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Research paper

Using Soft Computing and Chaos Theory in investigating the Deformed Stadium

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Abstract

This paper analyzed the dynamic system of billiards from a classic perspective. For this purpose, mapping and cross-section methods were first employed to study the behavior of this system and the results indicated that it was a chaotic one. Then a deformed stadium was introduced and its long-term behavior was analyzed. Considering changes in the behavior of this system following the slightest deformation at the boundaries, Poincaré map was used to demonstrate the occurrence of regular and irregular motions, indicating the completely chaotic behavior of the system. The shape of the cross-section of the regular motion shows that the points of contact with the boundary are located on a line in the phase space. On the other hand, the cross-sectional surface of a chaotic motion, the surface is covered with collision points and the empty spaces are surrounded by invariant curves. These spaces are also filled in case of $n \to \infty$ and they eventually disappear and the surface is covered with collision points, completely. This behavior is characteristic of chaotic systems

1. Introduction

"Chaos" has a Greek origin, denoting a gaping void or a chasm that existed before all things. Romans applied the word to the rough, shapeless mass from which architects of the world create order and harmony. In modern language, chaos is used to imply disorder and lawlessness. Upon the introduction of Newton's laws in 1687, scholars used them to solve copious problems. Due to the large variety of these problems, they believed that the subsequent states of the system could be reached at any other given point in time if the initial conditions existed.

By the late 19th century, Poincaré showed that the temporal evolution of some systems created by the Hamiltonian equations could have chaotic motions[1]. In 1963, Lorenz demonstrated that a simple set of three linear first-order differential equations could produce completely chaotic trajectories. He found one of the first examples of algebraic chaos in dissipative systems. Chaos can be detected in a non-linear system where dynamic rules uniquely determine its temporal evolution through initial conditions[2]. In recent years, new theoretical findings along with high-speed computers and experimental results have helped us realize that nature prevents these phenomena. Furthermore, non-linearity is a necessary but not sufficient condition for chaos to occur. Chaotic motions are not observed due to external noise sources, infinite degrees of freedom of the system, or uncertainty in quantum mechanics. However, the main source of irregularity is a property of nonlinear systems that exponentially separates initial paths that are very close to each other in the phase space[3]. Therefore, it is impossible to predict the behavior of such systems for long periods, as errors grow exponentially with the limited accuracy of the initial conditions. Lorenz called this sensitivity to initial conditions the butterfly effect because the results of equations can change by flaps of a butterfly's wings[4]. Billiards is a dynamic system studied in classic and quantum mechanics[5]. This paper aimed to analyze the dynamic system of billiards from a classic perspective. To this end, first dynamic systems were introduced and examined their properties. By analyzing the trajectories of these systems in the phase space, the classic properties of chaotic systems were introduced. Next, building on our knowledge of classic chaotic motions, the chaotic system of billiards was introduced, proposing a method for studying various motions of this system. Then different types of motion in the stadium billiards were examined. Finally, a specific billiard system was introduced and investigated its various observed motions.

2. Dynamic Systems

Newton's laws are employed to analyze dynamic systems, being the base from which describing equations of these systems are derived. However, the number of dynamic systems that can be fully analyzed to obtain explicit solutions is very limited. Most dynamic systems are non-integrable, and their behaviors must be studied through numerical methods.

Dynamic systems are characterized by two specific features: 1) the states of the system at each moment are determined by the values of N variables $x_1, x_2, ..., x_N$; 2) the evolution of the system is determined by N differential equations. In other words:

$$\frac{dx_i}{dt} = f_i\left(x_1, ..., x_N\right) \quad i = 1, ..., N \tag{1}$$

N is the order of the dynamic system, and N variables x_i, represent physical quantities such as position and velocity. If $\vec{X}$ is defined with $x_1, x_2, ..., x_N$ components and $\vec{F}$ with $f_1, f_2, ..., f_N$ components, the differential equations are written more simply as[6-9]:

$$d\vec{X}/dt = \vec{F}\left(\vec{X}\right) \tag{2}$$

This equation is completed with the following initial conditions:

$$\vec{X}\left(t = 0\right) = \vec{X}_0 \tag{3}$$

Its product would be an integral curve passing through $\vec{X}_0$. There are two states for the product of a dynamic system. First state: the overall product is explicitly written. In this case, the product is written as $X\left(a_1, a_2, ..., a_N, t\right)$, where *a*'s are integration constants. Second state: the overall product is not explicitly known, in which case, the product can be divided into two categories: First: The product is valid only within a limited time interval, such as calculating the positions of planets in the next few years, where direct numerical integration leads to the desired product. Second: The product is acceptable for a relatively long time, such as the long-term stability of the solar system. In such problems, the asymptotic behavior of the product in $t \to \infty$ is examined. The paths resulting from such problems are divided into two categories: 1) Nonreversible paths that never return to their initial position; and 2) Reversible paths that return to their initial position after a limited period[6].

