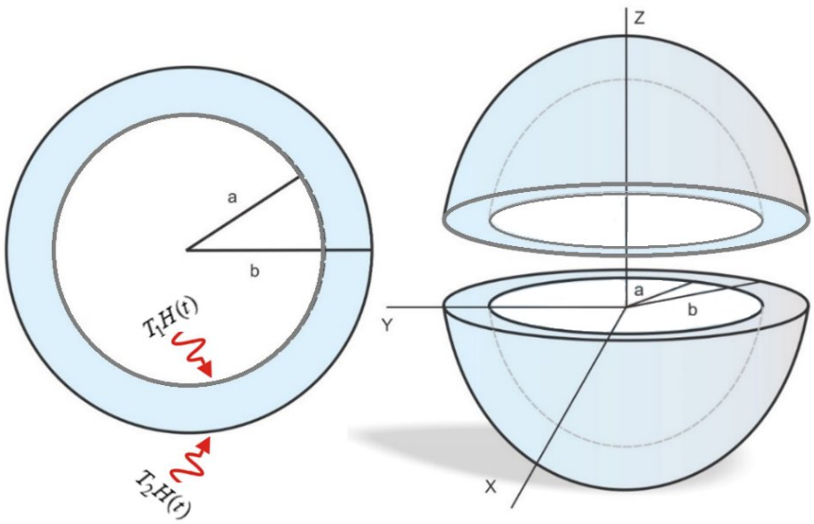

Fig. 1

Schematic sketch of a hollow sphere under thermal shock.

3.1 Governing equation of thermoelasticity

The linear theory of coupled thermoelasticity can be expressed using governing equations for motion without body forces and considering radial symmetry.

$$\frac{\partial}{\partial r}\sigma_{rr}+\frac{1}{r}(2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{\phi\phi})=\rho\frac{\partial^2 u}{\partial t^2} \tag{18}$$

The problem assumes spherical symmetry, with displacement and temperature components as functions of r and t only, and non-zero strain components in terms of displacement are given as

$$e_{rr}=\frac{\partial u}{\partial r}, e_{\theta\theta}=e_{\phi\phi}=\frac{u}{r} \tag{19}$$

Then, the non-zero stress components are obtained as

$$\sigma_{rr}=(\lambda+2\mu)\frac{\partial u}{\partial r}+2\lambda\frac{u}{r}-\gamma T \tag{20}$$

$$\sigma_{\theta\theta}=\sigma_{\phi\phi}=\lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+(\lambda+2\mu)\frac{u}{r}-\gamma T \tag{21}$$

From Eqs. (24)-(27), the equation of motion is given as

$$(\lambda+2\mu)\left(\frac{\partial^2 u}{\partial r^2}+2\frac{\partial u}{\partial r}-2\frac{u}{r^2}\right)=\rho\frac{\partial^2 u}{\partial t^2}+\gamma\frac{\partial T}{\partial r} \tag{22}$$

3.2 Non-dimensional parameter

For the sake of convenience, we introduce the following non-dimensional variable:

$$(\hat{r},\hat{u}) = c_0\eta(r,u),\ (\hat{t},\hat{\tau}_q,\hat{\tau}_T,\hat{\tau}_\upsilon) = c^2{}_0\eta(t,\tau_q,\tau_T,\tau_\upsilon),$$
$$\hat{T} = \frac{\gamma T}{\lambda+2\mu}, \hat{\sigma}_{rr} = \frac{\sigma_{rr}}{\lambda+2\mu}, \eta = \frac{\rho C_e}{\kappa}, c_0 = \sqrt{\frac{\lambda+2\mu}{\rho}} \tag{23}$$

Now, Eqs. (20), (24)-(28) are simplified by using Eq. (29) as a non-dimensional quantity and omitting the overhat prime for further calculations.

$$\left(D^0_{\tau_q} + \tau_q D^1_{\tau_q} + \frac{1}{2}\tau_q^2 D^2_{\tau_q}\right)\left(\rho C_e \frac{\partial^2 T}{\partial t^2} + p_1\gamma T_0 \frac{\partial^2 e}{\partial t^2} - p_1\dot{Q}\right)$$
$$= k\frac{\partial}{\partial t}\left(1 + b\frac{\partial}{\partial t}\right)\left(\tau_u^* + k\frac{\tau_T}{c_0^2\eta}D^1_{\tau_T}\right)\nabla^2 T + k^*\nabla^2 T \tag{24}$$

subjected to initial and boundary conditions

$$T(r,0) = \frac{\partial}{\partial t}T(r,0) = 0 \tag{25}$$

$$T(a,t) = T_1 H(t) \tag{26}$$

$$T(b,t) = T_2 H(t) \tag{27}$$

Applying the operator $(1/r^2)\partial/\partial r(r^2)$ in Eq. (28), one gets

$$\nabla^2 e - \nabla^2 T = \frac{1}{c_0^2}\frac{\partial^2 e}{\partial t^2} \tag{28}$$

The dimensionless stress components of Eq. (26) and (27) are

$$\sigma_{rr} = (1-\Omega^2)e + \Omega^2\frac{\partial u}{\partial r} - T \tag{29}$$

$$\sigma_{\theta\theta} = (1-\Omega^2)e + \Omega^2\frac{u}{r} - T = \sigma_{\phi\phi} \tag{30}$$

The inner and outer boundary surface of the sphere is unaffected by any mechanical load, resulting in traction-free conditions

$$\sigma_{rr}(a,t) = 0 \tag{31}$$

$$\sigma_{rr}(b,t) = 0 \tag{32}$$

Under the assumption of a quiescent state

$$\sigma_{rr}(r,0) = \frac{\partial}{\partial t}\sigma_{rr}(r,0) = 0 \tag{33}$$

$$u(r,0) = \frac{\partial}{\partial t}u(r,0) = 0 \tag{34}$$

and

$$\Omega^2 = \frac{\mu}{\lambda + 2\mu}, p_1 = \frac{\gamma}{\lambda + 2\mu}, p_2 = c_0^2 \eta \tau_u^*, \ p^* = k^* \frac{c_0 \gamma}{\lambda + 2\mu} \tag{35}$$

and the cubic dilation e is given by

$$e = \nabla \cdot u = \frac{1}{r^2} \frac{\partial (r^2 u)}{\partial r} = \frac{\partial u}{\partial r} + \frac{2u}{r} \tag{36}$$

4 SOLUTION OF THE PROBLEM

The Laplace transform [35] is defined by

$$\overline{f}(r,s) = \mathcal{L}[f(r,t)] = \int_0^\infty e^{-st} f(r,t)dt, \mathrm{Re}(s) > 0 \tag{37}$$

