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Ranking extreme and non-extreme efficient DMUs on the basis of MPSS in DEA

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Abstract

Finding units with the most productive scale size (MPSS) is very important. The use of MPSS in ranking is thus the main idea in this paper. We propose an algorithm in DEA that ranks all extreme and non-extreme efficient DMUs in a number of steps. In this method, units with the most productive scale size are identified in each step and are then ranked. We finally show the application of the method using a numerical example.

Keywords: Data envelopment analysis; Efficiency; Extreme efficient; ranking; productivity.

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1. Introduction

One of the aims of data envelopment analysis (DEA) is to evaluate the performance of decision-making units (DMUs) and determining the efficient in inefficient ones. Obviously, units are efficient if they obtain a score of 1, and are inefficient otherwise.

Charnes et al. [1] proposed a linear programming model for evaluating the DMUs, the radial CCR model. Later, Banker et al. [2] considered the variable return to scale (VRS) assumption to present the radial BCC model. In these radial models, the slack variables are not involved in the efficiency score values. Therefore, Charnes et al. [3] provided the non-radial additive model.

As the ranking of efficient units is of great importance, many papers have addressed this issue. The following are but a few of them.

Sexton et al. [4] presented the cross-efficiency method. They evaluated the efficiency of each DMU n times, using the weights obtained from the multiplier CCR model, and saved the data in a matrix. The main column efficiency would, then, be a criterion for ranking. This method, however, has some drawbacks. The main issue arises when the problem has multiple optimal solutions, in which ease selecting one of them for the calculations would not be easy.

Another important model use for ranking extreme efficient units was put forward by Anderson and petersen [5] (the AP model). In their method, the DMU under evaluation is removed from the set of observed DMUs and the DEA model is solved for the other DMUs. This method is unstable and infeasible in some cases and is unable to rank non-extreme DMUs.

To resolve the above-mentioned shortcomings, a lot of work has been done. For instance, Mehrabian et al. [6] improved the AP model by proposing the MAJ model. This method, too, might be infeasible in some cases. Saati et al. [7] modified the MAJ model and removed the infeasibility issue. Sinuany-stern et al. [8] presented the AHP\DEA model, which combines DEA and analytic Hierarchy process (AHP). Also, Jahanshahloo et al. [9] proposed a ranking system based on the effect of the DMUs on inefficient units. Some scholars have used certain norms; for example, Jahanshahloo et al. [10] used norm 1 for ranking efficient units and proved that the model is always feasible and stable. Most of the works in this area have been unable to rank non-extreme efficient units.

In this paper, we present a multi-step algorithm to rank all extreme and non-extreme efficient units on the basis of their highest productivity. Units with the most productive scale size (MPSS) in each step are ranked by the AP-Add model, which always feasible and stable.

This paper is organized as follows. Section 2 contains the necessary DEA background. The new ranking method is provided in section 3. A numerical example and an application with real data are given in section 4, and the conclusions constitute the last section.

2. DEA background

Consider n homogeneous observed $DMUs$, DMU_j($j = 1, \dots, n$)which produce the output vector Y_j ($j = 1, \dots, n$) using the input vector X_j ($j = 1, \dots, n$) , $X_j \in R^{m>0}$ ($j = 1, \dots, n$) and $Y_j \epsilon R^{S>0}$ ($j = 1, \dots, n$), meaning that X_j has m input elements and Y_j has s output elements. The production possibility set (PPS) T_c and T_v are defined as follows:

$$T_c = \left\{(x, y) \middle| x \geq \sum_{j=1}^{n} x_j\lambda_j, y \leq \sum_{j=1}^{n} y_j\lambda_j, \lambda_j \geq 0, \ j = 1, \dots, n\right\}$$

$$T_v = \left\{(x, y) \middle| x \geq \sum_{j=1}^{n} x_j\lambda_j, y \leq \sum_{j=1}^{n} y_j\lambda_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, \ j = 1, \dots, n\right\}$$

The input-oriented CCR model [1] that evaluates $DMUs$ over T_c is:

$$\theta_p^c = \text{Min} \quad \theta, \qquad (1)$$
$$\text{S.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{ip}, \quad i = 1, \dots, m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp}, \quad r = 1, \dots, s,$$
$$\lambda_j \geq 0, \quad j = 1, \dots, n.$$
$$s_i^- \geq 0, s_r^+ \geq 0.$$

