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Research Article

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Efficiency score in the presence of flexible factors, integer and fuzzy integer data

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Abstract
In traditional Data Envelopment Analysis (DEA) approaches, inputs and outputs are usually considered as exact and real values. The relative efficiency of the Decision-Making Units (DMUs) is evaluated and it is known that the factors are inputs and/or outputs. However, there are some conditions under which the efficiency of DMUs should be calculated while the data are integer and ambiguous. Therefore, various integer DEA models have been proposed to determine the performance of DMUs when integer data and fuzzy factors are available. In addition, there are cases where the efficiency of DMUs should be determined when integer data and flexible factors are available. Therefore, some integer DEA methods have been proposed to calculate the performance of DMUs and specify the role of flexible measures when some of the data are integer and flexible factors are available. However, there are some situations where there are integer data, fuzzy integer measures and flexible factors. Therefore, this paper sheds light on the nature of the model to determine the efficiency of DMUs when there are integer inputs and/or outputs, flexible factors and fuzzy integer measures, and determines the role of factors with uncertain inputs or outputs. In fact, slacks are addressed and a slack-based efficiency measure (SBM) is defined to compute the performance of DMUs in the presence of flexible factors, integer data and fuzzy integer measures. The proposed approaches are demonstrated and illustrated using an example.

Keywords: Data Envelopment Analysis; Efficiency; Slack-Based Efficiency Measure Model; Flexible Factor; Fuzzy Integer Data; Integer Data.

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1. Introduction

A mathematical programming approach called Data Envelopment Analysis (DEA) is used to evaluate the relative efficiency of Decision-Making Units (DMUs) with multiple inputs and outputs. CCR (Charnes, et al. [1]) and BCC (Banker, et al. [2]) are the two basic models of DEA based on the respective hypothesis of constant returns to scale (CRS) and variable returns to scale (VRS) of Production Possibility Set.

Traditional DEA models assume that inputs and outputs are continuous, real-valued data. Of course, in many applications, some of the inputs or outputs may be integer values, such as the number of doctors, nurses, books published, etc.

The role of some factors as inputs or outputs is not entirely clear in many real-world applications of DEA. In other words, a factor may be used as an input by some DMUs, while it may play an output role in other DMUS. These types of factors are called flexible factors. In traditional DEA models, the roles of the data are fixed. In practice, however, there are cases where the role of the data is unknown.

In classical DEA models, it is generally assumed that the inputs and outputs of the DMUs are specified by exact numerical values. However, in real applications, there are situations where the performance of DMUs must be evaluated in the presence of imprecise data. In the DEA literature, there are studies that deal with integer data and/or flexible factors.

Cook and Zhu [3] considered the flexible factors to evaluate the relative efficiency of DMUs and specified the role of flexible measures. Amirteimoori et al. [4] proposed a Slacks-based measure with flexible factors to evaluate the performance of DMUs. Tohidi and Matroud [5] introduced a method to define the role of flexible factors. Recently, Toloo et al. [6] proposed a non-radial directional distance model to categories the flexible measures. Kordrostami, et al. [7] presented methods to calculate the efficiency values of DMUs for which integer data and flexible factors are available. Lozano and Villa [8] presented models for analyzing integer data in DEA. Subsequently, Kuosmanen and Kazemi Matin [9] provided a new axiomatic basis for DEA models that are integer. Jie et al. [10] improved the model of Kuosmanen and Matin [9] and showed that the model indeed solves problems. In addition, Due et al. [11] provided methods that examine the slacks to estimate the relative efficiency and super efficiency values of DMUs when integer data are available.

Abolghasem et al. [12] proposed a cross-efficiency evaluation in the presence of flexible measures with an application to health systems. Sedighi Hassan Kiyadeh [13] proposed an improvement of models to determine the nature of flexible factors in data envelopment analysis. Toloo et al. [14] provided new DEA models for classifying flexible measures. Salehian, et al. [15] proposed flexible factors in categorized data for DEA.

Fuzzy set was used for the first time in [16]. Kao and Liu proposed fuzzy efficiency measures for DEA [17]. Hatami-Marbini et al. [18] provided fuzzy efficiency measures for DEA using a multi-objective lexicographic approach. Ebrahimnejad and Amani [19] proposed a fuzzy DEA analysis in the presence of unwanted output with ideal points. Peykani, et al. [20] proposed a fuzzy DEA with random constraints. Saati, et al. [21] introduced a fuzzy data envelopment analysis for clustering operating units with imprecise data. Hatami-Marbini, et al. [22] provided an efficiency analysis in two-level structures using fuzzy data envelopment analysis. In the framework of DEA, models with integer data and flexible factors can be found.

In addition, some studies have been conducted on models with integer data and fuzzy measures. There are also models with flexible factors and fuzzy measures.

