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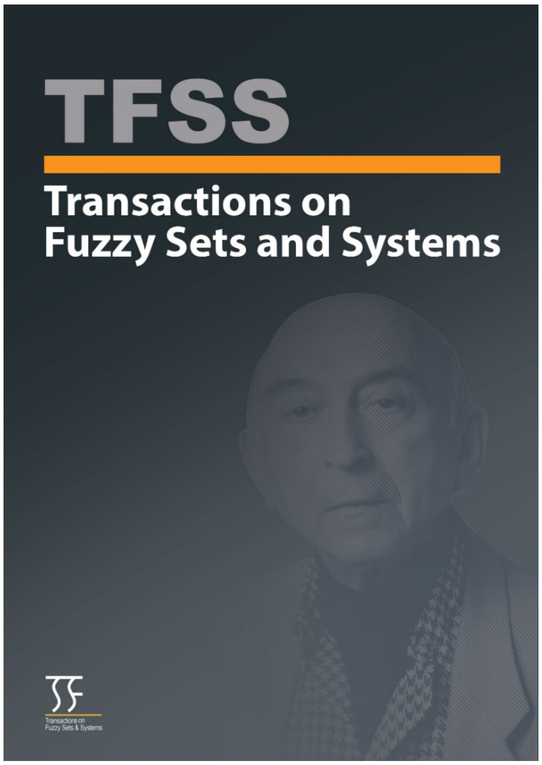

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Some Result on Fuzzy Integration

Reza Chaharpashlou*, **Ehsan Lotfali Ghasab**

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Abstract. In this article, we introduce a new concept of fuzzy measurement, the space of fuzzy measurable functions and fuzzy integral, which has a dynamic position and is different from previous approaches. With this concept, we create a new version of measurement theory and fuzzy integral. The main goal of this paper is to define the fuzzy integral in the fuzzy size space. First, we introduce fuzzy measurable functions and L^+ essential and related concepts in fuzzy space. In the continuation of the work, with the help of fuzzy measurable functions, we define the fuzzy integral in the fuzzy measurement space and examine the theorems related to it and the relationship between them in the fuzzy measurement space. The next step is to establish one of the fundamental convergence theorems with the uniform convergence theorem in the fuzzy measurement space and prove it. Finally, we prove Fatou's lemma as an application of the theorems raised in the fuzzy measurement space.

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Keywords and Phrases: Fuzzy measure space, Fuzzy measure, Fuzzy measurable function, Fuzzy integral.

1 Introduction

Here, we let $\Xi = [0,1]$, $\Upsilon = [0,\infty]$ and $\mho = (0,\infty)$.

Let us assume $\star : \Xi \times \Xi \to \Xi$ is a topological monoid with unit 1 and $\varrho \star \gamma \leq \varsigma \star \delta$ whenever $\varrho \leq \gamma$ and $\varsigma \leq \delta \quad (\varrho, \varsigma, \gamma, \delta \in \Xi)$. In this case, $\star$ is called a continuous t-norm and $\star_{i=1}^{\infty} = \lim_{n\to\infty} \star_{i=1}^{n}$. For some examples $\varrho \star \varsigma = \varrho.\varsigma$ and $\varrho \star \varsigma = \wedge(\varrho, \varsigma)$ are continuous t-norms.

Let us consider $U \neq \emptyset$, $\star$ is a continuous t-norm and ρ is a fuzzy set on $U^2 \times \mho$. Then $(U, \rho, \star)$ is said to be a fuzzy metric space where for arbitrary $\varepsilon, \upsilon, \eta \in U$ and $\tau, \theta > 0$,

(FM1) $\rho(\varepsilon, \upsilon, \tau) = 1$ for every $\tau \in \mho$ iff $\varepsilon = \upsilon$;

(FM2) $\rho(\varepsilon, \upsilon, \tau) = \rho(\upsilon, \varepsilon, \tau), \forall \varepsilon, \upsilon \in U, \quad \forall \tau \in \mho$;

