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Research paper

Introducing two classes of optimal codes derived from one-weight $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes

Narjes Mohsenifar[1] and Sadegh Sadeghi[2*]

[1] *Department of Electrical Engineering, Shahrekord Branch, Islamic Azad University, Shahrekord, Iran.*

[2] *Department of Mathematics, Shahrekord Branch, Islamic Azad University, Shahrekord, Iran.*

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**Corresponding Author's Email Address: sadegh.sadeghi@gmail.com*

Abstract

Let $\mathbb{F}_q$ be a finite field with q elements, where $q = p^m$, and $R = \mathbb{F}_q + u\mathbb{F}_q$ denotes the ring $\frac{\mathbb{F}_q[u]}{\langle u^2 \rangle}$. For positive integers α and β, a nonempty subset C of $\mathbb{F}_q^\alpha \times R^\beta$ is called an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code if C is an R-submodule of $\mathbb{F}_q^\alpha \times R^\beta$. In this paper, we obtain the generator matrix of these codes and the structure of their dual codes are given. we introduce Lee weight and homogenous weight over these codes. Also, we give some bounds on the minimum distance of these codes with respect to homogenous and Lee weights. At the end, we study one-weight codes and obtain $[q^2+q, 2, q^2]$ and $[2(q+1), 2, 2q]$ one-weight optimal codes over $\mathbb{F}_q$.

1. Introduction

A subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, where α and β are positive integers, is called a $\mathbb{Z}_2\mathbb{Z}_4$-additive code [5]. The studies on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and their algebraic structures have attracted many researchers; see [1, 3, 4, 5, 12]. In [12], Dougherty et al. described one weight $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. They described the structure and possible weights for all one weight $\mathbb{Z}_2\mathbb{Z}_4$-additive codes.

Later, these codes were generalized to $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes [2]. These codes have been studied with respect to Lee weight [6]. In particular, one-Lee weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes have been studied in [19]. Also, cyclic $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes have been studied [25].

Recently, $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes were generalized to $\mathbb{Z}_p\mathbb{Z}_p[u]$-additive codes, where p is a prime number [21]. Also, these codes have been studied with respect to Lee weight. In [21], the linear and cyclic structures of these codes were given. Among other results, some optimal codes were obtained from a subclass of these codes [21].

In this paper, we give a comprehensive study on $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes, where $q = p^m$ for some prime number p and a positive integer m. We study these codes with respect to Lee and homogenous weights. The structure of this paper is as follows.

In Section 2, we introduce some notations and basic facts which will be utilized later in our discussion. In Section 3, we introduce $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes and obtain the generator matrix of these codes in a case that q is a power of a prime number. Moreover, the structure of dual codes are given. In Section 4, we generalize the Lee weight over $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes. Also, we introduce another weight function, homogenous weight, over these codes. We obtain some bounds on minimum distance of these codes with respect to these weight functions. In Section 5, we study one-weight codes with respect to Lee and homogenous weights. In this section, by the Gray image of these codes, we obtain $[q^2+q, 2, q^2]$ and $[2(q+1), 2, 2q]$ one-weight optimal codes over $\mathbb{F}_q$. In Section 6, we introduce constacyclic codes over $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes and we obtain the structure of these codes.

2. Preliminaries

We begin with some definitions for codes over rings.

From now on, suppose that R is a finite commutative ring with identity. The Jacobson radical of R is denoted by $rad(R)$. For an R-module M, left or right, the socle of M, denoted by $soc(M)$, is the sum of all minimal nonzero submodules of M.

Definition 2.1. [17] Let R be a artinian ring. If $soc(R) \cong R/rad(R)$ as right R-modules and as left R-modules, then R is called a Frobenius ring.

Definition 2.2. [20] A ring R is called a finite chain ring, if its ideals are linearly ordered by inclusion.

Lemma 2.3. *If R be a chain ring, then R is a local ring.*

Proof. The proof is clear by the definition of chain ring. □

A code is a subset of R^n and a linear code over R is an R-submodule of R^n. In this case we say that the code has length n. For a code C, we define the rank of C, denoted by $rank(C)$, to be the minimum number of generators of C.