Kordrostami, et al. [23] presented a model for evaluating the efficiency values of DMUs and classifying the fuzzy measures. Saati and Imani [24] proposed a method for categorizing common factors using the fuzzy concept and determined the role of common factors.

Kordrostami, et al. [25] presented approaches to estimate the efficiency values of DMUs when integer data and fuzzy factors are available.

Then, parametric and nonparametric operations research models were found by Mirmozaffari [26,27]. In addition, Mirmozaffari, et al. [28] developed a novel integrated generalized DEA to evaluate hospital services. Furthermore, an integrated artificial intelligence model for evaluating businesses during the COVID-19 pandemic was presented by Mirmozaffari, et al. [29]. The application of DEA to emergency departments and the management of emergency situations was presented by Mirmozaffari, et al. [30]. Readers may also refer to [31,32,33].

However, as far as we know, there is no study that considers integer data, fuzzy integer measures, and flexible factors in DEA texts. In this paper, we develop an integer DEA model to evaluate the efficiency of DMUs and determine the status of flexible factors in the presence of integer data, flexible factors and fuzzy integer measures simultaneously. Namely, the efficiency is analyzed by taking a close look at the slacks. The status of the flexible factors is also indicated when the inputs are integer and the outputs are fuzzy integer measures. Moreover, the projection points of the inputs can be determined by the proposed approach. In the proposed model, the projection points of the integer data are integer measures. Triangular fuzzy numbers are used to evaluate uncertain outputs and flexible factors. The graded mean integration method is used to solve fuzzy integer DEA models. To illustrate the potential of the model, an example is examined. All inputs and flexible factors of the example are taken as integer data, all outputs as fuzzy integer measures.

The paper proceeds as follows. Section 2 introduces the notations used in this paper and explains the main ideas of DEA with integer data, the fuzzy integer model and the flexible slack-based model (FSBM). Section 3 introduces a new model for calculating efficiency with integer data, fuzzy integer measures and flexible factors. Section 4 presents an example. Section 5 contains a conclusion.

2. Preliminaries

2.1. Notations

Suppose we deal with n DMUs. Symbols are introduced as follows:

$j=1,...,n$: the set of DMUs

$i=1,...,m$: the set of inputs

$r=1,...,s$: the set of outputs

$k=1,...,K$: the set of flexible factors

DMU_j: the j-th DMU, $j=1,...,n$

DMU_o: the DMU under consideration

x_{ij} : the *i*-th input resource of *j*-th DMU

x_{io} : the input resource *i* of DMU_o

y_{rj} : the output product *r* of DMU_j

y_{ro} : the output product *r* of DMU_o

z_{kj} : the *k*-th flexible factor of DMU_j

z_{ko} : the *k*-th flexible factor of DMU_o

s_i : *i*-th input slacks for $i = 1, ..., m$

s_r : *r*-th output slacks for $r = 1, ..., s$

$g_k^{(3)}$: flexible factor slacks as the input for $k = 1, ..., K$

$g_k^{(4)}$: flexible factor slacks as the output for $k = 1, ..., K$

$d_k^{(1)}, d_k^{(2)}$: binary variables

λ_j : intensity vectors of DMU_j

I^I : the subset of inputs which are integer

O^I : the subset of outputs which are integer

K^I : the subset of flexible factors which are integer

$\tilde{x}_{ij}$: the *i*-th triangular fuzzy input of DMU_j

$\tilde{y}_{rj}$: the *r*-th triangular fuzzy output of DMU_j

$\tilde{x}_{io}$: the *i*-th triangular fuzzy input of DMU_o

$\tilde{y}_{ro}$: the *r*-th triangular fuzzy output of DMU_o

2.2. Integer DEA

Suppose we deal with *n* DMUs, DMU_j $(j = 1, ..., n)$ with *m* input resources x_{ij} $(i = 1, ..., m)$ and *s* output products $y_{rj}(r = 1, ..., s)$. In the traditional DEA methods, all data are regarded as real-valued measures. Thus, the performance of DMUs is measured while the reference points of DMUs obtain values which are real. However, in many real worlds, some inputs and/or outputs can only be integer measures. Assume $x_{ij}(i = 1, ..., m)$ and $y_{rj}(r = 1, ..., s)$ are integer data of DMU$_j$ $(j = 1,...,n)$, thus some DEA methods were developed and improved to get integer projections for integer measures. The suggested model is used to estimate the efficiency scores of DMUs in the presence of integer data, flexible factors and fuzzy integer measures.