(FM3) $\rho(\varepsilon, \eta, \tau + \theta) \geq \rho(\varepsilon, \upsilon, \tau) \star \rho(\upsilon, \eta, \theta), \quad \forall \varepsilon, \upsilon, \eta \in U, \quad \forall \tau, \theta \in \mho$;

(FM4) $\rho(\varepsilon, \upsilon, \cdot) : \mho \to (0,1]$ is continuous.([1, 2, 3, 4, 5, 6, 7, 8])

Definition 1.1. Suppose that $\mathfrak{X} \neq \emptyset$ and $\mathcal{C} \subseteq 2^{\mathfrak{X}}$ such that

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(i) $\emptyset \in \mathcal{C}$ and $\mathfrak{X} \in \mathcal{C}$;

(ii) if $\mathfrak{A} \in \mathcal{C}$, then $\mathfrak{A}^c \in \mathcal{C}$;

(iii) if $\mathfrak{A}_1, \cdots, \mathfrak{A}_n \in \mathcal{C}$, then $\cup_{i=1}^{n} \mathfrak{A}_i$ and $\cap_{i=1}^{n} \mathfrak{A}_i$ are in $\mathcal{C}$.

(iv) whenever $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ are in $\mathcal{C}$, then $\cup_{i=1}^{\infty} \mathfrak{A}_i$ and $\cap_{i=1}^{\infty} \mathfrak{A}_i$ are in $\mathcal{C}$.

So $\mathcal{C}$ is called a σ-algebra and $(\mathfrak{X}, \mathcal{C})$ is called a measurable space.

Definition 1.2. Let us assume that $\mathfrak{X} \neq \emptyset$ and $\mathcal{C} \subseteq 2^{\mathfrak{X}}$ a σ-algebra. A fuzzy function $\nu : \mathcal{C} \times \mho \to \Xi$ such that

(i) $\nu(\emptyset, \tau) = 1, \quad \forall \tau \in \mho$;

(ii) if $\mathfrak{A}_i \in \mathcal{C}$, for $i = 1, 2, \cdots$ are pairwise disjoint,

$$\nu(\cup_{i=1}^{\infty} \mathfrak{A}_i^{\infty}, \tau) = \star_{i=1}^{\infty} \nu(\mathfrak{A}_i, \tau), \quad \forall \tau \in \mho.$$

So ν is said to be a fuzzy measure and $(\mathfrak{X}, \mathcal{C}, \nu, \star)$ is said to be a fuzzy measure space.

Definition 1.3. Consider $(\mathfrak{X}, \mathcal{M})$ and $(\mathfrak{Y}, \mathcal{N})$ measurable spaces, a mapping $\jmath : \mathfrak{X} \to \mathfrak{Y}$ is called $(\mathcal{M}, \mathcal{N})$-measurable if $\jmath^{-1}(\mathfrak{E}) \in \mathcal{M}$ for all $\mathfrak{E} \in \mathcal{N}$. We know $\mathcal{B}_{\mathbb{R}}$ as σ-algebra on $\mathbb{R}$.

2 Main Results

Theorem 2.1. *Consider $\mathfrak{X}$ is a metric (or topological) space, then every cotinuous $\jmath : \mathfrak{X} \to \mathbb{R}$ is $(\mathcal{B}_{\mathfrak{X}}, \mathcal{B}_{\mathbb{R}})$-measurable.*

Proof. $\jmath$ is continuouse iff $\jmath^{-1}(U)$ in $\mathfrak{X}$ for every $U \subseteq \mathbb{R}$. $\square$

Theorem 2.2. *Consider $(\mathfrak{X}, \mathcal{M})$, $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ measurable spaces and $\jmath : \mathfrak{X} \to \mathbb{R}$, then the following statements are equivalence.*

(i) $\jmath$ *is* $\mathcal{M}$*-measurable;*

(ii) $\jmath^{-1}((q, \infty]) \in \mathcal{M}, \quad \forall q \in \mathbb{R}$;

(iii) $\jmath^{-1}([q, \infty]) \in \mathcal{M}, \quad \forall q \in \mathbb{R}$.