The Hamming distance between two vectors $u, v \in R^n$ is the number of coordinates in which u and v differ from one another and it is denoted by $d_H(u,v)$. The Hamming weight of a vector $u \in R^n$, denoted by $w_H(u)$, is the number of non-zero coordinates of u. The minimum Hamming distance of a code C is denoted by $d_H(C)$ and is given as: $d_H(C) = min\{d_H(u,v) : u,v \in C, u \neq v\}$. The minimum Hamming weight of a code C, denoted by $w_H(C)$, is the minimum value of $w_H(u)$ for $u \in C \setminus \{0\}$.

We define the inner product of vectors u and $v \in R^n$ as follows:

$$u.v = \sum_{i=1}^{n} u_i v_i.$$

Let C be a linear code over R. The dual code of C, denoted by $C^\perp$, is defined as follows:

$$C^\perp = \{v \in R^n |\, u.v = 0, \text{ for all } u \in C\}.$$

Lemma 2.4. [26, Theorem 3] *Let C be a code of length n over a finite Frobenius ring. Thus, $|C||C^\perp| = |R^n|$.*

The Gray maps, which are defined in each case, have been used as tools to linked codes over rings and codes over finite fields. Gray maps from $\mathbb{Z}_4^n$ to $\mathbb{Z}_2^{2n}$ were effectively used by Sloane, Calderbank, et al. in their work [14], as a tool to obtain the binary nonlinear Kerdock, Preparata, and Goethals codes as the Gray images of linear codes over $\mathbb{Z}_4$. Carlet, in [6], extended this map to $\mathbb{Z}_{2^k}$ with the homogeneous weight and used this to obtain the generalized Kerdock codes that were non-linear binary codes with large minimum distances. Several other authors, like Ling generalized the notion of Gray maps to more general rings with certain homogeneous weights defined on them in [18]. In this paper, we use the Gray map used in paper [16], which is defined for chain rings.

3. $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes

Throughout this paper, $\mathbb{F}_q$ denotes a finite field with q elements, where $q = p^m$ is a power of a prime number p. The ring $\mathbb{F}_q + u\mathbb{F}_q$ consists of all polynomials of degrees 0 and 1 in an indeterminate u over $\mathbb{F}_q$, and it is closed under polynomial addition and multiplication modulo u^2. Thus $\mathbb{F}_q + u\mathbb{F}_q = \frac{\mathbb{F}_q[u]}{\langle u^2 \rangle} = \{\delta + \theta u : \ \delta, \theta \in \mathbb{F}_q\}$. It is easy to see that $\mathbb{F}_q + u\mathbb{F}_q$ is a chain ring with the maximal ideal $\mathfrak{m} = \langle u \rangle$.

Also, we have the following ring homomorphism:

$$\tau : \mathbb{F}_q + u\mathbb{F}_q \longrightarrow \mathbb{F}_q, \ \delta + \theta u \longmapsto \delta.$$

Hence $\mathbb{F}_q$ is an $(\mathbb{F}_q + u\mathbb{F}_q)$-module, where for $\delta + \theta u \in \mathbb{F}_q + u\mathbb{F}_q$ and $c \in \mathbb{F}_q$, the scalar multiplication is defined as $(\delta + \theta u).c = \delta c \in \mathbb{F}_q$. From now on, we denote $\mathbb{F}_q + u\mathbb{F}_q$ by R.

In this section, we introduce and study basic facts of $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes. In particular, the structure of these codes and their dual codes are given.

Definition 3.1. Let α and β be two positive integers. A nonempty subset C of $\mathbb{F}_q^\alpha \times R^\beta$ is called an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code if C is an R-submodule of $\mathbb{F}_q^\alpha \times R^\beta$ with the following scalar multiplication, where for $r = \delta + \theta u \in R$ and $(a_\alpha, b_\beta) = (a_0, a_1, ..., a_{\alpha-1}, b_0, b_1, ..., b_{\beta-1}) \in C$,

$$r.(a_\alpha, b_\beta) = (\delta a_\alpha, r b_\beta) =$$
$$(\delta a_0, \delta a_1, ..., \delta a_{\alpha-1}, r b_0, r b_1, ..., r b_{\beta-1}).$$

By the same argument as [21, Theorem 1], the following theorem gives the generator matrix of $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes.