The following model is a slacks-based nonlinear model with the integer data to analyze the DMUs' performance where integer measures are present. In this part, a brief explanation of this model is provided.

$$\min \quad \rho = \frac{1-(m)^{-1}(\sum_{i\in I}\frac{s_i^-}{x_{io}})}{1+(s)^{-1}(\sum_{r\in O}\frac{s_r^+}{y_{ro}})} \tag{2.1}$$

$$
\begin{aligned}
s.t.\quad & \\
& \sum_{j=1}^{n}\lambda_j x_{ij} \le x_i, && i=1,...,m,\\
& x_i = x_{io} - s_i^-, && i=1,...,m,\\
& \sum_{j=1}^{n}\lambda_j y_{rj} \ge y_r, && r=1,...,s,\\
& y_r = y_{ro} + s_r^+, && r=1,...,s,\\
& \lambda_j \ge 0,\ \ , s_i^-, x_i \ge 0,\ \ s_r^+, y_r \ge 0,\ \ i=1,...,m,\ \ r=1,...,s,\ \ j=1,...,n, &&\\
& x_i \in Z,\ \ \forall i \in I^I, &&\\
& y_r \in Z,\ \ \forall r \in O^I. &&
\end{aligned}
$$

In aforementioned formula, s_i^-, s_r^+ show the non-radial slacks. $x_i \in Z^+, y_r \in Z^+$ are the integer projection points for inputs I^I and outputs O^I respectively. $\lambda_j (j = 1,...,n)$ are the intensity vectors.

In the next subdivision, a fuzzy integer number is determined.

2.3. Main concepts of fuzzy integer numbers

Let R be the set of real numbers and Z be the set of integer numbers.

Definition 2.3.a. Suppose $u : R \to [0,1]$ is a fuzzy set. It is called a fuzzy integer if its support is a closed integer interval (denoted as $\langle \underline{u}(0), \overline{u}(0)\rangle$) and satisfies the following:

1. u is normal; i.e., there exists $x' \in \langle \underline{u}(0), \overline{u}(0)\rangle$ such that $u(x') = 1$,
2. $u(x_i) \le u(x_j)$ for any $x_i, x_j \in \langle \underline{u}(0), x'\rangle$ with $x_i \le x_j$,
3. $u(x_i) \ge u(x_j)$ for any $x_i, x_j \in \langle x', \overline{u}(0)\rangle$ with $x_i \le x_j$.

Note that an interval, which is closed and integer, is showed by $\langle s_1, s_2 \rangle = \{x \in Z : s_1 \le x \le s_2\}$ for any $s_1, s_2 \in Z$ and $s_1 \le s_2$.

Definition 2.3.b. Suppose s_0, s_1, t_1 and $t_0 \in I$ with $s_0 \le s_1 \le t_1 \le t_0$, and $\underline{m}, \bar{m} \in Z$. If the fuzzy set $u : R \to [0,1]$ is determined as:

$$u(x) = \begin{cases} 1 & if \quad x \in \langle s_1, t_1 \rangle \\ \dfrac{x - s_0}{s_1 - s_0} & if \quad x \in \langle \underline{m}, s_1 \rangle \\ \dfrac{t_0 - x}{t_0 - t_1} & if \quad x \in \langle t_1, \overline{m} \rangle \\ 0 & if \quad x \in \langle \underline{m}, \overline{m} \rangle \end{cases}$$

where $s_0 \le \underline{m} \le s_1$ and $t_1 \le \bar{m} \le t_0$; then u is a trapezoidal fuzzy integer. A triangular fuzzy integer number can obtain, if $s_1 = t_1$. See Kordrostami, et al. [25].

Assume there exist *n* DMUs, that produce s outputs by consuming m inputs. The *j*-th DMU showed by DMU$_j$ $(j = 1,\dots,n)$, whose x_{ij} $(i = 1,\dots,m)$ and y_{rj} $(r = 1,\dots,s)$ are *i*-th input and *r*-th output, respectively. See Kordrostami, et al. [25]. The following model, referred to as the CCR model, was introduced by Charnes et al. [1] for estimating the entities' relative efficiency with data that are precise and real numbers.

$$\begin{aligned} \min \quad & \theta \\ s.t. \quad & y_{ro} \le \sum_{j=1}^{n} \lambda_j y_{rj}, \qquad r = 1,\dots,s, \\ & \theta x_{io} \ge \sum_{j=1}^{n} \lambda_j x_{ij}, \qquad i = 1,\dots,m, \\ & \lambda_j \ge 0, \qquad j = 1,\dots,n. \end{aligned} \tag{2.2}$$

θ shows the efficiency score. $\lambda_j (j = 1,\dots,n)$ indicate intensity vectors. In this model, x_{io} and y_{ro} are symbols of the inputs and outputs of DMU$_o$, respectively. $\tilde{x}_{ij} = (x_{ij1}, x_{ij2}, x_{ij3})$, $\tilde{y}_{rj} = (y_{rj1}, y_{rj2}, y_{rj3})$ are inputs and outputs which are triangular fuzzy numbers. Note that $\tilde{x}_{io}$ and $\tilde{y}_{ro}$ are inputs and outputs of DMU$_o$. The graded mean integration representation approach is used to calculate the DEA models with fuzzy data, as follows:

Definition 2.3.c Suppose $\tilde{A} = (a,b,c)$ is a triangular fuzzy number, the graded mean integration representation $\tilde{A}$ can be determined as $(a + 4b + c) / 6$.