Lemma 2.3. *Suppose $\jmath, \imath : \mathfrak{X} \to \mathbb{R}$ are $\mathcal{M}$-measurable so $\mathfrak{F} : \mathfrak{X} \to \mathbb{R} \times \mathbb{R}$ with $\mathfrak{F}(p) = (\jmath(d), \imath(p))$ is $\mathcal{M}$-measurable.*

Proof. We know $\mathcal{B}_{\mathbb{R}\times\mathbb{R}} = \mathcal{B}_{\mathbb{R}} \bigotimes \mathcal{B}_{\mathbb{R}}$. So $\mathfrak{F}$ is a $(\mathcal{M}, \mathcal{B}_{\mathbb{R}\times\mathbb{R}})$-measurable. $\square$

Theorem 2.4. *If $\jmath, \imath : \mathfrak{X} \to \mathbb{R}$ are $\mathcal{M}$-measurable, then $\jmath + \imath : \mathfrak{X} \to \mathbb{R}$ with $(\jmath + \imath)(p) = \jmath(p) + \imath(p)$ is a $\mathcal{M}$-measurable.*

Proof. Define $\mathfrak{F} : \mathfrak{X} \to \mathbb{R} \times \mathbb{R}$ with $\mathfrak{F}(p) = (\jmath(p), \imath(p))$, $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $\phi(z, w) = z + w$. Since $\mathcal{B}_{\mathbb{R}\times\mathbb{R}} = \mathcal{B}_{\mathbb{R}} \bigotimes \mathcal{B}_{\mathbb{R}}$, $\mathfrak{F}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}\times\mathbb{R}})$-measurable, wherease ϕ is $(\mathcal{B}_{\mathbb{R}\times\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$-measurable by theorem 2.1. Thus $\jmath + \imath = \phi \circ \mathfrak{F}$ is $\mathcal{M}$-measurable. $\square$

Theorem 2.5. *If $\jmath_j$ is a sequence of $\mathbb{R}$-valued measurable functions on $(\mathfrak{X}, \mathcal{M})$, then the functions $\imath_n(p) = \sup_{j\geq n} \jmath_j(p)$, $\imath(p) = \limsup \jmath_j(p)$, $\mathfrak{h}_n(p) = \inf_{j\geq n} \jmath_j(p)$, $\mathfrak{h}(p) = \liminf \jmath_j(p)$ are $\mathcal{M}$-measurable. If $\jmath(p) = \lim_{j\to\infty} \jmath_j(p)$ exists for every $p \in \mathfrak{X}$, then $\jmath$ is $\mathcal{M}$-measurable.*

Proof. We have $\imath_n^{-1}((q,\infty]) = \cup_{j=n}^{\infty} \jmath_j^{-1}((q,\infty]) \in \mathcal{M}$, $\mathfrak{h}_n^{-1}((q,\infty]) = \cap_{j=n}^{\infty} \jmath_j^{-1}((q,\infty]) \in \mathcal{M}$, more generally, if $\mathfrak{h}_k(p) = \sup_{j\geq n} \jmath_j(p)$ then $\mathfrak{h}_k$ is measurable for each k, so $\imath(p) = \inf_{k\geq 1} \mathfrak{h}_k(p)$ is measurable, and likewise for $\mathfrak{h}$. Finally, $\jmath$ exists then $\jmath = \imath = \mathfrak{h}$, so $\jmath$ is measurable. □

Corollary 2.6. *If $\jmath, \imath : \mathfrak{X} \to \mathbb{R}$ are fuzzy measurable, then so* $\max\{\jmath, \imath\}$ *and* $\min\{\jmath, \imath\}$.