Theorem 3.2. *Let $C \subseteq \mathbb{F}_q^\alpha \times R^\beta$ be an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code. Then C is permutation equivalent to an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code with the standard form matrix*

$$G = \left(\begin{array}{cc|ccc} I_{k_0} & A_1 & 0 & 0 & uB \\ 0 & A_2 & I_{k_1} & D_1 & D_2+uD_3 \\ 0 & 0 & 0 & uI_{k_2} & uD_4 \end{array}\right), \quad (3.1)$$

where I_{k_0}, I_{k_1} and I_{k_2} denote the $k_0 \times k_0$, $k_1 \times k_1$ and $k_2 \times k_2$ identity matrices. respectively, A_1, A_2, B, D_1, D_2, D_3 and D_4 are matrices over $\mathbb{F}_q$. Moreover, $|C| = q^{k_0+2k_1+k_2}$.

An $\mathbb{F}_q\mathbb{F}_q[u]$-additive code $C \subseteq \mathbb{F}_q^\alpha \times R^\beta$ with the generator matrix given in Equation 3.1 is said to be of type $(\alpha, \beta, k_0, k_1, k_2)$, where $k = k_0 + k_1 + k_2$ is called the rank of C and denoted by rank(C).

The inner product for two elements $x, y \in \mathbb{F}_q^\alpha \times R^\beta$ is defined as follows:

$$x.y := u(\sum_{i=0}^{\alpha-1} x_i y_i) + \sum_{i=\alpha}^{\alpha+\beta-1} x_i y_i.$$

For an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code C, $C^\perp$ is the dual of C with respect to the above inner product.

Proposition 3.3. *Let $C \subseteq \mathbb{F}_q^\alpha \times R^\beta$ be an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code. Then*

(1) $|C||C^\perp| = |\mathbb{F}_q^\alpha \times R^\beta|$.

(2) $(C^\perp)^\perp = C$.

Proof. (1) Define $\varphi : \mathbb{F}_q^\alpha \times R^\beta \longrightarrow R^\alpha \times R^\beta$ by $(x, y) \longmapsto (xu, y)$. It is easy to see that φ is an injective R-module homomorphism. Hence, $\varphi(C)$ is a linear code over R of length $\alpha + \beta$, where $|\varphi(C)| = |C|$. Now, let $\varphi(C)^\perp$ be the dual of $\varphi(C)$ with respect to the standard inner product over R. Then, $\varphi(C)^\perp = \{(a+bu, c+du) : (a, c+du) \in C^\perp\}$. Since $b \in \mathbb{F}_q^\alpha$, so $|\varphi(C)^\perp| = |C^\perp||\mathbb{F}_q|^\alpha = |C^\perp|q^\alpha$. Since R is a Frobenius ring, by Lemma 2.4, we have $|\varphi(C)^\perp||\varphi(C)| = |R|^{\alpha+\beta} = q^{2\alpha+2\beta}$. Thus, $|C^\perp|q^\alpha = \frac{q^{2\alpha+2\beta}}{|\varphi(C)|} = \frac{q^{2\alpha+2\beta}}{|C|}$. This shows that $|C^\perp||C| = \frac{q^{2\alpha+2\beta}}{q^\alpha} = q^{\alpha+2\beta} = |\mathbb{F}_q^\alpha \times R^\beta|$.

(2) Clearly $C \subseteq (C^\perp)^\perp$. Now, by part (1), $|C| = \frac{|\mathbb{F}_q^\alpha \times R^\beta|}{|C^\perp|}$ and $|C^\perp| = \frac{|\mathbb{F}_q^\alpha \times R^\beta|}{|(C^\perp)^\perp|}$. Thus $|C| = |(C^\perp)^\perp|$, which completes the proof. □

The following theorem gives the generator matrix of $C^\perp$.

Theorem 3.4. *Let C be an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code of type $(\alpha, \beta, k_0, k_1, k_2)$ with the standard form matrix defined in Equation 3.1. Then the generator matrix for $C^\perp$ is given by*

$$H = \left(\begin{array}{cc|ccc} -A_1^t & I_{\alpha-k_0} & -uA_2^t & 0 & 0 \\ -B^t & 0 & -(D_2+uD_3)^t + D_4^t D_1^t & -D_4^t & I_{\beta-k_1-k_2} \\ 0 & 0 & -uD_1^t & uI_{k_2} & 0 \end{array}\right).$$