The input-oriented BCC model [2] that evaluates $DMUs$ over T_v is:

$$\theta_p^v = \text{Min} \quad \theta \qquad (2)$$
$$\text{S.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_p, i = 1, \dots, m,$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_p, \; r = 1, \dots, s$$
$$\sum_{j=1}^{n} \lambda_j = 1,$$
$$\lambda_j \geq 0, \quad j = 1, \dots, n.$$
$$s_i^- \geq 0, s_r^+ \geq 0.$$

The additive model [3] over T_v for evaluating $DMUs$ is:

$$\theta_p^{Add} = \text{Max} \quad \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+$$
$$\text{S.t.} \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{ip}, i = 1, \dots, m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp}, \quad r = 1, \dots, s$$
$$\sum_{j=1}^{n} \lambda_j = 1, \qquad (3)$$
$$\lambda_j \geq 0, \qquad j = 1, \dots, n.$$
$$s_i^- \geq 0, s_r^+ \geq 0.$$

DMU_p is CCR-efficient and BCC-efficient if and only if $S_r^{+*} = 0 (r = 1, \dots, s)$, $S_i^{-*} = 0 (i = 1, \dots, m)$ at each optimal solution and $\theta^* = 1$ by models (1) and (2). It can be easily shown that DMU_p is BCC –efficient if and only if it is Add-efficient (for more details, see Cooper et al. [11]).

Definition: $DMU_P \epsilon T_v$ has $MPSS$ if and only if for each $\alpha > 0$ and $\beta > 0$ such that $(\alpha X_p, \beta Y_p) \epsilon T_v$ we have:

$$\frac{\beta}{\alpha} \leq 1.$$

Theorem: $DMU_P \epsilon T_v$ has $MPSS$ if and only if it is CCR-efficient (more details in [12]).

2.1. AP-Add model for ranking extreme efficient units

To evaluate DMU_p, Anderson and Petersen [5] removed the unit from the set of observation and solved the DEA model for the remaining $DMUs$. The optimal value obtained by the model is a criterion for ranking. Using the Additive model, we will have the following model.

$$\theta_p^{AP} = \text{Min} \quad \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+$$
$$\text{S.t.} \quad \sum_{j \neq p} \lambda_j x_{ij} - s_i^- \leq x_p, \; i = 1, \dots, m,$$
$$\sum_{j \neq p} \lambda_j y_{rj} + s_r^+ \geq y_p, r = 1, \dots, s$$
$$\sum_{j \neq p} \lambda_j = 1 \qquad (4)$$
$$\lambda_j \geq 0, \qquad j = 1, \dots, n.$$
$$s_i^- \geq 0, s_r^+ \geq 0.$$

The above model is always feasible. if $\theta_p^{AP} = 0$ ، DMU_P is inefficient or non-extreme efficient and if $\theta_P^{AP} > 0$ ، DMU_p is extreme efficient (more details in [13]). Some studies have been done in the field of find the extreme DMUs in DEA [14-18].

3. A method for ranking efficient units

In this section, we provide an algorithm for ranking all efficient units, either extreme or non-extreme, on the basis of their $MPSS$.

The general procedure in the algorithm is to find all units with $MPSS$, first, and rank them by model (4). First, extreme efficient units are ranked and then removed. Next, non-extreme efficient $DMUs$ are ranked.) In the next step, all these $DMUs$ are removed from the set of observations and the same procedure is repeated for the rest of $DMUs$ until all $DMUs$ are ranked. Note that the inefficient $DMUs$ are not ranked, but as they play a role in the ranking of the efficient units, they are considered in the set of observation throughout all the steps in solving model (4). Units that are ranked in one step have better ranks compared to those that are ranked in the next step, because they have higher productivity.

Suppose $M_0 = \{DMU_j | j = 1, \dots n\}$ and E_v is the set of all efficient units in evaluation by the BCC model (2), that is, $E_v = \{DMU_j | \theta_j^v = 1, S^{-*} = 0, S^{+*} = 0\}$, where θ_j^v is the optimal solution of model (2) in evaluating DMU_j. First, we set $i = 0, r = 0$, and $F_0 = M_0$.