Proof. We now discuss the functions that are building blocks for the theory of integration. Let $(\mathfrak{X}, \mathcal{M})$ be a measurable space. If $\mathfrak{E} \subseteq \mathfrak{X}$, the characteristic function $\chi_{\mathfrak{E}}$ of $\mathfrak{E}$ is given as

$$\chi_{\mathfrak{E}}(p) = \begin{cases} 1 & p \in \mathfrak{E}, \\ 0 & p \notin \mathfrak{E}. \end{cases}$$

$\chi_{\mathfrak{E}}$ is measurable iff $\mathfrak{E} \in \mathcal{M}$. A simple function on $\mathfrak{X}$ is a finite linear combination, with coefficients in $\mathbb{R}$, of characteristic functions of sets in $\mathcal{M}$. So $\jmath : \mathfrak{X} \to \mathbb{R}$ is simple iff $\jmath$ is measurable and the range of $\jmath$ is a finite subset of $\mathbb{R}$. Indeed, we have $\jmath = \sum_{j=1}^{n} z_j \chi_{\mathfrak{E}_j}$, where $\mathfrak{E}_j = \jmath^{-1}(\{z_j\})$ and $range(\jmath) = \{z_1, z_2, \cdots, z_n\}$, we call this the standard representation of $\jmath$. □

Theorem 2.7. *If $\jmath, \imath : \mathfrak{X} \to \mathbb{R}$ are simple functions, so then $\jmath + \imath$.*

Theorem 2.8. *Consider $(\mathfrak{X}, \mathcal{M})$ measurable space. If $\jmath : \mathfrak{X} \to \mathbb{R}$ is fuzzy measurable, there exist a sequence $\{\phi_n\}$ of fuzzy simple functions such that $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \jmath$ pointwise, and $\phi_n \to \jmath$ uniformly on $\mathfrak{X}$.*

Definition 2.9. Consider $(\mathfrak{X}, \mathcal{C}, \nu)$ measure space, define

$$L^+ = \{\jmath : \mathfrak{X} \times \mho \to [0,\infty) \mid \jmath \text{ is measurable function and increase with second component}\}.$$

Consider $\phi \in L^+$ a simple function by $\phi = \sum_{i=1}^{n} q_i \chi_{\mathfrak{E}_i}$, for every $q_i \geq 0$, we define the fuzzy integral of ϕ with respect to ν by

$$\int_{\mathfrak{X}} \phi(p) d\nu(p,\tau) = \star_{i=1}^{n} \nu(\mathfrak{E}_i, \frac{\tau}{q_i}),$$

for every $\tau \in \mho$. If $\mathfrak{A} \in \mathcal{C}$, then $\phi\chi_{\mathfrak{A}}$ is also simple function and define $\int_{\mathfrak{A}} \phi(p) d\nu(p,\tau)$ to be $\int_{\mathfrak{X}} (\phi\chi_{\mathfrak{A}})(p) d\nu(p,\tau)$.

Theorem 2.10. *Consider ϕ and ψ simple functions in L^+.*

(i) *If $c \in \Xi$ then $\int_{\mathfrak{X}} (c\phi)(p) d\nu(p,\tau) \geq c \int_{\mathfrak{X}} \phi(p) d\nu(p,\tau)$, if $c \in (1,\infty)$ then $\int_{\mathfrak{X}} (c\phi)(p) d\nu(p,\tau) \leq c \int_{\mathfrak{X}} \phi(p) d\nu(p,\tau)$, $\forall \tau \in \mho$;*

(ii) $\int_{\mathfrak{X}} (\phi+\psi)(p) d\nu(p,\tau) \geq \left(\int_{\mathfrak{X}} \phi(p) d\nu(p,\tau) \star \int_{\mathfrak{X}} \psi(p) d\nu(p,\tau)\right)$, $\quad \forall \tau \in \mho$*;*

(iii) *If $\phi \leq \psi$, then $\int_{\mathfrak{X}} \phi(p) d\nu(p,\tau) \geq \int_{\mathfrak{X}} \psi(p) d\nu(p,\tau)$, $\quad \forall \tau \in \mho$;*

(iv) *The map $\mathfrak{A} \to \int_{\mathfrak{A}} \phi(p) d\nu(p,\tau)$ is a measure on $\mathcal{C}$, $\quad \forall \tau \in \mho$.*

Proof.