Proof. Let $\widetilde{C}$ be the $\mathbb{F}_q\mathbb{F}_q[u]$-additive code, which is generated by H. If G be the generator matrix of C, it is easy to see that $HG^t = 0$. Hence, $\widetilde{C} \subseteq C^\perp$. From the generator matrices of C and $\widetilde{C}$, we have that $|C| = q^{k_0+2k_1+k_2}$ and $|\widetilde{C}| = q^{\alpha-k_0+2(\beta-k_1-k_2)+k_2} = q^{\alpha+2\beta-k_0-2k_1-k_2}$. By the proposition 3.3, $|C^\perp| = \frac{|\mathbb{F}_q^\alpha \times R^\beta|}{|C|} = \frac{q^{\alpha+2\beta}}{q^{k_0+2k_1+k_2}} = |\widetilde{C}|$. Hence, $\widetilde{C} = C^\perp$ and the proof is completed. □

4. Weight functions over $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes

In this section, we introduce two weight functions, homogenous weight and Lee weight, over $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes. First, note that R is a chain ring with the maximal ideal $\mathfrak{m} = \langle u \rangle$, nilpotency index 2, and residue field $R/\mathfrak{m} = \mathbb{F}_q$. A homogenous weight over R is defined as follows:

$$\omega_{hom}(t) = \begin{cases} q-1, & t \in R \setminus \mathfrak{m}; \\ q, & t \in \mathfrak{m} \setminus \{0\}; \\ 0, & t = 0. \end{cases}$$

Now, the weight function ω over $\mathbb{F}_q^\alpha \times R^\beta$ is defined as $\omega(x, y) = \omega_H(x) + \omega_{hom}(y)$, where $(x, y) \in \mathbb{F}_q^\alpha \times R^\beta$ and ω_H denotes the Hamming weight. Also, the distance between any two codewords is the weight of their difference; for $(x, y), (x', y') \in \mathbb{F}_q^\alpha \times R^\beta$, $d_\omega((x, y), (x', y')) = \omega(x - x', y - y')$. In partiqular, for an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code $C \subseteq \mathbb{F}_q^\alpha \times R^\beta$, the nonzero minimum distance between the codewords in C is denoted by $d_{hom}(C)$. By [16, Proposition 3.1], there exists a Gray map from (R^β, d_{hom}) to $(\mathbb{F}_q^{q\beta}, d_H)$, where d_H denotes the Hamming distance on $\mathbb{F}_q^{q\beta}$. Let φ_{hom} :

$R^\beta \longrightarrow \mathbb{F}_q^{q\beta}$ be the defined Gray map in [16]. Then the map $(id, \varphi_{hom}) : \mathbb{F}_q^\alpha \times R^\alpha \longrightarrow \mathbb{F}_q^{\alpha+q\beta}$ is an isometry which transforms the homogenous distance in $\mathbb{F}_q^\alpha \times R^\beta$ to the Hamming distance in $\mathbb{F}_q^{\alpha+q\beta}$. In this paper, we denote (id, φ_{hom}) by Φ_{hom}.

Example 4.1. Let C be an $\mathbb{F}_5\mathbb{F}_5[u]$-additive code of type $(6, 31, 1, 1, 0)$ with the following standard form matrix

$$G = \left(\begin{array}{cccccc|cccccccccc} 1 & 0 & 2 & 3 & 4 & 1 & 0 & \mathbf{0} & u & \mathbf{B} & \mathbf{B} & \mathbf{B} & \mathbf{B} & \mathbf{B} & \mathbf{B} \\ 0 & 1 & 4 & 3 & 2 & 4 & 1 & \mathbf{D} & 0 & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{u} & \mathbf{1+u} \end{array}\right),$$

where $\mathbf{B} = (u\ 2u\ 3u\ 4u)$ and $\mathbf{D} = (2\ 3\ 4\ u\ 1{+}u)$. By Theorem 3.2, $|C| = 5^{k_0+2k_1+k_2} = 5^3$. Also, we can see that $\Phi_{hom}(C)$ is a $[161, 3, 125]$ linear code over $\mathbb{F}_5$ with respect to Hamming weight.

Now, we introduce another weight function over these codes. In [21], the Lee weight over $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes, in the case that q is a prime number, is defined. We give this weight function on $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes in general.