Step 1: Solve the CCR model (1) for DMU_j ($j \epsilon M_i$) over the set of observations M_i set: $E_i := E_v \cap E_c^i$ and $E_c^i = \{DMU_j | \theta_j^c = 1, S^{-*} = 0, S^{+*} = 0, \ j \epsilon M_i\}$.

If $E_i = \emptyset$, go the step 5. otherwise, go to step 2.

Note1: E_i is the set of all units that have MPSS in the i th step, thus having better ranks than the units in the set E_{i+1}.

Step 2: If $|E_i| = 1$ ($|.|$ denotes the cardinal), then DMU_j ($j \epsilon E_i$) has the best rank in the i th step, go to step4. Otherwise (i.e., $|E_i| > 1$), go to step 3.

Step 3: this step has two parts and ranks the DMUs in E_i.

3-a) solve model (4) for DMU_j ($j \epsilon E_i$) over the set of observations F_r and create the following two sets:

$$NE_i^x = \{DMU_j | \theta_j^{AP} = 0\}$$

$$E_i^x = \{DMU_j | \theta_j^{AP} > 0\}$$

Note 2: NE_i^x is the set of non-extreme efficient DMUs and E_i^x is the set of extreme efficient DMUs.

If $|NE_i^x| \leq 1$, all the DMUs in have been ranked (θ_j^{AP} is a criterion for ranking), go to step 4. Otherwise, (i.e., if $|NE_i^x| > 1$), go to 3-b.

3-b) set $F_{r+1} = F_r \backslash E_i^x$ and $r := r + 1$ and go to 3-a.

Step 4: if $E_v \backslash (\cup_{j=0}^i E_j) = \emptyset$, all the efficient DMUs have been ranked and the algorithm terminates. Otherwise, set $M_{i+1} = M_i \backslash E_i$, $F_{r+1} = F_r \backslash E_i$ and $i := i + 1, r := r + 1$, and go to step 1.

Step 5: since $E_i = \emptyset$ and $E_c^i \neq \emptyset$, then $\forall \ DMU_j \epsilon E_c^i \quad \theta_j^v < 1$

Set $M_{i+1} = M_i \backslash E_c^l$, $F_{r+1} = M_{i+1} \cup A$ where $A = \{DMU_j | DMU_j \epsilon E_c^l \text{ s.t } E_l = \emptyset, l = 0, \dots, i\}$ and $i := i + 1, r := r + 1$ and go to step 1.

With regard to the following properties, the algorithm is valid.

1. The number of DMUs is finite ($|M_0| < \infty$).
2. The programing problems are linear and always feasible. As the AP-Add model has been employed, the method is stable.
3. In each step, the number of DMUs to be ranked decreases.

4. Numerical example

In this section, a numerical example and an application with real data are provided to demonstrate the utility of the algorithm for ranking all efficient DMUs.

consider six DMUs with one input and one output each, whose data are given in Table 1 and Fig 1.