(i) : If $c \in \Xi$ then

$$\begin{aligned} & \int_{\mathfrak{X}} (c\phi)(p) d\nu(p,\tau) \\ = & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} c a_i \chi_{\mathfrak{E}_i}(p) \right) d\nu(p,\tau) \\ = & \star_{i=1}^{n} \nu \left(\mathfrak{E}_i, \frac{\tau}{c a_i} \right) \\ \geq & \star_{i=1}^{n} \nu \left(\mathfrak{E}_i, \frac{\tau}{q_i} \right) \\ = & \int_{\mathfrak{X}} (c\phi)(p) d\nu(p,\tau) \\ \geq & c \int_{\mathfrak{X}} \phi(p) d\nu(p,\tau), \end{aligned}$$

if If $c \in (1, \infty)$ then

$$\begin{aligned} & \int_{\mathfrak{X}} (c\phi)(p) d\nu(p,\tau) \\ = & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} c a_i \chi_{\mathfrak{E}_i}(p) \right) d\nu(p,\tau) \\ = & \star_{i=1}^{n} \nu \left(\mathfrak{E}_i, \frac{\tau}{c a_i} \right) \\ \leq & \star_{i=1}^{n} \nu \left(\mathfrak{E}_i, \frac{\tau}{q_i} \right) \\ = & \int_{\mathfrak{X}} (c\phi)(p) d\nu(p,\tau) \\ \leq & c \int_{\mathfrak{X}} \phi(p) d\nu(p,\tau). \end{aligned}$$

(ii) :

$$\begin{aligned} & \int_{\mathfrak{X}} (\phi + \psi)(p) d\nu(p,\tau) \qquad (1) \\ = & \int_{\mathfrak{X}} \left(\left(\sum_{i=1}^{n} q_i \chi_{\mathfrak{E}_i}(p) \right) + \left(\sum_{j=1}^{m} b_j \chi_{\mathfrak{F}_j}(p) \right) \right) d\nu(p,\tau) \\ = & \int_{\mathfrak{X}} \left(\sum_{i,j} (q_i + b_j) \chi_{\mathfrak{E}_i \cap \mathfrak{F}_j}(p) \right) d\nu(p,\tau) \\ = & \star_{i=1}^{n} \star_{j=1}^{m} \nu \left((\mathfrak{E}_i \cap \mathfrak{F}_j), \frac{\tau}{(q_i + b_j)} \right) \end{aligned}$$

Since

$$
\begin{aligned}
& \left(\int_{\mathfrak{X}} \phi(p) d\nu(p,\tau) \star \int_{\mathfrak{X}} \psi(p) d\nu(p,\tau)\right) \\
= & \left(\int_{\mathfrak{X}} \left(\sum_{i=1}^{n} q_i \chi_{\mathfrak{E}_i}(p)\right) d\nu(p,\tau)\right) \star \left(\int_{\mathfrak{X}} \left(\sum_{j=1}^{m} b_j \chi_{\mathfrak{F}_j}(p)\right) d\nu(p,\tau)\right) \\
= & \left(\star_{i=1}^{n} \star_{j=1}^{m} \nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{q_i}\right)\right) \star \left(\star_{j=1}^{m} \star_{i=1}^{n} \nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{b_j}\right)\right) \\
= & \star_{i=1}^{n} \star_{j=1}^{m} \left(\nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{q_i}\right) \star \nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{b_j}\right)\right) \\
\leq & \star_{i=1}^{n} \star_{j=1}^{m} \left(\nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{(q_i+b_j)}\right) \star \nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{(q_i+b_j)}\right)\right) \\
\leq & \star_{i=1}^{n} \star_{j=1}^{m} \left(\nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{(q_i+b_j)}\right)\right)
\end{aligned}
\tag{2}
$$