2.1. Hamiltonian Systems

Hamiltonian systems are a special case of dynamic systems. The first feature of these systems is their even dimensionality, $N = 2n$ [6,7,10]. n is the number of degrees of freedom of the system and N dimensions of the phase space. The 2n variables that make up the phase space are $q_1, ..., q_n, p_1, ..., p_n$. The system is described by a 2n-dimensional function (instead of N functions in the general state), called the Hamiltonian $H\left(p_1, ..., q_n\right)$, and major differential equations for the variables are:

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i}\ ,\ \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}\quad i=1,...,n \tag{4}$$

q_i and p_i are called conjugate variables. Hamiltonian is an accessible integral and can be demonstrated H fixes on a path, using equations (2). Therefore, the order of the system is reduced to 2n-1[6,7]. By introducing a cross-section for the system, the problem is reduced to studying the system in a 2n-2-dimensional space. An appropriate method for reducing the dimension of the problem under study is to first eliminate a variable like p_i using the integral H, and then define the cross-section using the equation $q_i=0$. In this method, one pair of variables is eliminated, and n-1 pairs of conjugate variables determine the cross-section[6-13].

The motion of Hamiltonian systems can be divided into two categories: 1) Regular motion: These motions can be described using Newton's equations, such as the motion of a simple harmonic oscillator in one dimension and the motion of planets if disturbances from other planets are overlooked. In regular motion systems, paths with close initial conditions linearly diverge. 2) Irregular motion: Motions such as the motion of gas molecules when the molecules are confined to a plane and all molecules except one are fixed. In this case, a completely unpredictable motion with two degrees of freedom exists. In irregular motion systems, paths with roughly similar boundary conditions exponentially diverge and are highly sensitive to initial conditions. The difference between regular and irregular motions becomes apparent in the geometry of paths in phase space in the long-term[7,11].

2.2. Integrable Systems

Canonical transformations can help simplify the Hamiltonian concept. Using these transformations, variables $p_1,...,p_n$ and $q_1,...,q_n$ are transformed into new variables $P_1,...,P_n$ and $Q_1,...,Q_n$, and motion equations are derived from the new Hamiltonian, $H\left(p_1,...,q_n\right)\rightarrow H\left(P_1,...Q_n\right)$

If the canonical transformation is such that one of the variables does not appear in H, then the Hamiltonian will be simpler. If H does not depend on variable Q_i, then:

$$\frac{dp_i}{dt}=-\frac{\partial H}{\partial Q_i}=0 \tag{5}$$

As a result $P_i\left(t\right)=P_i\left(0\right)=\text{const}$. This constant value is a parameter that, if known, the Hamiltonian will depend on 2n-2 variables; meaning that there are n-1 conjugate variables and the degrees of freedom of the system decrease by two. Whenever there exists a canonical transformation under which the Hamiltonian does not depend on any Q_i s, that is: $H\left(P_1,...,P_n\right)$ then, P_i s are considered actions, Q_i s. angles,

$$P_i\left(t\right)=C_i\qquad i=1,...,n \tag{6}$$

$$\frac{dQ_i}{dt}=\frac{\partial H}{\partial P_i}=\omega\left(C_1,...,C_n\right) \tag{7}$$

$$Q_i\left(t\right)=\omega_i t+D_i \tag{8}$$

where C_i s and D_i s are 2n constants of integration. If the above conditions hold for the Hamiltonian of a system, the system is considered to be in a normal state. If a system's Hamiltonian can be brought into the normal state, the system is integrable. $P_1,...,P_n$ actions are integrals of the system and have constant values along any path. Conversely, if there are n specified integrals in a Hamiltonian system, there exists a canonical transformation whose resulting P_i s will be the integrals of the system[6-12]. For a system with a time-independent Hamiltonian, the Hamiltonian itself is an integral of the system. All systems with n=1 and a time-independent Hamiltonian are integrable[14].

3.2. Phase Space

In solving Hamiltonian equations for q and p as a function of time, given the initial conditions q_1 and p_1 at time t_1, the trajectory of the motion for any time t_2 can be determined. This p-q space is called the phase space of the system. A good way to present a dynamic system is by using phase space. Each state of the system at each point of time is presented by a point in the phase space. This point evolves concerning time and its velocity is $\vec{F}$, whose components are determined by equation (1). The geometric location of the points corresponding to the transformation of a system forms a curve in the phase space, where the velocity vector is at a tangent at every point in time. Therefore, by drawing the velocity vector in the phase space without integration, the trajectory can be determined because the equation (2) is independent of time and the first order[6-10]. The integral curves of equation (2) create a flux in the phase space, only one of the curves being a solution to condition (3). The created flux in the phase space has the following properties: 1) The time evolution of each path is uniquely determined as a function of the initial conditions; 2) The equation (2) is also integrable in time reversal; that is, two different paths never collide; and 3) Paths limited to a boundary in a region of the phase space remain limited to the boundary over time[14].