By utilizing the convolution theorem, it becomes possible to employ the Laplace transform to the memory-derivative $D_{\tau_i}^p$ [34], satisfying the property

$$\mathcal{L}[\tau_i D_{\tau_i}^p f(r,t)] = \mathcal{L}\left[\int_{t-\tau_i}^t K(t-\xi) f^p(r,\xi) d\xi\right] = s^{p-1} G(s,\tau_i) \mathcal{L}[f(r,t)] \tag{38}$$

and we define the kernel function as

$$K(t-\xi) = 1 - \frac{2f}{\tau}(t-\xi) + \frac{e^2}{\tau^2}(t-\xi)^p = \begin{cases} 1, & \text{if } e = f = 0, p = 0 \\ 1-(t-\xi), & \text{if } e = 0, f = \frac{\tau}{2}, p = 0 \\ \left(1 - \frac{(t-\xi)}{\tau_i}\right)^2, & \text{if } e = 1, f = 1, p = 1 \end{cases} \tag{39}$$

where is the delay time τ ; e and f are constants, and $p \in \mathbb{R}$, respectively
Taking Laplace transform on Eqs. (30)-(36), one obtains

$$(\nabla^2 - l_2)\overline{T} = l_3(\gamma T_0 s\overline{e} - \overline{Q}) \tag{40}$$

$$[\nabla^2 - (1/c_0^2)s^2]\overline{e} = \nabla^2 \overline{T} \tag{41}$$

$$\overline{\sigma}_{rr} = (1-\Omega^2)\overline{e} + \Omega^2 \frac{\partial \overline{u}}{\partial r} - \overline{T} \tag{42}$$

$$\overline{\sigma}_{\theta\theta} = (1-\Omega^2)\overline{e} + \Omega^2 \frac{\overline{u}}{r} - \overline{T} \tag{43}$$

where

$$\begin{aligned} l_0 &= ks(1+bs)[p_2 + k(\tau_T / c_0^2 \eta) G(s,\tau_T)] + k^*, \\ l_1 &= [1 + G(s,\tau_q) + (1/2)s\, G(s,\tau_q)] / l_0, \\ l_2 &= l_1(\rho C_e s^2), \ l_3 = l_1 p_1 s, \end{aligned} \tag{44}$$

$$G(s,\tau_i)=\begin{cases}1-e^{-s\tau_i}, & K(t-\xi)=1\\ 1-\dfrac{1-e^{-s\tau_i}}{s}, & K(t-\xi)=1-(t-\xi)\\ \left(1-\dfrac{2}{s\tau_i}\right)+\dfrac{2(1-e^{-s\tau_i})}{s^2\tau_i^{\,2}}, & K(t-\xi)=\left(1-\dfrac{(t-\xi)}{\tau_i}\right)^2\end{cases} \tag{45}$$

and s is the Laplace parameter and subscript $i=T,q,u$.
Now $\overline{e}$ eliminating from Eqs. (46) and (47), one gets

$$(\nabla^4-A_1\nabla^2+A_2)\overline{T}=0 \tag{46}$$

Since A_1 and A_2 are real positive numbers, then Eq.(52) becomes

$$(\nabla^2-m_1^2)(\nabla^2-m_2^2)\overline{T}=0 \tag{47}$$

where m_1^2 and m_2^2 are the roots of the characteristics equation

$$m^4-A_1m^2+A_2=0 \tag{48}$$

where

$$A_1=l_2+l_4{}^2+l_3\gamma T_0 s,\, A_2=l_2l_4{}^2,\, l_4=s/c_0 \tag{49}$$

and

$$m_1^2, m_2^2=\frac{A_1\pm\sqrt{A_1^2-4A_2}}{2} \tag{50}$$

The solution of Eq.(47) is given as

$$\overline{T}=\frac{1}{\sqrt{r}}\sum_{i=1}^{2}[G_iI_{1/2}(m_i r)+H_iK_{1/2}(m_i r)] \tag{51}$$

From Eqs. (40) and (51) one gets the dilation $\overline{e}$ in the Laplace domain as follows

$$\overline{e}=\frac{1}{l_3\gamma T_0 s\sqrt{r}}\sum_{i=1}^{2}(m_i^2-l_2)[G_iI_{1/2}(m_i r)+H_iK_{1/2}(m_i r)] \tag{52}$$

Using Eq.(52) in (36) and using

$$\int r^{3/2}I_{1/2}(m_i r)dr=\frac{r^{3/2}I_{3/2}(m_i r)}{m_i},\qquad \int r^{3/2}K_{1/2}(m_i r)dr=\frac{-r^{3/2}K_{3/2}(m_i r)}{m_i} \tag{53}$$

one obtains the displacement $\overline{u}$ as

$$\overline{u}=\frac{1}{l_3\gamma T_0 s\sqrt{r}}\sum_{i=1}^{2}\frac{(m_i^2-l_2)}{m_i}[G_iI_{3/2}(m_i r)-H_iK_{3/2}(m_i r)] \tag{54}$$

By differentiating Eq. (54) with respect to r, we get

$$\frac{d\bar{u}}{dr} = \frac{1}{l_3 \gamma T_0 s \sqrt{r}} \sum_{i=1}^{2} \left\{ \begin{array}{l} (m_i^2 - l_2)[G_i I_{1/2}(m_i r) + H_i K_{1/2}(m_i r)] - \\ \dfrac{2(m_i^2 - l_2)}{r\, m_i}[G_i I_{3/2}(m_i r) - H_i K_{3/2}(m_i r)] \end{array} \right\} \tag{55}$$

Using Eqs. (51)-(51) and (54)-(55), the thermal stresses $\bar{\sigma}_{rr}$ and $\bar{\sigma}_{\theta\theta}$ that appeared in Eqs. (42) and (43) can be expressed as

$$\begin{aligned} \bar{\sigma}_{rr} = \sum_{i=1}^{2} \Bigg\{ & \left[\frac{(m_i^2 - l_2)}{l_3 \gamma T_0 s \sqrt{r}} + \frac{1}{\sqrt{r}} \right] [G_i I_{1/2}(m_i r) + H_i K_{1/2}(m_i r)] \\ & - \frac{2\Omega^2}{l_3 \gamma T_0 s r \sqrt{r}} \frac{(m_i^2 - l_2)}{m_i} [G_i I_{3/2}(m_i r) - H_i K_{3/2}(m_i r)] \Bigg\} \end{aligned} \tag{56}$$

$$\begin{aligned} \bar{\sigma}_{\theta\theta} = \sum_{i=1}^{2} \Bigg\{ & \left[\frac{(1-\Omega^2)(m_i^2 - l_2)}{l_3 \gamma T_0 s \sqrt{r}} - \frac{1}{\sqrt{r}} \right] [G_i I_{1/2}(m_i r) + H_i K_{1/2}(m_i r)] \\ & + \frac{\Omega^2}{l_3 \gamma T_0 s r \sqrt{r}} \frac{(m_i^2 - l_2)}{m_i} [G_i I_{3/2}(m_i r) - H_i K_{3/2}(m_i r)] \Bigg\} \end{aligned} \tag{57}$$