The Gray map defined on R can be expressed as follows:

$$\phi' : R \longrightarrow \mathbb{F}_q^2,\ \phi'(a + bu) = (b, a + b).$$

The Lee weight is $\omega_L(a + bu) = \omega_H(b, a + b)$. The map ϕ' and the Lee weight ω_L generalize over R^β naturally; for all $c := (a_1 + b_1u, \cdots, a_\beta + b_\beta u) \in R^\beta$, we have $\phi'(c) = (b_1, \cdots, b_\beta, a_1 + b_1, \cdots, a_\beta + b_\beta)$ and $\omega_L(c) = \omega_H(\phi(c))$.

A weight function ω' over $\mathbb{F}_q^\alpha \times R^\beta$ is defined as $\omega'(x, y) = \omega_H(x) + \omega_L(y)$, where $(x, y) \in \mathbb{F}_q^\alpha \times R^\beta$. Also, the distance between any two codewords is defined by the argument as homogenous weight. Now, the following map is an isometry which transforms the Lee distance in $\mathbb{F}_q^\alpha \times R^\beta$ to the Hamming distance in $\mathbb{F}_q^{\alpha+2\beta}$.

$$\Phi_{Lee} : \mathbb{F}_q^\alpha \times R^\beta \longrightarrow \mathbb{F}_q^{\alpha+2\beta},\ (x, y) \longmapsto (x, \phi'(y)).$$

From now on, we call this weight function, Lee weight [21]. In addition, If $C \subseteq \mathbb{F}_q^\alpha \times R^\beta$ is an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code, we denote the non-zero minimum distance between the codewords in C by $d_L(C)$.

Example 4.2. Let C be an $\mathbb{F}_3\mathbb{F}_3[u]$-additive code of type $(4, 2, 1, 1, 0)$ with the following standard form matrix

$$G = \left(\begin{array}{cccc|cc} 1 & 0 & 1 & 1 & 0 & u \\ 0 & 1 & 1 & 2 & 1 & 0 \end{array}\right).$$

By Theorem 3.2, $|C| = 3^{k_0+2k_1+k_2} = 3^3$. Also, one can see that $\Phi_{Lee}(C)$ is a $[8, 3, 2]$ linear code over $\mathbb{F}_3$ with respect to Hamming weight.

Lemma 4.3. [24, Theorem 1] *Let C be a q-ary code of parameters (n, M, d_H), where M is the size of C. Then, the Singleton bound is as follows:*

$$d_H(C) \leq n - \log_q |M| + 1.$$

The following theorem gives some bounds on $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes with respect to homogenous and Lee weights.

Theorem 4.4. *Let C be an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code of type $(\alpha, \beta, k_0, k_1, k_2)$. Then*

(1) $\frac{d_{hom}(C)-1}{q} \leq \frac{\alpha}{q} + \beta - \frac{k_0+2k_1+k_2}{q}$;

(2) $\lfloor \frac{d_{hom}(C)-1}{q} \rfloor \leq \alpha + \beta - (k_0 + k_1 + k_2)$;

(3) $\frac{d_L(C)-1}{2} \leq \frac{\alpha}{2} + \beta - \frac{k_0+2k_1+k_2}{2}$;

(4) $\lfloor \frac{d_L(C)-1}{2} \rfloor \leq \alpha + \beta - (k_0 + k_1 + k_2)$,

where $\lfloor x \rfloor$ refers to the greatest integer less than or equal to x.

Proof. *(1)* Consider the Gray map $\Phi_{hom} : \mathbb{F}_q^\alpha \times R^\beta \longrightarrow \mathbb{F}_q^{\alpha+q\beta}$. Since $\Phi_{hom}(C) \subseteq \mathbb{F}_q^{\alpha+q\beta}$, thus $\Phi_{hom}(C)$ is a code over $\mathbb{F}_q$ of length $\alpha + q\beta$. Now by the Singleton bound, we have $d_H(\Phi_{hom}(C)) \leq \alpha + q\beta - \log_q |\Phi_{hom}(C)| + 1$. But $d_H(\Phi_{hom}(C)) = d_{hom}(C)$ and $|\Phi_{hom}(C)| = |C| = q^{k_0+2k_1+k_2}$. This completes the proof.