In fact, abovementioned models are used because of the easiness and rational calculation. Therefore, by considering Definition 2.3.c, the CCR model with fuzzy factors can be changed with the model (2.3) as follows:

$$\begin{aligned} \min \quad & \theta && (2.3) \\ s.t. \quad & \frac{1}{6}(4y_{ro2}+y_{ro1}+y_{ro3}) \le \frac{1}{6}\sum_{j=1}^{n}(4y_{rj2}+y_{rj1}+y_{rj3})\lambda_j, \qquad r=1,\ldots,s, \\ & \frac{\theta}{6}(4x_{io2}+x_{io1}+x_{io3}) \ge \frac{1}{6}\sum_{j=1}^{n}(4x_{ij2}+x_{ij1}+x_{ij3})\lambda_j, \qquad i=1,\ldots,m, \\ & \lambda_j \ge 0, \qquad j=1,\ldots,n. \end{aligned}$$

Definition 2.3.c will be true wherever integer variables in the fuzzy linear programming are present.

Nevertheless, model (2.3) is not appropriate to assess DMUs' efficiency scores where integer measures and fuzzy factors are present. Indeed, as non-integer values may be the reference point of a DMU with integer data. The aim of preparing model (2.3) is to compare its outcomes with the models with integer measures and fuzzy factors.

2.4. SBM model with Flexible factor (FSBM)

Amirteimoori, et al. [4] provided the following model in terms of computing the DMUs' efficiency where the flexible measures are present:

$$\pi_o^* = \min \quad \frac{1-(m+K)^{-1}\left(\sum_{i=1}^{m}\frac{s_i}{x_{io}}+\sum_{k=1}^{K}\frac{g_k^{(1)}}{z_{ko}}\right)}{1+(s+K)^{-1}\left(\sum_{r=1}^{s}\frac{q_r}{y_{ro}}+\sum_{k=1}^{K}\frac{g_k^{(2)}}{z_{ko}}\right)} \qquad (2.4)$$

$$\begin{aligned} s.t. \quad & \\ & x_{io} = \sum_{j=1}^{n}\lambda_j x_{ij} + s_i, && i=1,\ldots,m, \\ & y_{ro} = \sum_{j=1}^{n}\lambda_j y_{rj} - q_r, && r=1,\ldots,s, \\ & z_{ko} = \sum_{j=1}^{n}\lambda_j z_{kj} + g_k^{(1)} - g_k^{(2)}, && k=1,\ldots,K, \\ & g_k^{(1)}.g_k^{(2)} = 0, && k=1,\ldots,K, \\ & \lambda_j, g_k^{(1)}, g_k^{(2)}, q_r, s_i \ge 0, \quad i=1,\ldots,m, \quad r=1,\ldots,s, \quad j=1,\ldots,n, \quad k=1,\ldots,s. \end{aligned}$$

Aforementioned model evaluates the maximum of DMUs' efficiency scores and determines the role of the flexible measures. Then, they changed model (2.4) into MILP model (2.5) using Charnes and Cooper's transformation [34] and some variables substitutions:

$$\pi_0^* = \min \qquad \rho - (m+k)^{-1}\left(\sum_{i=1}^{m} \frac{s_i}{x_{io}} + \sum_{k=1}^{K} \frac{g_k^{(1)}}{z_{ko}}\right) \tag{2.5}$$

$$
\begin{aligned}
&s.t.\\
&\rho + (s+k)^{-1}\left(\sum_{r=1}^{s} \frac{q_r}{y_{ro}} + \sum_{k=1}^{K} \frac{g_k^{(2)}}{z_{ko}}\right) = 1\\
&\rho x_{io} = \sum_{j=1}^{n} \lambda_j x_{ij} + s_i && i = 1,\dots,m,\\
&\rho y_{ro} = \sum_{j=1}^{n} \lambda_j y_{rj} - q_r && r = 1,\dots,s,\\
&\rho z_{ko} = \sum_{j=1}^{n} \lambda_j z_{kj} + g_k^{(1)} - g_k^{(2)}, && k = 1,\dots,K,\\
&0 \le g_k^{(1)} \le M d_k^{(1)}, && k = 1,\dots,K,\\
&0 \le g_k^{(2)} \le M d_k^{(2)}, && k = 1,\dots,K,\\
&d_k^{(1)} + d_k^{(2)} = 1, && k = 1,\dots,K,\\
&\lambda_j, g_k^{(1)}, g_k^{(2)}, q_r, s_i, \ge 0, \quad d_k^{(1)}, d_k^{(2)} \in \{0,1\}, && \forall i, j, k, r,
\end{aligned}
$$

That $\left(1 + (s+K)^{-1}\left[\sum_{i=1}^{m} \frac{q_r}{y_{ro}} + \sum_{k=1}^{K} \frac{g_k^{(2)}}{z_{ko}}\right]\right) \wedge \{-1\} = \rho$ and M is a large positive number.