Using the conditions given by Eqs. (26)-(27), and (31)-(32) as in Laplace domain

$$\bar{T}(a,s) = T_1 / s \tag{58}$$

$$\bar{T}(b,s) = T_2 / s \tag{59}$$

$$\bar{\sigma}_{rr}(a,s) = 0 \tag{60}$$

$$\bar{\sigma}_{rr}(b,s) = 0 \tag{61}$$

in Eqs. (51) and (56), one get

$$\sum_{i=1}^{2} [G_i I_{1/2}(m_i a) + H_i K_{1/2}(m_i a)] = \frac{\sqrt{a} T_1}{s} \tag{62}$$

$$\sum_{i=1}^{2} [G_i I_{1/2}(m_i b) + H_i K_{1/2}(m_i b)] = \frac{\sqrt{b} T_2}{s} \tag{63}$$

$$\begin{aligned} \sum_{i=1}^{2} \Bigg\{ & a[(m_i^2 - l_2) + l_3 \gamma T_0 s][G_i I_{1/2}(m_i a) + H_i K_{1/2}(m_i a)] \\ & -2\Omega^2 \frac{(m_i^2 - l_2)}{m_i} [G_i I_{3/2}(m_i a) - H_i K_{3/2}(m_i a)] \Bigg\} = 0 \end{aligned} \tag{64}$$

$$\begin{aligned} \sum_{i=1}^{2} \Bigg\{ & b[(m_i^2 - l_2) + l_3 \gamma T_0 s][G_i I_{1/2}(m_i b) + H_i K_{1/2}(m_i b)] \\ & -2\Omega^2 \frac{(m_i^2 - l_2)}{m_i} [G_i I_{3/2}(m_i b) - H_i K_{3/2}(m_i b)] \Bigg\} = 0 \end{aligned} \tag{65}$$

On solving this system of Eqs. (62)-(65), one gets the value of the parameter G_1, H_1, G_2, H_2, and hence, we get the temperature, displacement and stress components of the medium in the Laplace transform domain.

5 FOURIER SERIES METHOD FOR THE NUMERICAL INVERSION

The inverse formula for the Laplace transform of Eq. (37) may be expressed as

$$g(x,t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \exp(st)\,\overline{g}(x,s)ds \tag{66}$$

where d is an arbitrary real number greater than real parts of singularities of $\overline{g}(x,s)$.
Taking $s = d + i\,w$, the preceding integral takes the form

$$g(x,t) = \frac{\exp(dt)}{2\pi i} \int_{-\infty}^{\infty} e^{itw}\overline{g}(x, d+iw)dw \tag{67}$$

Expanding the function $h(x,t) = e^{-dt} g(x,t)$ in a Fourier series in the interval [0, 2T], we obtain the approximate formula [36]

$$g(x,t) = g_\infty(x,t) + E_D \tag{68}$$

where

$$g_\infty(x,t) = \frac{1}{2}C_0 + \sum_{k=1}^{\infty} C_k \quad \text{for } 0 \le t \le 2T \tag{69}$$

and

$$C_k = \frac{\exp(dt)}{T}\left\{\exp\left[ik\pi t / T\,\overline{f}\left(x, d + i\frac{k\pi t}{T}\right)\right]\right\} \tag{70}$$

The discretization error E_D can be made arbitrarily small by choosing d large enough [44]. Since the infinite series in Eq. (70) can be summed up to a finite number N of terms, the estimated value of $g(x,t)$ becomes

$$g_N(x,t) = \frac{1}{2}C_0 + \sum_{k=1}^{N} C_k \quad \text{for } 0 \le t \le 2T \tag{71}$$

Using the preceding formula to evaluate $g(x,t)$, we introduce a truncation error E_D that must be added to the discretization error to produce a total approximation error.
To reduce the total error, we first use the "Korrektur" method to reduce the discretization error. Next, the ε-algorithm is used to accelerate the convergence [36]. The method uses the following formula to evaluate $g(x,t)$:

$$g(x,t) = g_\infty(x,t) - \exp(-2dT)\,g_\infty(x, 2T+t) + E'_D \tag{72}$$

where the discretization error $|E'_D| \ll |E_D|$.
Thus, the approximate value of $g(x,t)$ becomes

$$g_{N_K}(x,t) = g_N(x,t) - \exp(-2dT) g_{N'}(x, 2T+t) \tag{73}$$

where N' is an integer such that $N' > N$. Secondly, we shall now describe the ε-algorithm that is used to accelerate the convergence of the series in Eq. (71). Let $N = 2q+1$, where q is a natural number and let $S_m = \sum_{k=1}^{m} C_k$ be the sequence of partial sums of series in (71). We define the ε-sequence by $\varepsilon_{0,m} = 0, \ldots \varepsilon_{1,m} = S_m$ and $\varepsilon_{p+1,m} = \varepsilon_{p-1,m+1} + 1/(\varepsilon_{p,m+1} - \varepsilon_{p,m}), p = 1,2,3,\ldots.$. It can be shown [36] that the sequence $\varepsilon_{1,1}, \varepsilon_{3,1}, \varepsilon_{5,1}, \ldots \varepsilon_{N,1}$ converges to $g(x,t) + E_D - C_0/2$ faster than the sequence of partial sums $s_m\,(m = 1,2,3,..)$. The actual procedure used to invert the Laplace transform consists of using Eq. (73) together with the ε-algorithm. The values of *d* and *T* are chosen according to the criterion outlined in [37].

6 NUMERICAL RESULTS, DISCUSSION AND REMARKS

The computations were done for various kernel functions in the Laplace domain as

$$G(s,\tau_i) = \begin{cases} 1 - e^{-s\tau_i}, & K(t-\xi) = 1 \\ 1 - \dfrac{1 - e^{-s\tau_i}}{s}, & K(t-\xi) = 1 - (t-\xi) \\ \left(1 - \dfrac{2}{s\tau_i}\right) + \dfrac{2(1 - e^{-s\tau_i})}{s^2 \tau_i^2}, & K(t-\xi) = \left(1 - \dfrac{(t-\xi)}{\tau_i}\right)^2 \end{cases} \tag{74}$$