(2) Define the map $\rho : \mathbb{F}_q \longrightarrow R$ by $a \longmapsto au$. It is easy to see that ρ is well defined and injective. Now for $b + cu \in R$, $\rho((b + cu).a) = \rho(ba) = bau = (b + cu).au = (b + cu).\rho(a)$. Hence ρ is an R-module homomorphism. Also, denote the natural generalization of ρ on $\mathbb{F}_q^\alpha$ by ρ. Thus the map $(\rho, id) : \mathbb{F}_q^\alpha \times R^\beta \longrightarrow R^{\alpha+\beta}$ is an injective R-module homomorphism. Now $\omega_H(a) = 1 \leq q = \omega_{hom}(au)$ for all $a \in \mathbb{F}_q$. Therefore $(\rho, id)(C)$ is a linear code with $d_{hom}(C) \leq d_{hom}((\rho, id)(C))$. If A is the maximum weight of elements in R, then by [22, Theorem 3.7], $\lfloor \frac{d_{hom}(C)-1}{A} \rfloor \leq \lfloor \frac{d_{hom}((\rho,id)(C))-1}{A} \rfloor \leq \alpha + \beta - \text{rank}((\rho, id)(C)) = \alpha + \beta - \text{rank}(C) = \alpha + \beta - (k_0 + k_1 + k_2)$. But $A = q$ which completes the proof.

(3) Consider the Gray map $\Phi_{Lee} : \mathbb{F}_q^\alpha \times R^\alpha \longrightarrow \mathbb{F}_q^{\alpha+2\beta}$ instead of the Gray map Φ_{hom} in part (1). By the same argument as part (1), the result is followed.

(4) Consider the map $(\rho, id) : \mathbb{F}_q^\alpha \times R^\beta \longrightarrow R^{\alpha+\beta}$ in part (2). Then $\omega_H(a) = 1 \leq 2 = \omega_H(a, a) = \omega_L(au)$

for all $a \in \mathbb{F}_q$. Also, the maximum Lee weight of elements in R is equal to $A = 2$. Now, by the same argument as part (2), the result is obtained. $\square$

5. One-weight $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes

An $\mathbb{F}_q\mathbb{F}_q[u]$-additive code is said to have one-Lee (or homogenous) weight if every non-zero codeword has the same Lee (or homogenous) weight. In this section, we study the properties of one-Lee weight and one-homogenous weight $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes. We give the exact structure of some large classes of one-weight codes. In particular, we introduce two classes of one-weight codes that their Gray images are $[q^2+q, 2, q^2]$ and $[2(q+1), 2, 2q]$ one-weight optimal codes over $\mathbb{F}_q$.

One-homogenous weight codes

Now, we study one-homogenous weight $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes.

Lemma 5.1. *Let $C \subseteq \mathbb{F}_q^\alpha \times R^\beta$ be an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code. Then*

$$\sum_{c\in C} \omega(c) = \frac{|C|(q-1)}{q}(\alpha + q\beta).$$

Proof. We write the codewords of C as rows of a matrix G. Consider the column j of G, where $\alpha+1 \le j \le \alpha+\beta$. Let J be the ideal of R generated by all elements of the column j. Since R is a chain ring and $\mathfrak{m}$ is the maximal ideal of R, so $J = \mathfrak{m}$ or $J = R$. Now, we consider the following two cases:

Case 1 ($J = \mathfrak{m} = \langle u\rangle$). Since C is an R-submodule, any element of J is an element of the column j. Now we show that any two elements of J have the same repetition number in the column j. Let au and bu be two elements of J with the repetition numbers n_a and n_b, respectively. Then $au = ab^{-1}(bu)$ and hence $n_a \ge n_b$. By the same argument, $n_b \ge n_a$. So $n_a = n_b$. Therefore, the sum of the weights of all elements of the column j is equal to

$$\frac{|C|}{|J|}(\sum_{s\in J}\omega_{hom}(s)) = \frac{|C|}{|J|}(q|J\setminus\{0\}|) =$$