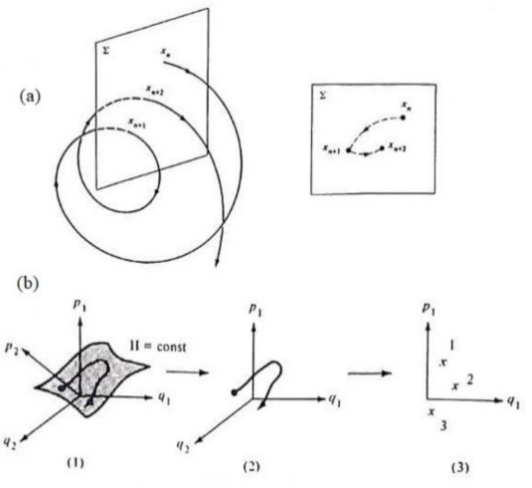

Figure 1Motion in the phase space and definition of the Poincaré cross-section: a) Points of collision of the path with the cross section; b) Motion with two degrees of freedom. 1) Four-dimensional phase space, 2) Image of the path in the volume (q_1,q_2,q_3), 3) Points of successive collisions of the path with the cross section $q_2=\text{const}$.

2.4. Cross section and Mapping

Since the aim of this study is analyzing the long-term and asymptotic behavior of the path of the dynamic system, following the path continuously is not needed; instead, the path discretely point by point is traversed in time. This idea is based on the cross section method. If N=4 is considered as figure. 1, then a four-dimensional phase space was existed. In this space, the cross section surface is a two-dimensional plane Σ. The consecutive intersection points of the path with this plane are denoted as $x_1,x_2,\ldots$. Since the path is reversible, there are points $x_0,x_{-1},\ldots$ in the return path. Using this property, if a point x_i is known, the point x_{i+1} can be determined. By following the path from the point x_i by integrating the differential equations until it collides with the plane Σ again, a new point x_{i+1} is obtained. This gives us a G mapping of the plane Σ to itself, called the Poincaré map. In general state:

$$G:\Sigma\to\Sigma \quad \text{and} \quad x_{i+1}=G(x_i) \tag{9}$$

in general

$$x_{i+j}=G^j(x_i) \tag{10}$$

Since the path can be followed in both time directions, the inverse mapping is defined as follows:

$$x_{i-1}=G^{-1}(x_i) \tag{11}$$

Overall, the equation (10) can be considered for positive and negative j. For any N, in an N-dimensional phase space, a N-1 dimensional subspace is considered and the collision points with $\ldots,X_{-2},X_{-1},X_0,X_1,X_2,\ldots$ are denoted The M=N-1 subspace should be called cross space, but for similarity with the case of N=4, called the cross section as well. In this space, the $x_1,x_2,\ldots,x_M$ coordinate system is introduced and the coordinates of each point X_i with $x_{i1},x_{i2},\ldots,x_{iM}$ is denoted The G mapping of this space is as follows:

$$x_{i+1,1}=g_1(x_{i,1},\ldots,x_{i,M}),\ldots,x_{i+1,M}=g_M(x_{i,1},\ldots,x_{i,M}) \tag{12}$$

In this method, consecutive points of X_i and overlook other details of the path are considered. These consecutive points are specified using G mapping and not using differential equations; thus, these equations are left out. Cross section and mapping methods are preferred for the following reasons: 1) the inherent properties of the dynamic system can be seen in the mapping and cross section equations. For example, a simple periodic path that returns to the initial point after one round corresponds to a fixed point in the G mapping; that is, the periodic path is stable if and only if the fixed point is constant; $X_i = G^j(X_i)$ 2) the new problem is much simpler because instead of differential equations, mapping equations are examined. Moreover, in an N-dimensional case, the investigated space has N-1 dimensions, making theoretical and numerical studies easier; 3) the inherent properties of the system are clearly shown in the long-term behavior, but the details of the short-term evolution are omitted. Therefore, the cross section should not be used to study the system in short periods; 4) Graphical display of results is much easier. For example, N=4 state is a two-dimensional cross section and its representation is much simpler than four-dimensional space.[6,7]

2.5. Ergodic Systems

In a Hamiltonian system, the path in the phase space is confined to a fixed subspace, $H = \text{const}$. This subspace is called the energy level. In an ergodic system, each path fills its energy level and the collision points cover all surfaces in the cross sectional space[6,8,10].

2.6. Chaotic Systems

Integrability is an exceptional property for Hamiltonian systems with over two degrees of freedom. Integrable systems are so rare that it is impossible to approximate a non-integrable system with a series of integrable ones[15]. Therefore, in most problems, numerical methods are used to obtain the solution of Hamiltonian equations, where small changes in initial conditions lead to significant changes in the obtained solutions. There is a class of dynamic systems where the particle passes through every point in the phase space. These systems are called chaotic systems. In a chaotic system, small changes in initial conditions cause paths to exponentially diverge, whereas paths diverge linearly in integrable systems. In the cross section of chaotic systems, chaotic regions can be seen, which are separated from each other by invariant and regular curves. However, the presence of these empty areas in the cross section does not contradict the ergodicity of the system, as these areas disappear with the mapping for $N \to \infty$, and the entire cross section surface is filled[6].