The thermoelastic material used for this purpose is copper, and the values of the various physical constants are as follows [38]:

$$\begin{aligned} &k = 386\,\mathrm{kg\,m\,K^{-1}s^{-3}}, \alpha_t = 1.78\times10^{-5}\,\mathrm{K^{-1}}, \rho = 895\,\mathrm{kg\,m^{-3}}, T_0 = 293\,\mathrm{K}, \\ &c_v = 383.1\,\mathrm{m^2\,K^{-1}s^{-2}}, \mu = 3.86\times10^{10}\,\mathrm{kg\,m^{-1}s^{-2}}, \lambda = 7.76\times10^{10}\,\mathrm{kg\,m^{-1}s^{2}} \end{aligned} \tag{75}$$

and the other non-dimensional values for the problems are

$$\begin{aligned} &\tau_q = 0.2, \tau_T = 0.3, \tau_\upsilon^* = 0.6, b = 0.2, k = 0.1, k^* = 0.2, c_0 = 0.4, \\ &\mu = 0.7, \eta = 0.5, \gamma = 0.6, \lambda = 0.85, \rho = 0.3, C_e = 0.75, K = 0.63 \end{aligned} \tag{76}$$

The main aim of the present literature review is to illustrate the changes in thermal field, displacement, and thermal stress on an elastic solid sphere within the confines of a memory-dependent nonsimple heat conduction model, with consideration given to a variety of thermal loadings. For the sake of brevity, we have considered point impulsive heat supply $g(t) = \delta(t)$ for Figures 2-5 to demonstrate the influence of kernel functions on the fluctuation of thermophysical properties. The numerical inversion of the Laplace transform was computed using the MATHEMATICA 10 software, specifically for the Fourier series algorithm. Figure 2 illustrates how, given positive values of phase lag and MDD in the radial direction, both thermodynamic and conductive temperatures progressively rise from the inner to the outer curved surface, resulting in a finite speed of thermal wave propagation. One possible explanation for the temperature reduction that occurs linearly from the outer curved surface of the sphere toward its inner curved surface is the sectional point impulsive heat supply, which is represented by Eq. (76). This heat source creates compressive strain and a lower temperature magnitude at the inner curved surface of the sphere and tensile strain and a bigger temperature magnitude at the outer curved surface.

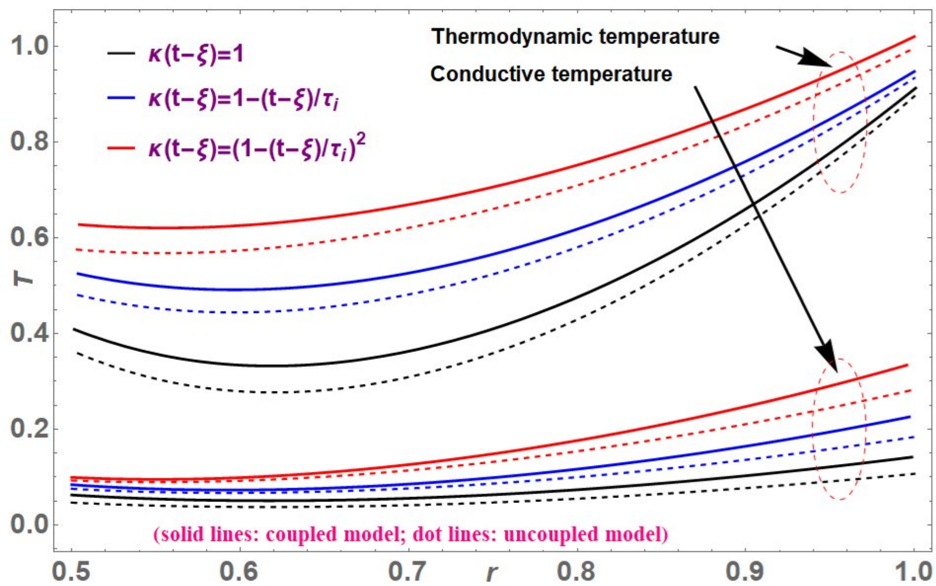

Fig. 2
Distribution of temperature along the radial direction.

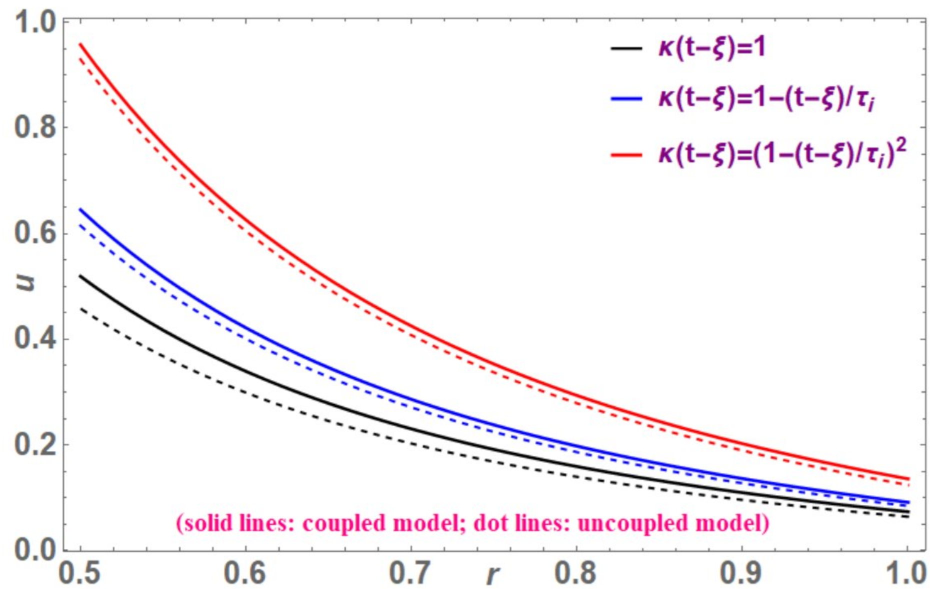

Fig. 3
Distribution of displacement along the radial direction.

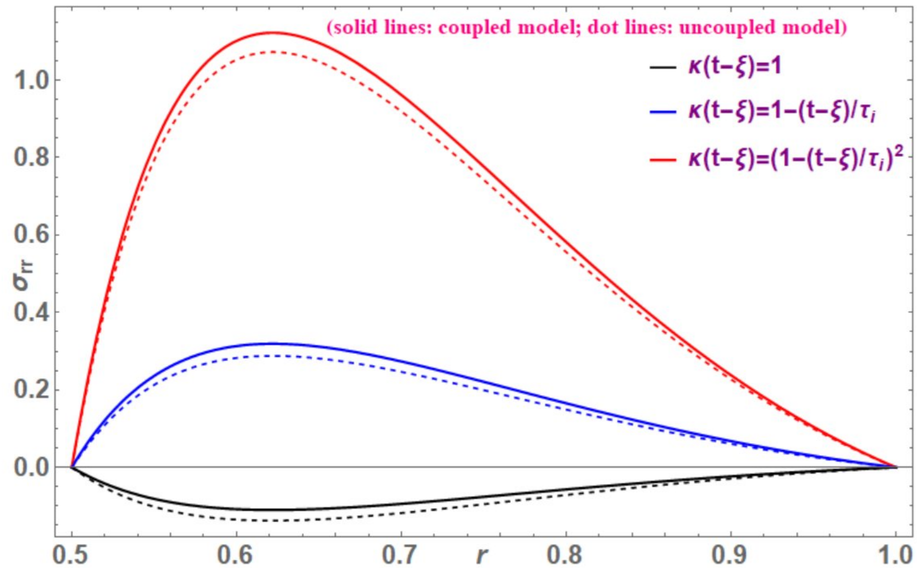

Fig. 4
Distribution of radial stress along the radial direction.