(iii) :If $\phi \leq \psi$, then $q_i \leq b_j$ whenever $\mathfrak{E}_i \cap \mathfrak{E}_j = \emptyset$, so

$$
\begin{aligned}
& \int_{\mathfrak{X}} \phi(p) d\nu(p,\tau) \\
= & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} q_i \chi_{\mathfrak{E}_i}(p)\right) d\nu(p,\tau) \\
= & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} q_i \chi_{\mathfrak{E}_i \cup (\cap_{j=1}^{\infty} \mathfrak{E}_j)}(p)\right) d\nu(p,\tau) \\
= & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} \sum_{j=1}^{m} q_i \chi_{\mathfrak{E}_i \cap \mathfrak{E}_j}(p)\right) d\nu(p,\tau) \\
= & \star_{i=1}^{n} \star_{j=1}^{m} \nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{q_i}\right) \\
\geq & \star_{i=1}^{n} \star_{j=1}^{m} \nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{b_j}\right) \\
\geq & \star_{j=1}^{m} \nu\left(\mathfrak{F}_j, \frac{\tau}{b_j}\right) \\
= & \int_{\mathfrak{X}} \psi(p) d\nu(p,\tau).
\end{aligned}
$$

(iv) : Assume that $\mathfrak{A}_k \in \mathcal{C}$ is a disjoint sequence and $\mathfrak{A} = \cup_{k=1}^{\infty}\mathfrak{A}_k$ where,

$$\begin{aligned}
& \int_{\mathfrak{A}} \phi(p)d\nu(p,\tau) \\
= & \int_{\mathfrak{X}} (\phi\chi_{\mathfrak{A}})(p)d\nu(p,\tau) \\
= & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} q_i\chi_{\mathfrak{E}_i}\right)\chi_{\mathfrak{A}}(p)d\nu(p,\tau) \\
= & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} q_i\chi_{\cup_{k=1}^{\infty}(\mathfrak{A}_k\cap\mathfrak{E}_i)}\right)(p)d\nu(p,\tau) \\
= & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} q_i\sum_{k=1}^{\infty}\chi_{\mathfrak{A}_k\cap\mathfrak{E}_i}\right)(p)d\nu(p,\tau) \\
= & \int_{\mathfrak{X}} \left(\sum_{i,k} q_i\chi_{\mathfrak{A}_k\cap\mathfrak{E}_i}\right)(p)d\nu(p,\tau) \\
= & \star_{i=1}^{n}\star_{k=1}^{\infty}\,\nu\left((\mathfrak{E}_i\cap\mathfrak{A}_k)\,,\frac{\tau}{q_i}\right) \\
= & \star_{k=1}^{\infty}\int_{\mathfrak{X}} (\phi\chi_{\mathfrak{A}_k})(p)d\nu(p,\tau) \\
= & \star_{k=1}^{\infty}\int_{\mathfrak{A}_k} \phi(p)d\nu(p,\tau).
\end{aligned}$$

Now we will have the definition of integral with expansion for all functions $\jmath \in L^+$ as follows

$$\int_{\mathfrak{X}} \jmath(p)d\nu(p,\tau) = \inf\left\{\int_{\mathfrak{X}} \phi(p)d\nu(p,\tau) \mid \quad 0\le\phi\le\jmath, \quad \phi \text{ is a simple}\right\}. \tag{3}$$

$\square$

It is obvious from the definition that theorem 2.10 satisfying for every $\jmath,\imath \in L^+$.

Theorem 2.11. *Consider $\jmath_n \in L^+$ such that $\jmath_i \le \jmath_{i+1}$ for every i, and $\jmath = \lim \jmath_n (= \sup_{n\in\mathbb{N}} \jmath_n)$, then $\int_{\mathfrak{X}} \jmath(p)d\nu(p,\tau) = \lim_{n\to\infty}\int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau)$.*

Proof. The sequence $\{\jmath_n(p)\}$, for every $p \in \mathfrak{X}$, is an increasing sequence of numbers, therefore $\lim_{n\to\infty}\jmath_n(p) = \jmath(p)$, moreover $\jmath_n(p) \le \jmath_n(p)$, for every $n \in \mathbb{N}$, so

$$\int_{\mathfrak{X}} \jmath(p)d\nu(p,\tau) \le \int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau),$$

then

$$\lim_{n\to\infty}\int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau) \ge \int_{\mathfrak{X}} \jmath(p)d\nu(p,\tau). \tag{4}$$