2.7. Liapunov Exponent

In a chaotic motion, the mapping points $x_{n+1} = G(x_n)$ diverge exponentially. The Liapunov exponent defines this divergence. As shown in the Figure 2:

$$\varepsilon \exp\left(N\lambda(x_0)\right) = \left|G^N(x_0+\varepsilon) - G^N(x_0)\right| \tag{13}$$

$$\lambda(x_0) = \lim_{N\to\infty}\lim_{\varepsilon\to\infty}\frac{1}{N}\log\left|\frac{G^N(x_0+\varepsilon) - G^N(x_0)}{\varepsilon}\right| \tag{14}$$

$$\lambda(x_0) = \lim_{N\to\infty}\frac{1}{N}\log\left|\frac{dG^N(x_0)}{dx_0}\right| \tag{15}$$

That is, $\exp(\lambda(x_0))$ is the average amount by which the distance between two neighboring points changes after one iteration[16]. Depending on the value of $\lambda(x_0)$, there are three different situations: 1) If it is positive, the two paths diverge exponentially; 2) If it is negative, the two paths converge; and 3) If it is zero, the distance between the two paths remains constant[17].

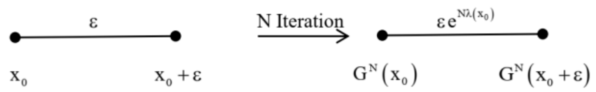

Figure 2 Liapunov Exponent

3. Billiards

Billiards is an important class of dynamic systems. By definition, billiards is a dynamic system with a closed environment (usually two-dimensional) that includes a free and dimensionless particle [6,11,15,16,18]. In billiards, the particle undergoes elastic collision with the boundary, therefore, energy conservation results velocity conservation. The particle behaves like a geometric ray with uniform angle of incidence and reflection at each collision point. A small group of billiards has interesting features. In this group, all velocity components are reversed at the reflection point. This type of reflection is Andreev reflection and billiards with this reflection is called Andreev billiards. Andreev billiards are of interest in condensed matter physics, especially in superconductivity [19]. To examine billiards, they are classified into two categories: 1) Integrable billiards: systems with n motion constants like rectangular and circular billiards; 2) Non-integrable billiards: systems where only energy conservation holds. These types of billiards are ergodic and chaotic systems [10]. Sinai and Bunimovich billiards are examples of these billiards. By defining billiards in a 2D state, Sinai proved that billiards is an ergodic and chaotic system and the moving particles of billiards behave chaotically[18]. Bunimovich also defined billiards with two degrees of freedom called the stadium billiards and proved that the stadium is an ergodic and chaotic system [20,21].

3.1. Birkhoff Mapping

In billiards, the movement of the particle starts from a point on the boundary, and the path is extended according to the differential equations describing the path until the particle hits the boundary at another point and reflects. This point is the end of the previous path and the beginning of the new one. Therefore, with the coordinates of this point, the path of the motion can be determined and it seems that the billiard wall is a suitable cross section to describe the motion of the particle. To investigate the behavior of the Hamiltonian billiard system, the state of the system with the coordinates of the reflection points on the boundary and in the direction of the motion is described. Thus, the phase space is determined as a mapping between consecutive and discrete collisions. The location of the reflection point on the billiard boundary is determined by the length of the arc S along the boundary and the direction of the motion after reflection by the angle Ψ, the angle of the velocity vector, $\vec{V}$, and the tangent vector on the boundary. If the total length of the boundary is L, the length of the arc is normalized to $s = \frac{S}{L}$. The conjugate coordinate s, is the tangential component of momentum, $P = \cos\Psi$. If the magnitude of the velocity is considered 1 in the selection of units, $0 \le \psi \le 180^\circ$ then the cross-sectional area is limited to a rectangle with the following dimensions:

$$-1 \le p \le 1 \quad , \quad 0 \le s \le 1 \tag{16}$$

s and p are defined coordinates of Birkhoff and the defined mapping, called the Birkhoff mapping, which is a class of Poincaré mappings[22,23,24]. Paths obtained are divided into three categories: 1) A finite set of N points, $(s_0, p_0), (s_1, p_1), \ldots, (s_{N-1}, p_{N-1})$, is obtained, which correspond to a closed path, and since for each closed path, there is:

$$(s_{n+N}, p_{n+N}) = M^N (s_n, p_n) = (s_n, p_n) \tag{17}$$

each of these N points is a fixed point of the mapping; 2) Repeating (s_0, p_0) forms a smooth and well-behaved curve in the phase space. Thus, the curve is called an invariable curve because, under the M mapping, each point on the curve returns to a point on the curve. This behavior can be seen in integrable systems, where there is a motion constant as a function $F(s,p)$ that: $F(s_0, p_0) = F(s_1, p_1)$ and each invariant curve is a contour of $F(s,p)$; 3) The repetition of (s,p) fills a certain level in the phase space, and in this case, the path is not limited by any constant quantity and is very sensitive to initial conditions (s_0, p_0). All three types of paths are observed in the study of a billiard dynamic system[19,23]. By comparing the solution method of differential equations and numerical integration and the mapping method, it becomes apparent that using the mapping method, the volume of computations and errors decreases significantly. Therefore, studying billiards using mapping is a suitable method for investigating dynamic systems.

3.2. Stadium

Stadium is an example of non-integrable and chaotic billiards (Bunimovich billiards). The stadium consists of two semicircles with the same radius, which are separated from each other by two parallel line segments of length a. The parameter $\eta = \frac{a}{R}$ is called the characteristic parameter of the stadium. Due to the above change, a stadium is perturbed by circular billiards, which turns into a stadium for non-zero values of η [16,20,23]. In the stadium, in addition to observing chaotic motions, in some of the initial conditions, completely regular and predictable motions are observed, which may disappear and be replaced by chaotic motions upon a small change in the initial conditions.

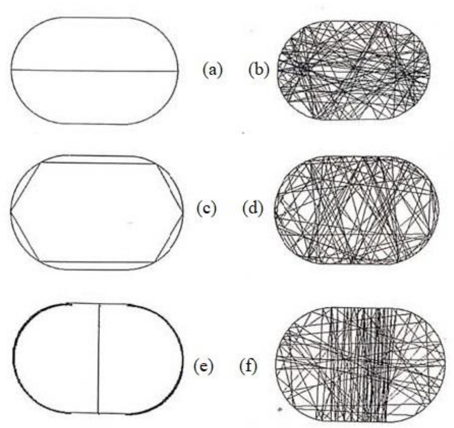

Figure 3 Regular and irregular motion in stadium. (a) Singular regular motion $\psi_0 = 90^\circ$,(b) chaotic motion $\psi_0 = 89^\circ$, (c) Singular regular motion $\psi_0 = 30^\circ$,(d) chaotic motion $\psi_0 = 29^\circ$, (e) Non-singular regular motion $\psi_0 = 90^\circ$,(f) chaotic motion $\psi_0 = 89^\circ$,

Figure 3 shows three examples of regular motion states of a particle in the stadium. With a small change in the initial conditions, the trajectory undergoes a major change and covers the entire surface of the stadium. Regular motions are divided into two categories: 1) Singular regular motion: there is a regular path corresponding to certain initial conditions, figure 3(a) and (c); 2) Non-singular regular motion: there is a set of regular trajectories corresponding to certain initial conditions, figure 3(e). Singular and non-singular paths are both unstable, and a small change in the location or momentum can cause the path to exponentially diverge from the regular state and cover the entire energy surface[10]. In examining the cross section area of the motion of the particle in the stadium using Birkhoff mapping, the obtained cross-sectional area can be seen in two forms: 1) The collision points lie on a one-dimensional curve shown in figure 4(a), which corresponds to the regular motion. 2) The collision points cover a two-dimensional surface that corresponds to the irregular motion of the particle in the stadium, shown in figure 4(b).