DMU$_o$ is efficient if and only if $\pi_o^* = 1$.

If $g_k^{(1)} = g_k^{(2)} = 0$, the factor which is flexible, can be input or output.

To assess the DMUs' relative efficiency where integer data and flexible measures are present, Kordrostami et al. [7] suggested the following model:

$$\max \quad \sum_{i \notin I^I} \frac{s_i^-}{x_{io}} + \sum_{i \in I^I} \frac{\tilde{s}_i^-}{x_{io}} + \sum_{k \in K^I} \frac{g_k'^{(1)}}{z_{ko}} + \sum_{k \notin K^I} \frac{\overline{g}_k^{(1)}}{z_{ko}} + \sum_{r \notin O^I} \frac{s_r^+}{y_{ro}} + \sum_{r \in O^I} \frac{\tilde{s}_r^+}{y_{ro}} + \sum_{k \in K^I} \frac{g_k'^{(2)}}{z_{ko}} + \sum_{k \notin K^I} \frac{\overline{g}_k^{(2)}}{z_{ko}}$$

$$
\begin{aligned}
&s.t.\\
&x_{io} - s_i^- = \sum_{j=1}^{n} \lambda_j x_{ij} && i \notin I^I, \qquad (2.6)\\
&y_{ro} + s_r^+ = \sum_{j=1}^{n} \lambda_j y_{rj} && r \notin O^I,\\
&\tilde{x}_{io} \ge \sum_{j=1}^{n} \lambda_j x_{ij}, \quad x_{io} - \tilde{s}_i^- = \tilde{x}_{io} && i \in I^I,\\
&\tilde{y}_{ro} \le \sum_{j=1}^{n} \lambda_j y_{rj}, \quad y_{ro} + \tilde{s}_r^+ = \tilde{y}_{ro} && r \in O^I,\\
&z_{ko} - \overline{g}_k^{(1)} + \overline{g}_k^{(2)} = \sum_{j=1}^{n} \lambda_j z_{kj}, && k \notin K^I,\\
&0 \le \overline{g}_k^{(1)} \le M d_k^{(1)}, \quad 0 \le \overline{g}_k^{(2)} \le M d_k^{(2)},
\end{aligned}
$$

$$\tilde{z}_{ko} - \hat{g}_k^{(1)} + \hat{g}_k^{(2)} = \sum_{j=1}^{n} \lambda_j z_{kj}, \quad z_{ko} - g_k'^{(1)} + g_k'^{(2)} = \tilde{z}_{ko} \qquad k \in K^I,$$
$$0 \le g_k'^{(1)} \le Md_k^{(1)}, \ \ 0 \le g_k'^{(2)} \le Md_k^{(2)}, \ \ 0 \le \hat{g}_k^{(1)} \le Md_k^{(1)}, \ \ 0 \le \hat{g}_k^{(2)} \le Md_k^{(2)},$$
$$d_k^{(1)} + d_k^{(2)} = 1,$$
$$\tilde{x}_{io}, \tilde{y}_{ro}, \tilde{z}_{ko} \in Z^+, \qquad d_k^{(1)}, d_k^{(2)} \in \{0,1\}, \qquad i \in I^I, k \in K^I, r \in O^I, \lambda_j \ge 0$$
$$s_i^- \ge 0, i \notin I^I, \ \ s_r^+ \ge 0, r \notin O^I, \ \ \tilde{s}_i^- \ge 0, i \in I^I, \ \ \tilde{s}_r^+ \ge 0, r \in O^I,$$
$$g_k'^{(1)}, g_k'^{(2)} \ge 0, \hat{g}_k^{(1)}, \hat{g}_k^{(2)} \ge 0, \ \ k \in K^I, \ \ \bar{g}_k^{(1)} + \bar{g}_k^{(2)} \ge 0, \ \ k \notin K^I$$

3. DEA models with integer data, flexible factors and fuzzy integer measures

Suppose n DMUs (i.e. DMU$_j$ $(j = 1,...,n)$) is present that produces s outputs (i.e. $y_{rj}(r = 1,...,s)$) by consuming m inputs (i.e. $x_{ij}(i = 1,...,m)$). The following model is provided to assess the DMUs' performance wherever integer data, the flexible factors and fuzzy integer measures are present and specify the role of flexible factors.