Now, Consider ϕ a simple function with $0 \le \phi \le \jmath$ and $\mathfrak{E}_n = \{p \mid \jmath_n(p) \ge \phi(p)\}$, then $\mathfrak{E}_n$ is a measurable set. We claim $\mathfrak{E}_n \subseteq \mathfrak{E}_{n+1}$ and $\cup_{n\in\mathcal{N}}\mathfrak{E}_n = \mathfrak{X}$, sine for any $p \in \mathfrak{E}_n$

$$\phi(p) \le \jmath_n(p) \le \jmath_{n+1}(p),$$

so $p \in \mathfrak{E}_{n+1}$. If $p \in \mathfrak{X}$ and $p \notin \cup_{n\in\mathcal{N}}\mathfrak{E}_n$, we have $\jmath_n(p) \le \phi(p)$, for all $n \in \mathcal{N}$, so $\jmath(p) \le \phi(p)$, it is a contradiction. Then $\cup_{n\in\mathcal{N}}\mathfrak{E}_n = \mathfrak{X}$ and there is $m \in \mathcal{N}$ such that $p \in \mathfrak{E}_m$, we have

$$\int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau) \le \int_{\mathfrak{E}_n} \jmath_n(p)d\nu(p,\tau) \le \int_{\mathfrak{E}_n} \phi(p)d\nu(p,\tau).$$

$\lim_{n\to\infty}\int_{\mathfrak{E}_n}\phi(p)d\nu(p,\tau) = \int_{\mathfrak{X}}\phi(p)d\nu(p,\tau)$ and hence $\lim_{n\to\infty}\int_{\mathfrak{X}}\jmath_n(p)d\nu(p,\tau) \le \int_{\mathfrak{X}} phi(p)d\nu(p,\tau)$. By taking the infimum over all simple $0 \le \phi \le \jmath$, we get

$$\lim_{n\to\infty}\int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau) \le \int_{\mathfrak{X}} \jmath(p)d\nu(p,\tau). \tag{5}$$

Frome (4) and (5) we have

$$\lim_{n\to\infty}\int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau) = \int_{\mathfrak{X}} \lim_{n\to\infty} \jmath_n(p)d\nu(p,\tau) = \int_{\mathfrak{X}} \jmath(p)d\nu(p,\tau).$$

□

3 Application

In this section, as an application of the theorems raised in the previous section, we prove Fatou's lemma fuzzy measure space.

Theorem 3.1. *If $\jmath_n$ is any sequence in L^+, then*

$$\liminf \int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau) \le \int_{\mathfrak{X}} \liminf \jmath_n(p)d\nu(p,\tau).$$

Proof. For each $k \ge 1$, $\inf_{n\ge k}\jmath_n \le \jmath_j$ for $j \ge k$, hence

$$\int_{\mathfrak{X}} \inf_{n\ge k} \jmath_n(p)d\nu(p,\tau) \ge \int_{\mathfrak{X}} \jmath_j(p)d\nu(p,\tau),$$

for $j \ge k$, hence

$$\int_{\mathfrak{X}} \inf_{n\ge k} \jmath_n(p)d\nu(p,\tau) \ge \inf_{j\ge k}\int_{\mathfrak{X}} \jmath_j(p)d\nu(p,\tau).$$

Now let $k \to \infty$ and apply the monotone convergence theorem

$$\lim \int_{\mathfrak{X}} \inf_{n\ge k} \jmath_n(p)d\nu(p,\tau) = \int_{\mathfrak{X}} \liminf \jmath_n(p)d\nu(p,\tau) \ge \liminf \int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau).$$

□

4 Conclusion

We worked on a new concept of fuzzy measurement. We define a new type of fuzzy measure with distance functions. With this concept, we introduced a new version of measurement theory and fuzzy integral and addressed theorems about it. As a continuation of this research, by defining the fuzzy outer measure, a new concept of fuzzy measurement can be defined and using it, new theorems in the fuzzy measure theory can be proposed.

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