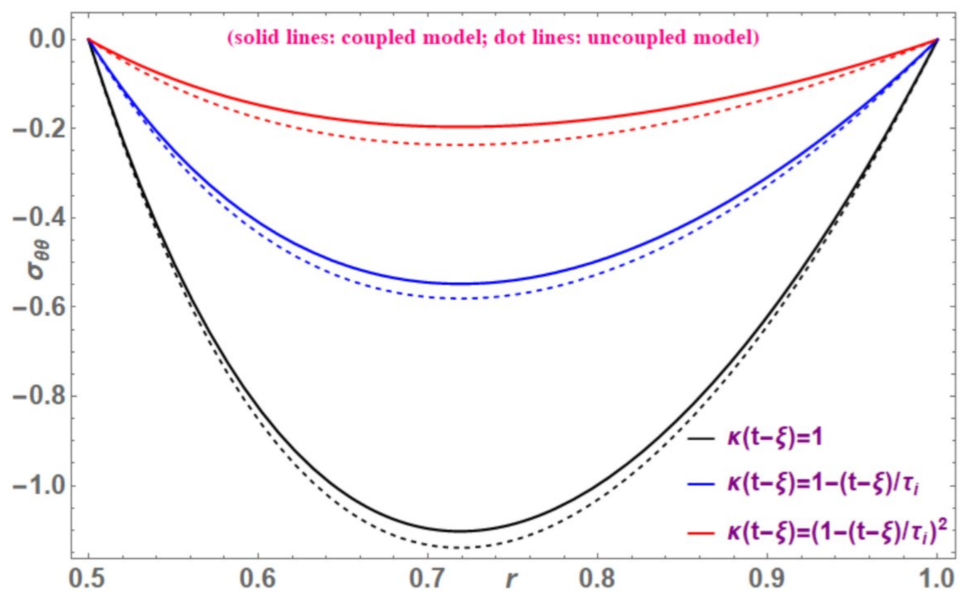

Fig. 5
Distribution of hoop stress along the radial direction.

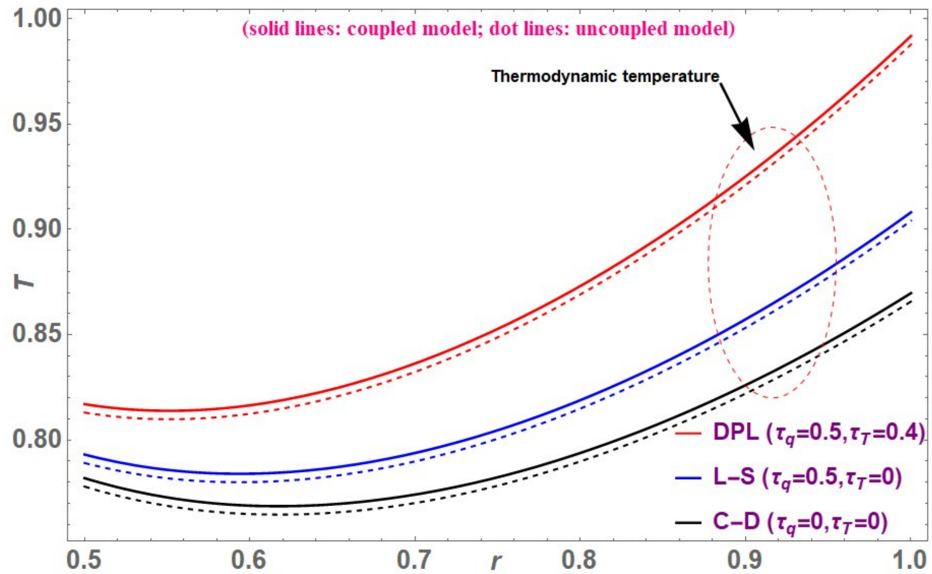

Fig. 6
Distribution of temperatures predicted by various thermoelastic theories .

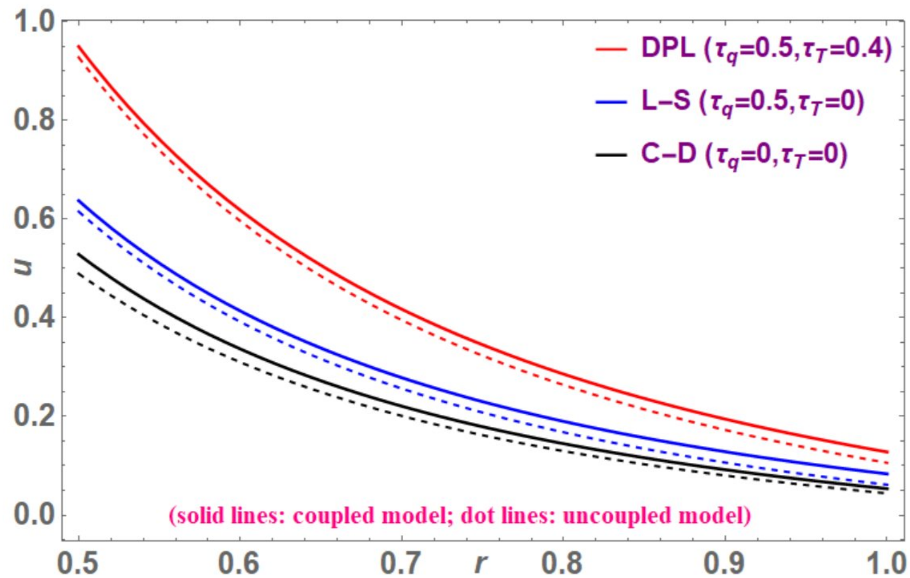

Fig. 7
Displacement profile for different models along the radial direction.