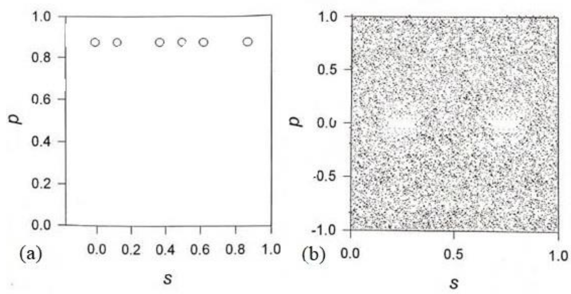

Figure 4 Collision cross section (a) regular, (b) irregular motion

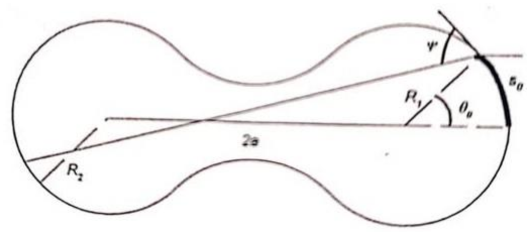

Figure 5 deformed stadium

3.3. Deformed Stadium

To better investigate the chaotic behavior of billiards, the effect of changing various parameters and changing the shape of the billiard boundary, a deformed stadium that specifies two arcs of a circle representing the boundary instead of two line segments (dumbbell shape) is introduced. To investigate this problem from a classic point of view, the free motion of a particle in billiards is considered. Using the Birkhoff map, points of collision of the particle with the boundary are determined and the cross section is plotted. The path corresponding to the cross section at the billiard level can also be determined. For this purpose, collision points are obtained using repeated calculations of obtained mapping equations. This eliminates the need for solving differential equations, integration, or dealing with errors resulting from these stages. To calculate the collision points of the particle with the billiard boundary, the relationship between p_n and p_{n+1} or (x_n, y_n) and (x_{n+1}, y_{n+1}) can be determined. Assuming the starting point of the movement is point (x_0, y_0) at Cartesian coordinates and (s_0, p_0) in the mapping coordinates, where $p_0 = \cos\psi_0$. Given the coordinates of (x_0, y_0) θ_0 can be defined as the polar angle relative to the positive x-axis, shown in figure 5. The motion equations for the particle are as follows,

$$x = x_0 + v_{0x}t \quad , \quad y = y_0 + v_{0y}t \tag{18}$$

where the velocity $v = \left(v_{0x}^2 + v_{0y}^2\right)^{1/2}$ is a constant value. By removing the time parameter from the equations (18), the path equation is obtained as follows:

$$y = y_0 + (x - x_0)\frac{v_{0y}}{v_{0x}} \tag{19}$$

quantity v_{0y}/v_{0x} represents the slope of the particle's path. Therefore, for two consecutive points, there are:

$$y = y_0 + (x - x_0)\tan\beta \quad , \quad \beta = \frac{\pi}{2} + \theta_0 + \psi_0$$

$$x = x_0 - (y - y_0)\tan(\theta_0 + \psi_0) \tag{20}$$

for two points in a row

$$x_{n+1} = x_n - (y_{n+1} - y_n)\tan(\theta_n + \psi_n) \tag{21}$$

for the collision point on one of the two arcs, the y_{n+1}-coordinate boundary is determined by calculating the collision point of the path with the equation of boundary, and for the collision point on the two ends:

$$y_{n+1} = R_1 \sin\theta_{n+1} \quad , \quad y_{n+1} = R_2 \sin\theta_{n+1} \tag{22}$$

using the above equations, the mapping equations are obtained and the coordinates and collision points are determined by repetitive computer calculations. The particle's trajectory and the collision cross-section are thus determined without solving differential equations describing the path.

4. Results

Figure 6(a) depicts the trajectory of a regular motion, whereas figure 6(b) displays the trajectory of a chaotic motion resulting from changes in the initial conditions. The cross-sectional surfaces of these two motions are also shown in figure 7. The shape of the cross-section of the regular motion shows that the four points of contact with the boundary are located on a line in the phase space. Figure 7(a) presents the cross-sectional surface of a chaotic motion, where the surface is covered with collision points and the empty spaces are surrounded by invariant curves. These spaces are also filled in case of $n \to \infty$ and they eventually disappear. In Figure 8, 9, 10 and 11, the cross-section of the motion are determined with different initial conditions, from which the following results can be obtained: 1) As shown in

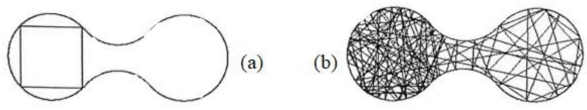

Figure 6 (a) regular motion $\psi = 45^\circ, n = 100$ **(b) chaotic motion** $\psi = 46^0, n = 100$

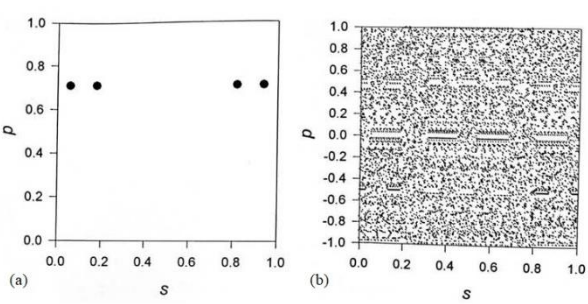

Figure 7 (a) cross section of 6(a), (b) cross section of 6(b).

figures 6 and 7, like the stadium, for some initial conditions, regular paths are obtained, and with a slight change in the initial conditions, these paths disappear, and irregular paths fill the entire billiard surface, covering the entire cross-sectional surface in phase space; 2) In contrast to the stadium case, where there are only two empty areas around two fixed points,

$$\left(s=\frac{3}{4},p=0\right),\ \left(s=\frac{1}{2},p=0\right)$$

which correspond to the non-singular motion perpendicular to two line segments, in the deformed stadium, the number and size of the empty areas in the cross-section depend on the parameters defining the shape of the boundary. For example, with the increase of the angle α, the angle of the perpendicular line to the point of contact of two parts of the boundary with the positive direction of the axis y, the number of empty areas increases and they are obtained around points,

$$\left(\frac{1}{2}+\frac{\pi R_2}{2L},0\right),\ \left(\frac{\pi R_1}{2L},0\right),\ \left(1-\frac{\pi R_1}{2L},0\right),...$$

which are surrounded by invariant curves.