$$\min \quad \frac{1 - (m+K)^{-1}\left(\sum_{i=1}^{m} \frac{s_i^-}{\tilde{x}_{io}} + \sum_{k=1}^{K} \frac{g_k^{(3)}}{\tilde{z}_{ko}}\right)}{1 + (s+K)^{-1}\left(\sum_{r=1}^{s} \frac{s_r^+}{\tilde{y}_{ro}} + \sum_{k=1}^{K} \frac{g_k^{(4)}}{\tilde{z}_{ko}}\right)} \tag{3.1}$$

$s.t.$

$$\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \le x_i, \quad x_i = \tilde{x}_{io} - s_i^-, \qquad \forall i,$$
$$\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \ge y_r, \quad y_r = \tilde{y}_{ro} + s_r^+, \qquad \forall r,$$
$$\sum_{j=1}^{n} \lambda_j \tilde{z}_{kj} = z_k - g_k^{(1)} + g_k^{(2)}, \qquad k \in K^I,$$
$$z_k = \tilde{z}_{ko} - g_k^{(3)} + g_k^{(4)}, \qquad k \in K^I,$$
$$0 \le g_k^{(1)} \le Md_k^{(1)}, \ \ 0 \le g_k^{(2)} \le Md_k^{(2)}, \ \ 0 \le g_k^{(3)} \le Md_k^{(1)}, \ \ 0 \le g_k^{(4)} \le Md_k^{(2)},$$
$$d_k^{(1)} + d_k^{(2)} = 1,$$
$$x_i, y_r, z_k \in Z \ , \ \ i \in I^I, k \in K^I, r \in O^I, \ \ d_k^{(1)}, d_k^{(2)} \in \{0,1\},$$
$$s_i^- \ge 0, \ \ s_r^+ \ge 0, \ \ g_k^{(1)}, g_k^{(2)}, g_k^{(3)}, g_k^{(4)} \ge 0, \ \ \lambda_j \ge 0, \ \ x_i \ge 0, \ \ , y_r \ge 0, \ \ \forall j, i, r$$

$\lambda_j (j = 1,...,n)$ are intensity vectors. When imprecise data as triangular fuzzy numbers are present (i.e. $\tilde{x}_{ij} = (x_{ij1}, x_{ij2}, x_{ij3}), \tilde{y}_{rj} = (y_{rj1}, y_{rj2}, y_{rj3})$ in the model (3.1) that $x_{ij1} \ge 0$ and $y_{rj1} \ge 0$), a fuzzy model should be used to assess the DMUs' efficiency score. In above model $\tilde{x}_{io}$ and $\tilde{y}_{ro}$ are inputs and outputs of DMU$_o$ The data of DMU$_o$ are fuzzy integer measures.

Consider that in this case inputs and outputs of DMU$_o$ are showed by $\tilde{x}_{io}$ and $\tilde{y}_{ro}$, respectively.

The graded mean integration method is used to calculate the DEA models with fuzzy data and handle fuzzy factors. Based on Definition 2.3.c, the model with integer data, fuzzy integer measures and flexible factors can be written as follows:

$$\min \quad \frac{1-\dfrac{6}{m+K}\left(\sum_{i=1}^{m}\dfrac{s_i^-}{4x_{io2}+x_{io1}+x_{io3}}+\sum_{k=1}^{K}\dfrac{g_k^{(3)}}{4z_{ko2}+z_{ko1}+z_{ko3}}\right)}{1+\dfrac{6}{s+K}\left(\sum_{r=1}^{s}\dfrac{s_r^+}{4y_{ro2}+y_{ro1}+y_{ro3}}+\sum_{k=1}^{K}\dfrac{g_k^{(4)}}{4z_{ko2}+z_{ko1}+z_{ko3}}\right)} \tag{3.2}$$

$$\begin{aligned}
s.t.\quad & \\
& \frac{1}{6}\sum_{j=1}^{n}\lambda_j(4x_{ij2}+x_{ij1}+x_{ij3})\le x_i, && \forall i,\\
& x_i=\frac{1}{6}(4x_{io2}+x_{io1}+x_{io3})-s_i^-, && \forall i,\\
& \frac{1}{6}\sum_{j=1}^{n}\lambda_j(4y_{rj2}+y_{rj1}+y_{rj3})\ge y_r, && \forall r,\\
& y_r=\frac{1}{6}(4y_{ro2}+y_{ro1}+y_{ro3})+s_r^+, && \forall r,\\
& \frac{1}{6}\sum_{j=1}^{n}\lambda_j(4z_{kj2}+z_{kj1}+z_{kj3})=z_k-g_k^{(1)}+g_k^{(2)}, && \forall k,\\
& z_k=\frac{1}{6}(4z_{ko2}+z_{ko1}+z_{ko3})-g_k^{(3)}+g_k^{(4)}, && \forall k,\\
& 0\le g_k^{(1)}\le Md_k^{(1)},\;\; 0\le g_k^{(2)}\le Md_k^{(2)},\;\; 0\le g_k^{(3)}\le Md_k^{(1)},\;\; 0\le g_k^{(4)}\le Md_k^{(2)}, &&\\
& d_k^{(1)}+d_k^{(2)}=1,\;\; d_k^{(1)},d_k^{(2)}\in\{0,1\}, &&\\
& x_i,y_r,z_k\in Z\;\;,\;\; i\in I^I, r\in O^I, k\in K^I, &&\\
& s_i^-\ge 0,\;\; s_r^+\ge 0,\;\; g_k^{(1)},g_k^{(2)},g_k^{(3)},g_k^{(4)}\ge 0,\;\; x_i\ge 0,\;\; y_r\ge 0,\;\; \lambda_j\ge 0,\;\; \forall j,i,r &&
\end{aligned}$$