Table 1: Data in numerical example 1

DMU	A	B	C	D	E	F
I	1	2	6	8	3	7
O	1	4	6	7	4	2

Table 2: The results of ranking

Efficient units in Example 1	DMU_A	DMU_B	DMU_C	DMU_D
Ranking	2	1	3	4

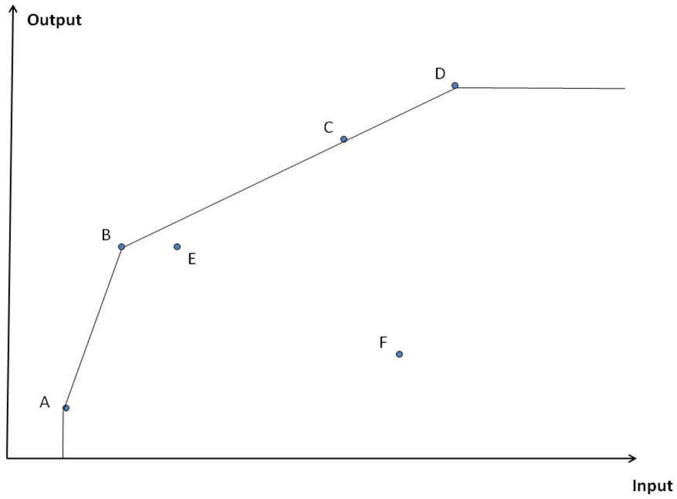

Figure 1: Data set in BCC model

Here, $F_0 = M_0 = \{DMU_A, \dots, DMU_F\}$. By solving Model (2), $E_v = \{A, B, C, D\}$. By Model (1) we have: $E_c^i = \{DMU_B\}$, $E_0 = \{DMU_B\}$ and $|E_0| = 1$ .so, DMU_B has the highest rank in the first step. Note that the DMUs that are ranked in one step have better ranks than those ranked in the next steps.
$M_1 = M_0 \backslash E_0 = \{DMU_A, DMU_C, DMU_D, DMU_E, DMU_F\}$ and $F_1 = F_0 \backslash E_0 = \{DMU_A, DMU_C, DMU_D, DMU_E, DMU_F\}$, $i = 1, r = 1$, and we go to step2. $E_c^1 = \{DMU_E\}$ and $E_1 = \emptyset$, thus we go to step 5: $M_2 = M_1 \backslash E_c^i = \{DMU_A, DMU_C, DMU_D, DMU_F\}$ and $A = \{DMU_E\}$. Therefore $F_2 = \{DMU_A, DMU_C, DMU_D, DMU_E, DMU_F\}$ and $r = 2, i = 2$. We go to step 1: $E_c^2 = \{DMU_A, DMU_C\}$, $E_2 = \{DMU_A, DMU_C\}$. since $|E_2| > 1$, we go to step 3: By solving Model (4), $\theta_A^{AP} = 2$ and $\theta_C^{AP} = 0.2$ and $NE_2^x = \{\emptyset\}$ and $E_2^x = \{DMU_A, DMU_C\}$, in the second step DMU_A is ranked higher than DMU_C . Since $|NE_2^x| = 0$, we go to step 4. $M_3 = M_2 \backslash E_2 = \{DMU_D, DMU_F\}$ and $F_3 = F_2 \backslash E_2 = \{DMU_D, DMU_E, DMU_F\}$, and $i = 3, r = 3$. We go to step 1: $E_c^3 = \{DMU_D\}$ and $E_3 = \{DMU_D\}$, thus DMU_D has the highest rank in the third step. Because $E_v \backslash (\cup_{j=0}^{3} E_j) = \emptyset$, the algorithm terminates. So, the ranking of the efficient DMUs in example 1 is as follows.

5. Case study

In this section, we consider the data for a privately owned hospital taken in Iran. Since human health is a strategic priority for all societies, investment in this sector will be very important. The purpose of this research is to evaluate the efficiency of hospitals and their ranking, and to provide a vision for dynamic managers in this field. Since organizations must have a clear vision of continued profitability in their activities in order to be accepted in the capital market; The researchers tried to measure the efficiency of the hospitals, so that they could choose the hospitals with the conditions to be admitted to the market by separating the efficient and inefficient hospitals. DEA technique, input-oriented CCR and BCC models were used to measure efficiency. The data includes the input and output of government hospital operations, so that the inputs include the number of active beds, the number of personnel, and the outputs include the number of inpatient admissions, the number of outpatient admissions. According to the model of DEA for the efficiency of inefficient units, it is possible to reach the efficiency limit by changing the inputs, but it seems that in order to make sustainable changes, changes should be made in the policies and macro-strategies of the health sector, which can be used to self-governance of hospitals, He pointed out the integration of efficient and non-efficient hospitals or the formation of a holding company from them and planning for the entry of hospitals into the capital market. Each hospital has two entrances and two exits. The data is given in Table 3. The efficiency scores of hospitals in constant and variable returns to scale are given in the last two columns of Table 3.

Table 3: Hospital data

DMU	I_1	I_2	O_1	O_2	CCR efficiency	BCC efficiency
1	150	0.2	14000	3500	1	1
2	400	0.7	14000	21000	1	1
3	320	1.2	42000	10500	1	1
4	520	2	28000	42000	1	1
5	350	1.2	19000	25000	0.98	1
6	320	0.7	14000	15000	0.87	0.9

Table 4: The results of ranking in the case study

Efficient units in case study	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
Ranking	3	4	2	1	5

The ranking of hospitals is given in Table 4. Hospital 4 has the best performance among hospitals. By using the proposed algorithm, we can provide a suitable ranking for hospitals. In relation to the hospitals that were placed at the top of ranking, that is, very good results and outputs are obtained for the data that is spent in the hospital, it is necessary to maintain their position or plan to increase the level of activities based on the capacity of the hospital.