The variance in the displacement component u, for different kernels is shown in Figure 3. The inner curved surface is where the radial displacement reaches its highest, and it diminishes toward the outer surface. For different kernel functions, the tensile deformations progressively diminish towards the vicinity of the outer spherical boundary surface. The radial stress profile is seen in Figure 4, where it first shows a modest upward trend from the positive radial stress value before peaking. Subsequently, it undergoes an exponential decline, ultimately aligning with radius r as mentioned in the boundary condition given by Eq. (26) for the kernel function

$K(t-\xi)=1-(t-\xi)$ and $K(t-\xi)=\left(1-(t-\xi)/\tau_i\right)^2$. The radial stress exhibits an increase to a maximum when the value of time ranges from $0.5 < r \le 0.6$, eventually reaches peak value at $r = 0.6$ followed by a progressive decrease from $r > 0.6$, before reaching to zero while for the kernel function in radial stress slightly toward the negative side and then reaching zero at the outer curved surface of the sphere. Figure 5 displays that there is a decrease in hoop stress from the inner curved surface up to $r = 0.7$, and later, it increases to an outer curved surface with a slightly upward trend due to tensile force till it reaches zero, satisfying the boundary condition as traction traction-free condition at the curved surface of the sphere. Figures 6 to 9 illustrate the curves that are predicted by three distinct theories of thermoelasticity and produced as a special example of the dual-phase-lag mode represented by the graphs. For a single time value, t = 0.6, and many parameter values $0 \le \tau_u \le \tau_T \le \tau_q$ $(\tau_q = 0.5, \tau_T = 0.4)$ are shown in Figure 6. These calculations were performed in the Lord Shulman thermoelastic theory (L-S) with phase-lag as $\tau_q = 0.5, \tau_T = 0$, in the coupled thermoelastic theory (C-D) with $\tau_q = 0, \tau_T = 0$. The temperature, displacement, radial stress, and hoop stress distributions vs. radius r are compared at three different times (t = 0.2, 0.4, and 0.6) in Figures 10–13. together, these data show that all values are climbing with increasing time.

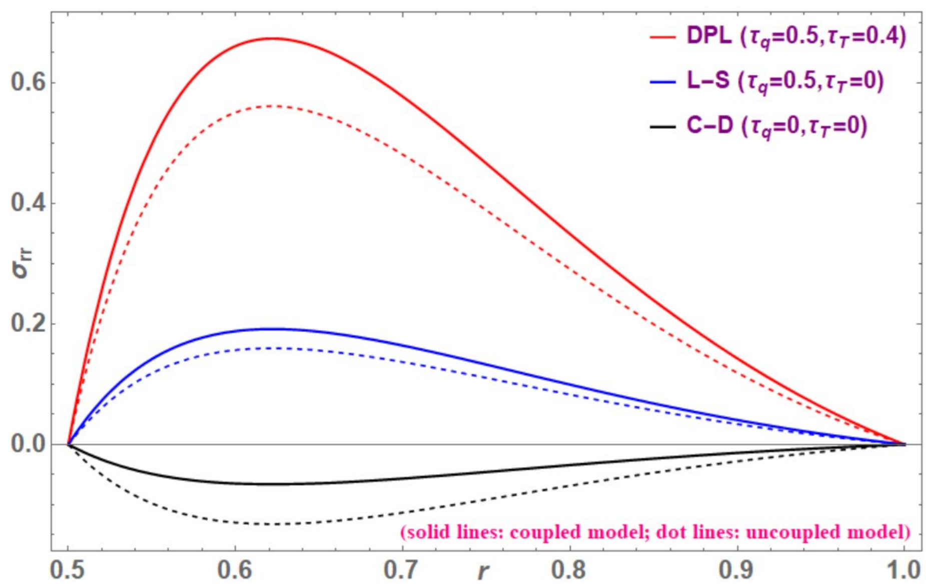

Fig. 8
Radial stress for different models along the radial direction.

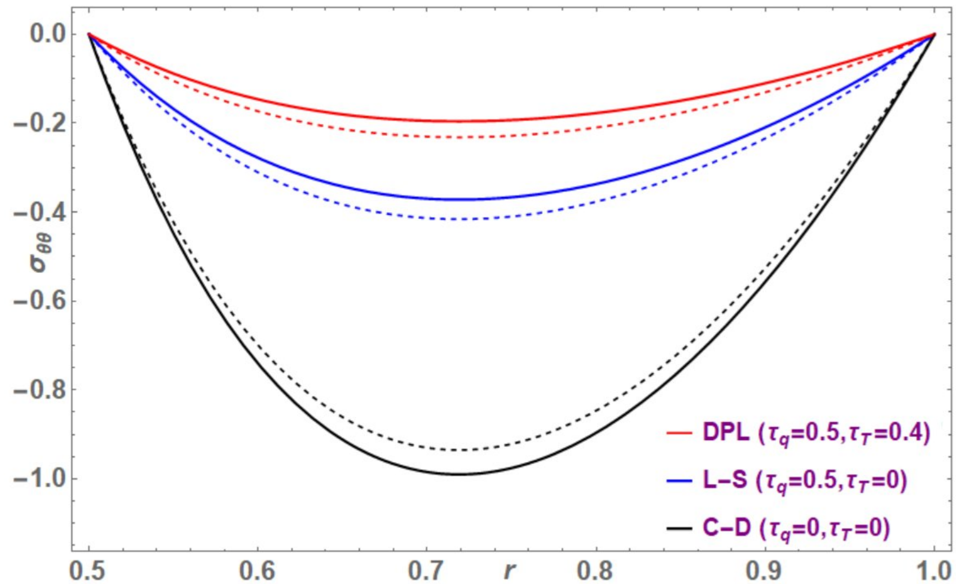

Fig. 9
Hoop stress for different models along the radial direction.

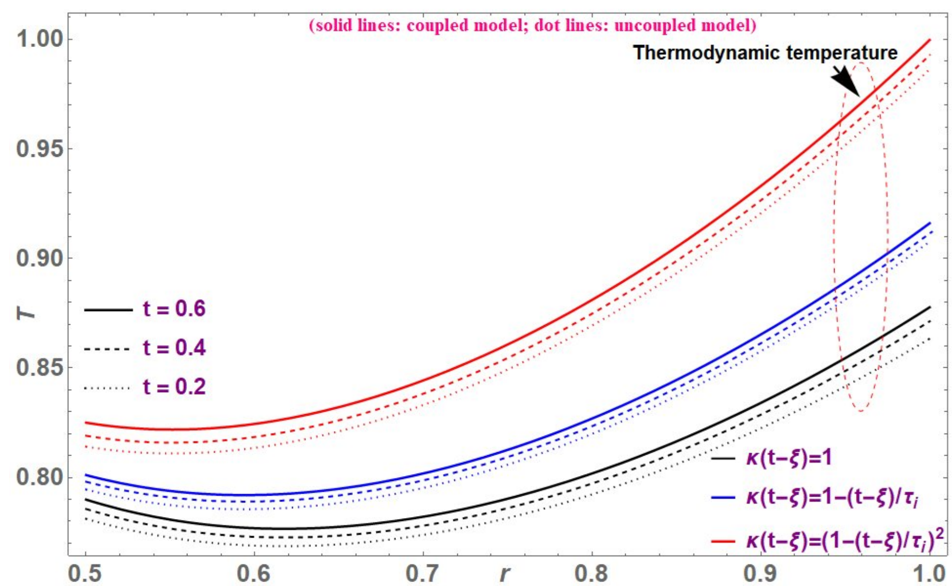

Fig. 10
Temperature distribution for different time t along the radial direction.