In aforementioned formula, λ_j indicates intensity weights. M is a large positive number. x_i, y_r, z_k are variables that are positive and integer. They represent integer projection points for data that are integer. Moreover, if $g_k^{(3)}>0$ then z_k is considered as an input, and it will be an output if $g_k^{(4)}>0$. On the other hands if $d_k^{(1)}=0$, the flexible factor is called output and otherwise (i.e. $d_k^{(2)}=0$), it is considered as an input.

In fact, fuzzy sets $\tilde{x}_{ij}, \tilde{y}_{rj}$ and $\tilde{z}_{kj}$ are replaced with $(4x_{ij2}+x_{ij1}+x_{ij3})/6$, $(4y_{rj2}+y_{rj1}+y_{rj3})/6$ and $(4z_{kj2}+z_{kj1}+z_{kj3})/6$, respectively. The pessimistic and optimistic targets are used to show the fuzzy produced outputs and the fuzzy consumed inputs

Table 1. Data of an example

DMU	(x_1) **Beds number**	(x_2) **Doctors Number**	(x_3) **Nurses Number**	y_1	y_2	**(Flexible measure) Nurse Trainees Number**
1	158	44	138	(7000,9500,8280)	(2000,900,1918)	15
2	62	22	81	(2000,3000,1000)	(300,250,770)	12
3	139	41	142	(4300,2700,1100)	(700,900,860)	60
4	263	70	412	(9400,5700,7748)	(2500,3000,2486)	50
5	245	69	286	(8000,4900,10500)	(1700,1250,1046)	30
6	129	40	202	(3900,3000,5400)	(820,500,1140)	15
7	123	21	169	(2700,2050,1550)	(500,750,370)	20
8	222	41	143	(2000,2000,1400)	(320,120,442)	30
9	123	68	160	(4800,5300,3100)	(850,1000,838)	30
10	61	25	90	(2750,1200,1030)	(400,550,196)	34
11	100	28	113	(3600,5000,5800)	(730,1200,848)	24
12	32	22	38	(800,820,1380)	(170,100,210)	15
13	117	35	121	(4100,6000,5500)	(750,750,1212)	52
14	103	41	121	(5100,6000,3600)	(920,1050,826)	14
15	32	14	70	(800,300,700)	(75,120,180)	36
16	50	31	114	(9500,9000,10600)	(80,150,412)	60
17	119	38	86	(2700,3000,2600)	(600,850,290)	55
18	38	21	75	(1500,2000,1000)	(270,300,396)	13

based on [35]; that is $w_1x_{ij2}+w_2x_{ij1}+w_3x_{ij3}$, $w_1y_{rj2}+w_2y_{rj1}+w_3y_{rj3}$ and $w_1z_{kj2}+w_2z_{kj1}+w_3z_{kj3}$ where $w_1+w_2+w_3=1.$

As explained in [35], x_{ij3} and y_{rj3} are too optimistic and x_{ij1} and y_{rj1} are too pessimistic. Therefore, we use the weights $w_1=1/6, w_2=4/6$ and $w_1=1/6, w_2=4/6$ that can be substituted subjectively. Thus, these boundary values provide us boundary solutions.

Notice that (a,b,c) is a triangular fuzzy integer number, while $(a+4b+c)/6$ is gained as non-integer value, we will round it to the closest integer value. Indeed, we consider $\lfloor (a+4b+c)/6 \rfloor$ in terms of the effect of rounding $(a+4b+c)/6$ is almost insignificant.