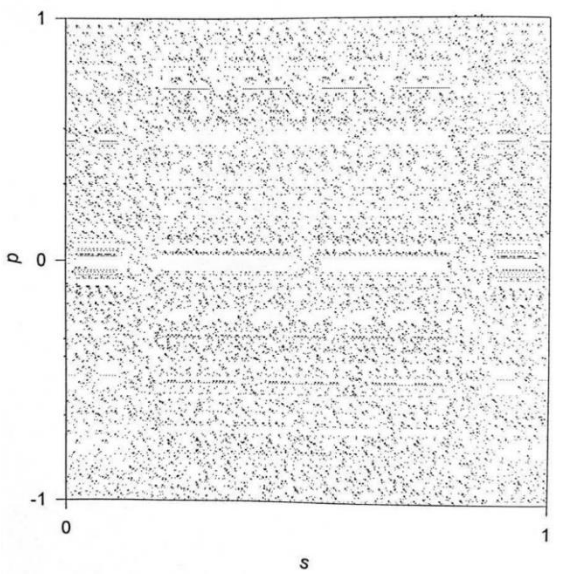

Figure 8 chaotic motion in the deformed stadium for initial conditions $s=0.1,\Psi=30^{\circ},\alpha=55^{\circ}$.

Conclusions

A deformed stadium that specifies two arcs of a circle representing the boundary instead of two line segments (dumbbell shape) is introduced to investigate the chaotic behavior of billiards. The results show:

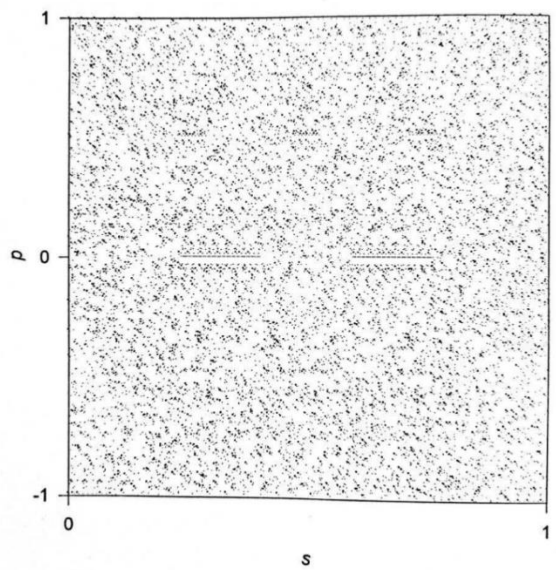

Figure 9 chaotic motion in the deformed stadium for initial conditions $s=0.1,\Psi=30^{\circ},\alpha=10^{\circ}$.

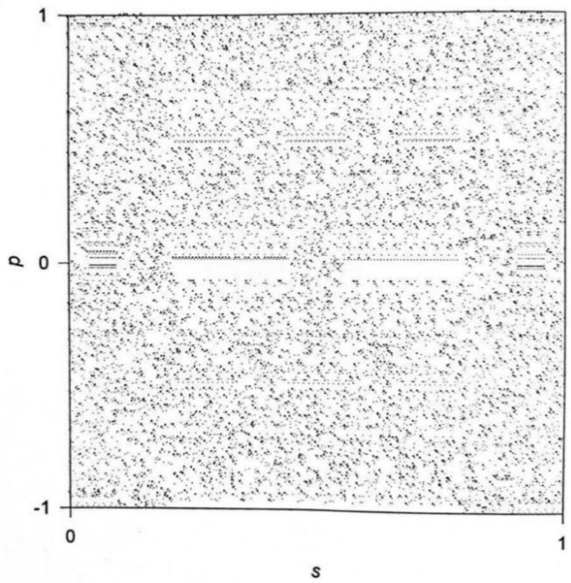

Figure 10 chaotic motion in the deformed stadium for initial conditions $s=0,\Psi=45^{\circ},\alpha=35^{\circ}$.

. The shape of the cross-section of the regular motion is located on a line in the phase space.

1) The cross-sectional surface of a chaotic motion is covered with collision points and the empty spaces are surrounded by invariant curves.

2) The empty spaces are also filled in case of $n\rightarrow\infty$ and they eventually disappear.