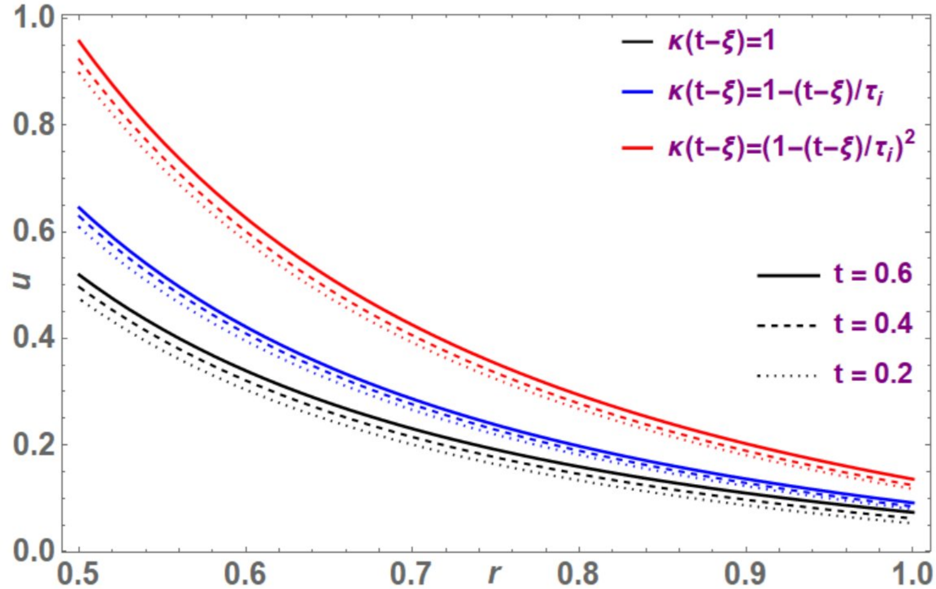

Fig. 11
Displacement profile for different time t along the radial direction.

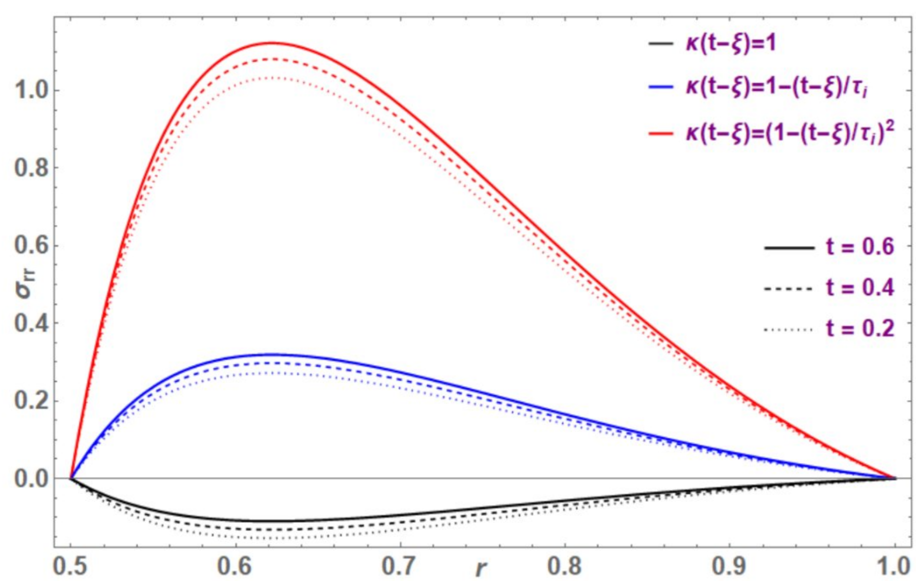

Fig. 12
Radial stress distribution for different time t along the radial direction.

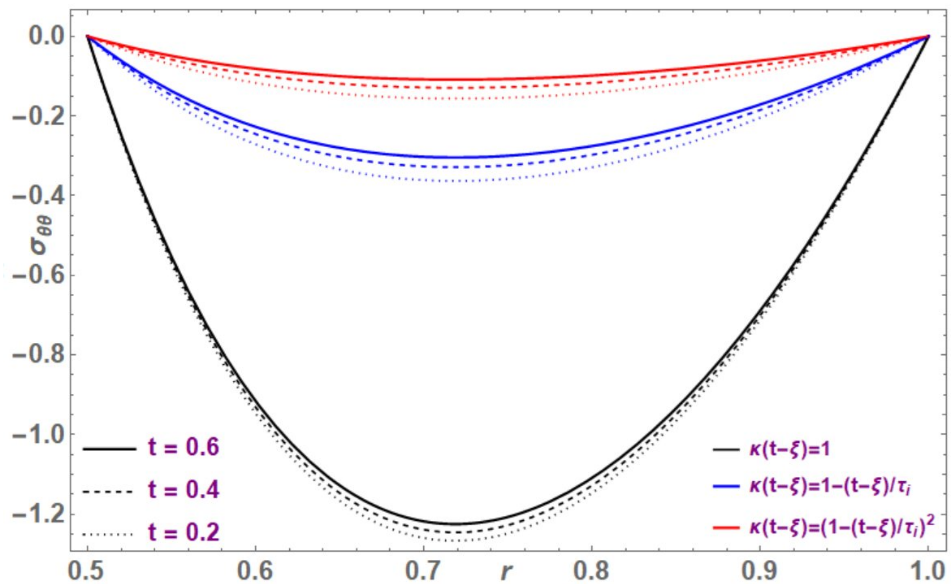

Fig. 13
Hoop stress distribution for different time t along the radial direction.