4. Numerical example

This example is about estimating the efficiency of 18 hospitals in an Iranian province, and determining the role of flexible factors. The inputs and flexible factors were taken from [7]. Since enormous resources are used in the health sector such as hospitals, it is necessary to evaluate the efficiency of these facilities. In this case study, the number of beds, doctors and nurses are considered as inputs. In addition, the number of nursing trainees is assumed to be a flexible variable. The columns in Table 1 show the inputs, outputs and flexible factors. Columns 2,3 and 4 show the first, second and third inputs while columns 5 and 6 show the outputs and column 7 the flexible factors. As you can see, all inputs and flexible factors are integer data, and the outputs are triangular fuzzy integers. Model (3.2) is used to evaluate the

Table 2. Efficiency scores and optimal slacks

DMU	Efficiency	s_1^{*-}	s_2^{*-}	s_3^{*-}	s_1^{*+}	s_2^{*+}	$d_k^{(1)}$	$d_k^{(2)}$
1	0.59	70	18	35	20.420	0	0	1
2	0.61	27	10	34	43.985	0	0	1
3	0.41	60	19	53	1600	3.429	1	0
4	1	0	0	0	0	0	0	1
5	0.49	100	27	100	3527.800	0	0	1
6	0.38	61	14	100	2884.269	0	0	1
7	0.51	62	4	100	1150	6.286	1	0
8	0.41	100	3	0	0	601.567	1	0
9	0.56	21	31	17	3559.944	0	0	1
10	0.39	17	12	40	1107.208	2.012	1	0
11	1	0	0	0	0	0	1	0
12	0.33	19	17	21	0	6.791	1	0
13	0.74	35	7	13	0	3.773	1	0
14	0.65	10	12	8	40	0	0	1
15	0.19	20	10	57	223.894	1.304	1	0
16	1	0	0	0	0	0	0	1
17	0.52	52	19	11	474.168	0	1	0
18	0.53	0	5	14	2209.202	0	0	1

performance of hospitals and determine the status of the student nurse factor. From the efficiency scores in Table 2, it can be seen that three hospitals are efficient with 4,11 and 16. In addition, column 2 of Table 2 shows that DMU 15 is the most inefficient DMU in this study with an efficiency score of 0.19. As shown in columns 8 and 9 of Table 2, nine hospitals (3,7,8,10,11,12,13,15,17) consider the student nurse factor as input and nine hospitals (1,2,4,5,6,9,14,16,18) as output. The optimal slacks are also listed in columns 3-7 of Table 2. As shown, all optimal slacks of the optimal hospital are zero. Furthermore, the projection

points determined with model (3.2) are shown in Table 3. It can be seen that all the targets of the integer factors are given as integer values.

Note that, first, we used a method to defuzzify the fuzzy numbers. Second, we evaluated the performance of the DMUs. Therefore, the reference points are evaluated as continuous and unique values.

Recall that all integer measures in model (3.2), have integer projections. However, in Lozano and Villa [8], it is pointed out that the rounding of the real reference point may not be appropriate. Therefore, the proposed model (Model (3.2)) is obviously suitable under the conditions of integers, fuzzy integer measures and flexible factors.

5. Conclusions

In conventional DEA models, all data are usually considered as exact and real numbers. Classification of data is an important issue when analyzing performance. In real applications, there are situations where the efficiency of DMUs needs to be evaluated in the presence of integer data, fuzzy integer measures and flexible factors. Therefore, in this paper a DEA model was developed to evaluate the efficiency of DMUs and specify the role of flexible factors, where all input data and flexible factors are integer numbers, and the output data are fuzzy integer measures.

Table 3. Projection points

DMU	Projection Beds number	Projection Doctors Number	Projection Nurses Number	Projection y_1	Projection y_2
1	88	26	103	4953.753	973
2	35	12	47	2710.652	336.667
3	79	22	89	4033.333	896.762
4	263	70	412	6041.333	2914.333
5	145	42	186	9361.133	1216
6	68	26	102	6284.269	606.667
7	61	17	69	3116.667	692.952
8	122	38	143	9266.667	795.234
9	102	37	143	8493.277	973
10	44	13	50	2278.875	493.012
11	100	28	113	5133.333	1141.333
12	13	5	17	913.333	125.124
13	82	28	108	5916.667	830.773
14	93	29	113	5640.794	1012.667
15	12	4	13	590.560	131.304
16	50	31	114	9266.667	193.667
17	67	19	75	3407.501	756.667
18	38	16	61	4042.536	316

Some studies investigated the efficiency scores of DMUs when integer data and fuzzy integer measures are available. In addition, some studies investigated the efficiency of DMUs when integer data and flexible factors are available. In this paper, a Slacks-based nonlinear programming problem was proposed to estimate the efficiency values of DMUs and specify the role of flexible factors in the presence of integer data, flexible factors and fuzzy integer measures. The method of graded mean integration representation was applied to fuzzy data in order to defuzzify them. An example was given to describe and illustrate the approaches. The extension of the proposed model for situations where some data are integer and flexible factors and fuzzy numbers are also present, are interesting topics for future research. Further research can also rank DMUs when integer data, flexible factors and fuzzy integer measures are present.

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