3) Like the stadium, for some initial

conditions, regular paths are obtained, and with a slight change in the initial conditions, these paths disappear, and irregular paths fill the entire billiard surface, covering the entire cross-sectional surface in phase space

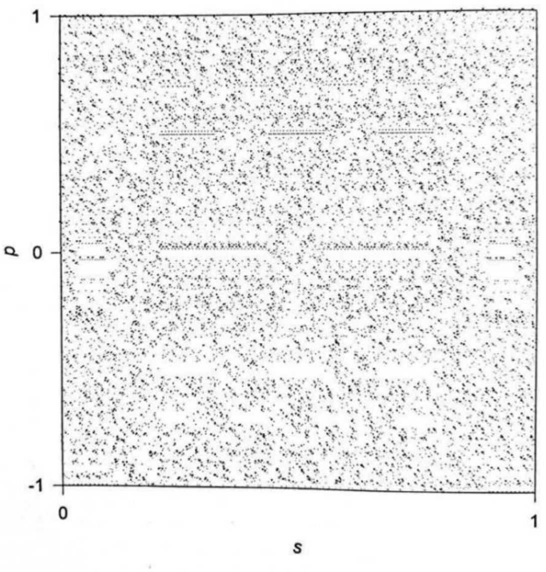

Figure 11 chaotic motion in the deformed stadium for initial conditions $s = 0.5, \Psi = 40^{\circ}, \alpha = 35^{\circ}$

4) In contrast to the stadium case, where there are only two empty areas around which correspond to the non-singular motion perpendicular to two line segments, in the deformed stadium, the number and size of the empty areas in the cross-section depend on the parameters defining the shape of the boundary.

The observation of chaotic motions in classic mechanics prompts the question of how this randomness manifests in quantum mechanics. To find the answer to this question, the wave equation for these systems shall be studied in the following.

References

[1] H. Poincaré, "New methods of celestial mechanics" *National Aeronautics and Space Administration*, 1967.

[2] J. Gleick, "Chaos: Making a new science" *Penguin* 2008.

[3] S. H. Strogatz, "Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering" *CRC press* 2018.

[4] E. N. Lorenz, "Deterministic nonperiodic flow" *Journal of atmospheric sciences*, Vol *20*(2), pp.130 1963.

[5] E. Ott, "Chaos in dynamical systems" *Cambridge university press* 2002.

[6] R. C. Hilborn, "Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers" *Oxford University Press*, 2000.

[7] J. R. C. Piqueira, C. E. N. Mazzilli, C. P. Pesce, and G. R. Franzini, "Lectures on Nonlinear Dynamics" *Springer International Publishing AG*, 2024.

[8] T. Huang, L. Dai, and H. Zhang, "An approach combining periodicity ratio and secondary Poincaré map for characteristics diagnosis of nonlinear oscillatory systems" *Nonlinear Dynamics*, vol 84, pp. 959, 2016.

[9] A. Shahhosseini, M. H. Tien, and K. D'Souza "Poincare maps: a modern systematic approach toward obtaining effective sections" *Nonlinear Dynamics*, vol 111(1), pp. 529 2023.

[10] A. Lichtenberg, A. M. Liberman, "Regular and stochastic Motion" *Springer-Verlag*, New York, 1983.

[11] M. Berry, "Semiclassical Mechanics of Regular and Irregular Motion" *North-Holland* 1983.

[12] H. Goldstein, C. Poole and J. Safko, "Classical Mechanics" *Addison-Wesley*, 2001.

[13] M, C. Gutzwiller, "Chaos in Classical and Quantum Mechanices" *Springer-Verlag*, 1990.

[14] B. L. Hao, "Directions in Chaos" *World Scientific*, Vol. 1, 1987.

[15] B. L. Hao, "Chaos II" *World Scientific*, 1990.

[16] H. G. Schuster and W. Just, "Deterministic Chaos: An Introduction 4th Revised and Enlarged Edition" *Wiley-VCH*, 2005.

[17] H. Haken, "At least one Lyapunov exponent vanishes if the trajectory of an attractor does not contain a fixed point." *Phys. Lett. A.* vol. 94(2), pp. 71-2, Feb 1983.

[18] Y. G. Sinai, "Dynamical systems with elastic reflections." *Russ. Math. Surv.* Vol. 25(2), pp. 137 Apr 1970.

[19] I. Kosztin, D. L. Maslov and P. M. Goldbart "Chaos in Andreev billiards." *Phys. Rev. Let.* vol. 75(9), pp. 1735 Aug 1995.

[20] L. A. Bunimovich, "On ergodic properties of certain billiards." *Functional Analysis and Its Applications.* Vol. 8(3), pp. 254-5, Jul 1974.

[21] L.A. Bunimovich, "On the ergodic properties of nowhere dispersing billiards" *Commun. Math. Phys.* vol. 65, pp. 295-312, Oct 1979.

[22] Z. Shahriari, S. D. Algar, D. M. Walker, and M. Small "Ordinal Poincaré sections: Reconstructing the first return map from an ordinal segmentation of time series" *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol 33(5) 2023.

[23] M. V. Berry, "Regularity and chaos in classical mechanics, illustrated by three deformations of a circular billiard" *Eur. J. Phys.* Vol. 2(2), pp. 91 Apr 1981.

[24]W. Schlei, K. C. Howell, X. Tricoche, and C. Garth,. "Enhanced visualization and autonomous extraction of poincaré map topology". *The Journal of the Astronautical Sciences*, Vol 61, pp.170, 2014.