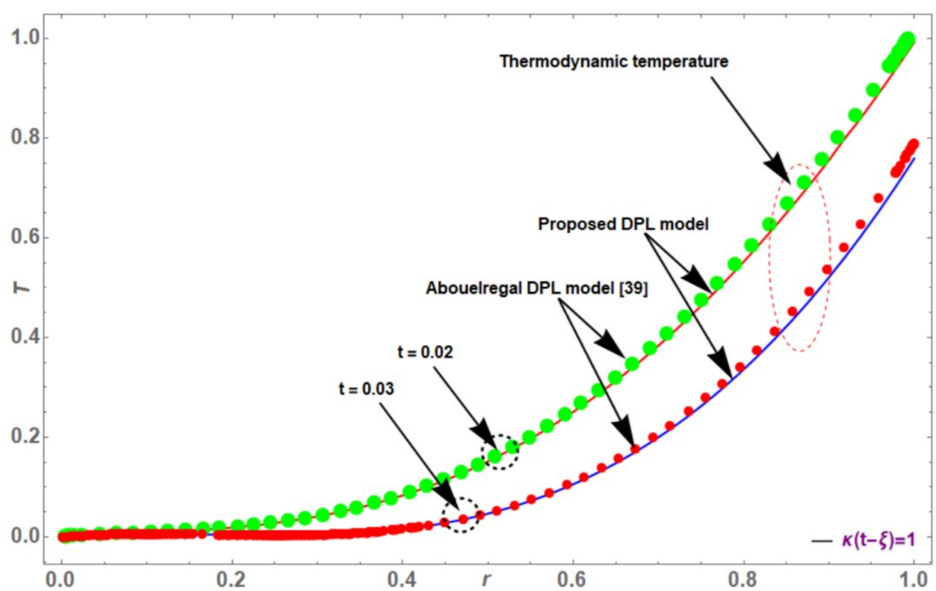

Fig. 14
Temperature distribution t along the radial direction.

Figures 14 depict the comparison curves between the DPL model suggested by Abouelregal [39] and the proposed model. As an example of the dual-phase-lag model, we have used identical parameters to test the disparity of two underlying temperature distributions. The Kolmogorov-Smirnov test indicates a high degree of similarity between two curve distributions, with a (dis-)similarity value of 0.11 (approaching zero), indicating the curves are "alike". The study validated a novel nonsimple thermoelastic DPL model using a memory-dependent effect, which may be more effective than fractional models. The model, with its unique form and clearer physical meaning based on the essence of MDD's definition, is represented by integer-order differential and integral equations, making it more convenient for numerical calculations compared to fractional equations. Lastly, the model allows for arbitrary choices of the Kernel function, temperature discrepancy factor, and time delay, providing more flexibility in describing the material's practical response. It makes the new model more adaptable in applications compared to fractional models, where the fractional order parameter is the main variable of significance.

7 CONCLUSION

This paper introduces the use of MDD as an alternative to fractional derivatives due to its uniqueness and adaptability. MDD uses integer-order differentiation and integration, providing more reliable numerical approximations. Its kernel function, time delay, and temperature discrepancy parameters can be adjusted to meet specific applications, demonstrating a higher level of adaptability compared to fractional-order variables. The use of Laplace transformation in MDD can provide accurate solutions for thermoelastic materials under various heat conduction situations. This paper suggests a TPL model that satisfies the stability requirements according to the well-posed problem in a hollow sphere with a spherically symmetry system subjected to thermal shock on the inner and outer curved surface. In order to meet initial and boundary constraints in a two-temperature, three-phase-lag heat conduction model for all MDD kernel functions, the study numerically estimates thermal fluctuations in conductive temperature, thermodynamic temperature, displacement distribution, radial stress, and hoop stress. When all three delays τ_q, τ_T and τ_u are zero, the generalized coupled thermoelasticity with MDD approaches to classical uncoupled theory results closely resemble the generalized LS theory of thermoelasticity. Thus, in the two-temperature theory, the suggested fractional DPL heat conduction model satisfactorily recovers linked and extended thermoelasticity theories. Using MDD and applied phase delays, the TPL model for generalized coupled thermoelasticity with the MDD model, which is based on two temperature-generalized thermoelasticity theories can be used scientifically to classify materials according to their capacity for heat conduction. The study also introduces an innovative generalized concept for TPL heat, conducting Fourier's law, integrating time delay variables and kernel functions. The time delay parameter is crucial for characterizing behavioural patterns of physical field variables and can be used to create an innovative categorization framework for materials. Increased time delay significantly impacts temperature and stress distribution, leading to smoother physical field distribution. Different processes require different kernels to depict memory effects accurately. A constant kernel function can increase MDD's influence on results. Modifying time delays, temperature discrepancy factor, and selecting different kernels can alter MDD's memory effect, flattening response curves while maintaining a constant memory-dependent parameter. MDD captures the memory effect more closely, potentially improving materials' heat-handling ability.
The numerical results provide several key insights:

- Laplace transformation in MDD for thermoelastic materials
  - The Laplace transformation offers exact solutions for various heat conduction scenarios in thermoelastic materials.
  - The TPL model for a spherically symmetric elastic sphere under thermal shock is proposed, enhancing the understanding of thermal responses.
  - The dynamic coupled thermoelasticity with MDD shows a close resemblance to the LS theory of thermoelasticity, suggesting potential improvements in modeling techniques.
  - The approach successfully recovers linked and extended thermoelasticity theories within the nonsimple theory framework.
- Three-phase lag model for generalized coupled thermoelasticity
  - Utilizes MDD and applied phase delays, categorizing materials based on their heat conduction capacity.
  - Introduces a TPL heat concept integrating Fourier's law with time delay variables, providing a more comprehensive heat transfer model.
  - The time delay parameter characterizes the behavioral patterns of physical field variables, with increased delays impacting temperature and stress distributions.
- Practical applications
  - The flexibility in modifying time delays and kernel functions allows for the design of materials with improved heat-handling capabilities, which is crucial in aerospace, automotive, and electronics industries.

   - The proposed models can be applied to design materials and structures with enhanced resistance to thermal shocks, benefiting applications in high-temperature environments.
- Advanced simulation and modeling
   - The use of MDD and TPL models in simulations can lead to more accurate predictions of thermal and mechanical responses, aiding in the development of advanced engineering systems.
   - These models can be used to optimize manufacturing processes that involve rapid thermal cycling, such as welding and additive manufacturing.
- Academic and industrial research
   - The results provide a benchmark for future research in thermoelasticity, helping to refine and develop new models.
   - The findings can be extended to other fields, such as geophysics and biomechanics, where understanding thermoelastic behavior is crucial.

By connecting the findings to these practical applications, the study underscores the potential real-world impact and usefulness of the proposed models.

NOTATIONS

The notation to be active in this study is as follows:

q	heat conduction vector
T	thermodynamic temperature
Φ	conductive temperature
T_0	initial temperature
∇T	temperature gradient
∇u	thermal displacement gradient
k	thermal conductivity
τ_q , τ_T , τ_φ	phase lags of heat flux, temperature, conductive temperature
τ_u	phase lags of thermal displacement gradient
σ_{ij}	components of the stress tensor
e_{ij}	components of strain tensor
F_i	external forces per unit mass
ρ	density assumed independent of time
δ_{ij}	Kronecker's delta function
C_e	specific heat
∇^2	Laplacian operator
$Y^{(m)}$	order derivative

$H(t)$	Heaviside function
λ, μ	Lame's constants
γ	coupling parameter
α_t	thermal expansion factor
$I_{1/2}$	first kind modified Bessel function order 1/2
$K_{1/2}$	second kind modified Bessel function order 1/2

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