6. Conclusion

Several methods have been presented so far for ranking efficient MUs . Most of these methods, however, suffer from shortcoming such as infeasibility, instability, and inability to rank extreme efficient DMUs (as is the case with the AP model) two of the factors that do not receive attention in ranking are the returns to scale and productivity of each DMU. Since the DMUs with the most productive scale size (MPSS) are more important than other DMUs, they are ranked higher by the ranking method proposed in this paper. Our method identifies the DMUs with MPSS in each step and ranks them using the AP-Add model, which is always feasible and stable. Thus, the model resolves all the above-mentioned shortcomings.

References

[1] Charnes A, Cooper WW, Rhodes E. Measuring the efficiency of decision-making units. European journal of operational research. 1978 Nov 1; 2(6): 429-44.

[2] Banker RD, Charnes A, Cooper WW. Some models for estimating technical and scale inefficiencies in data envelopment analysis. Management science. 1984 Sep; 30(9): 1078-92.

[3] Charnes A, Cooper WW, Golany B, Seiford L, Stutz J. Foundations of data envelopment analysis for Pareto–Koopmans efficient empirical production functions', Journal of Econometrics. 1985 30: 91–107.

[4] Sexton TR, Silkman RH, Hogan AJ. Data envelopment analysis: critique and extensions. In: Silkman, R.H. (ed.) Measuring efficiency: an assessment of data envelopment analysis. 1986 73-105.

[5] Andersen P, Petersen NC. A procedure for ranking efficient units in data envelopment analysis. Management science. 1993 Oct; 39(10): 1261-4.

[6] Mehrabian S, Alirezaee MR, Jahanshahloo, GR, A complete efficiency ranking of decision-making units in data envelopment analysis. Computational Optimization and Applications, 1999 14: 261-266.

[7] Saati MS, Zarafat Angiz M, Jahanshahloo GR. A model for ranking decision-making units in data envelopment analysis. Ricerca Operativa. 2001 31(97): 4759.

[8] Sinuany-stern Z, Mehrez A, Hadad Y. An AHP/DEA methodology for ranking decision-making units, International Transactions in Operation Research. 20007 109-124.

[9] Jahanshahloo GR, junior HV, Lotfi FH, Akbarian D. A new DEA ranking system based on changing refrence set, European Journal of Operational Research. 2007 181: 331-337.

[10] Jahanshahloo GR, Hosseinzadeh Lotfi F, Shoja N, Tohidi G, Razavian S. Ranking using norm in data envelopment analysis. Applied mathematics and computational. 2004 153 (1): 215-224.

[11] Cooper WW, Li S, Seiford LM, Tone T. Data Envelopment Analysis: A Comprehensive Text with Models, Applications, Reverences and DEA solver Software, Kluwer Academic Publisher, Norwell, Mass. 1999.

[12] Rajiv Banker D. Estimating most productive scale size using data envelopment analysis. European Journal of Operational Research. 1984 17: 35-44.

[13] Juan Du, Liang Liang, Joe Zhu, A slacks-based measure of super-efficiency in data envelopment analysis: A comment. European Journal of Operational Research, 2010 204: 694–697.

[14] Gerami J, Mozaffari MR, Wanke PF. A multi-criteria ratio-based approach for two-stage data envelopment analysis. Expert Systems with Applications. 2020 158: 113508.

[15] Gerami J, Kiani Mavi R, Farzipoor Saen R, Kiani Mavi N. A novel network DEA-R model for evaluating hospital services supply chain performance. Annals of Operations Research. 2020, 324, 1–2: 1041–1066.

[16] Gerami J. An interactive procedure to improve estimate of value efficiency in DEA. Expert Systems with

Applications. 15 December 2019, 137 29-45.

[17] Gerami J, Mozaffari MR, Wanke PF, Correa H. A novel slacks-based model for efficiency and super-efficiency in DEA-R. Operations Research. 2021 22, 4: 3373–3410.

[18] Gerami J. Strategic alliances and partnerships based on the semi-additive production technology in DEA. Expert Systems with Applications1 October 2024 251: 123986.