$$\frac{|C|}{q}(q(q-1)) = |C|(q-1).$$

Case 2 ($J = R$). In this case, there exists an invertible element c_j in the column j. If c'_j is an element of J, then $c'_j = (c'_j c_j^{-1})c_j$. Thus any element of $J = R$ is an element of the column j. Now, let $c_1 = a_1 + b_1u$ and $c_2 = a_2 + b_2u$ be two elements of J. We show that c_1 and c_2 have the same repetition number in the column j. If a_1 and a_2 are non-zero elements, then c_1 and c_2 are invertible. Hence $c_1 = (c_1c_2^{-1})c_2$ and $c_2 = (c_2c_1^{-1})c_1$. This shows that c_1 and c_2 have the same repetition number. If $a_1 = a_2 = 0$, then by the same argument as case 1, $n_{c_1} = n_{c_2}$. Now, let a_1 be a non-zero element and $a_2 = 0$. Then c_1 is invertible. We have that $c_2 = (c_2c_1^{-1})c_1$ which proves $n_{c_2} \ge n_{c_1}$. Also, $c_1 = (a_1c_j^{-1})c_j + (b_1b_2^{-1})c_2$ which shows that $n_{c_1} \ge n_{c_2}$. Thus, the elements of J have the same repetition number $\frac{|C|}{|J|} = \frac{|C|}{q^2}$.

Therefore, the sum of the weights of all elements of the column j is equal to

$$\frac{|C|}{q^2}(\sum_{s\in R\setminus\mathfrak{m}}\omega_{hom}(s) + \sum_{s\in\mathfrak{m}\setminus\{0\}}\omega_{hom}(s)) =$$

$$\frac{|C|}{q^2}((q^2-q)(q-1) + (q-1)q) = |C|(q-1).$$

If $1 \le j \le \alpha$, then the ideal J of R generated by all elements of the column j is equal to $\mathbb{F}_q$. Since all elements of $\mathbb{F}_q$ are invertible, they have the same repetition number. Hence, the sum of the weights of all elements of the column j is equal to

$$\frac{|C|}{|\mathbb{F}_q|}(\sum_{s\in\mathbb{F}_q}\omega_H(s)) = \frac{|C|}{q}(q-1).$$

Therefore,

$$\sum_{c\in C}\omega(c) = \frac{|C|(q-1)}{q}(\alpha + q\beta).$$

$\square$

Theorem 5.2. *Let $C \subseteq \mathbb{F}_q^\alpha \times R^\beta$ be a one-homogenous weight $\mathbb{F}_q\mathbb{F}_q[u]$-additive code with weight m. Then there exists a unique positive integer λ such that $m = \lambda\frac{|C|}{q}$ and $\alpha + q\beta = \lambda(\frac{|C|-1}{q-1})$.*

Proof. By the lemma 5.1, we have

$$\sum_{c\in C}\omega(c) = \frac{|C|(q-1)}{q}(\alpha+q\beta).$$

On the other hand, the sum of the weights of all codewords is $(|C|-1)m$. Hence, $\frac{|C|(q-1)}{q}(\alpha+q\beta) = (|C|-1)m$. Since $|C| = q^{k_0+2k_1+k_2}$, $\gcd(\frac{|C|}{q}, (|C|-1)) = 1$. Therefore, there exists a positive integer λ such that $m = \lambda\frac{|C|}{q}$, and hence $(q-1)(\alpha+q\beta) = \lambda(|C|-1)$. $\square$

The following theorem, determines a class of one-homogenous weight codes.

Theorem 5.3. *Let $C \subseteq \mathbb{F}_q^\alpha \times R^\beta$ be a one-homogenous weight $\mathbb{F}_q\mathbb{F}_q[u]$-additive code with weight m. If m is odd, and $q = 2^s$ for some positive integer s, then* $C = \{(\underbrace{\theta, ..., \theta}_{\alpha}, \underbrace{\theta u, ..., \theta u}_{\beta}) : \theta \in \mathbb{F}_q\}$.

Proof. Clearly, $C = \{(\theta, ..., \theta, \theta u, ..., \theta u) : \theta \in \mathbb{F}_q\}$ is a one-homogenous weight code of weight $\alpha + q\beta$. Now, we show that any one-homogenous weight code has this form. By the Theorem 5.2, $m = \lambda\frac{|C|}{q}$. Since m is odd and $q = 2^s$, λ should be an odd integer and $\frac{|C|}{q} = 1$. Hence, $m = \lambda = \alpha + q\beta$. Let $(a, b) = (a_1, ..., a_\alpha, b_1, ..., b_\beta)$ be a codeword in C. If $a_i = 0$ for some i, or $b_j \notin \mathfrak{m}$, then $\omega(a, b) < \alpha + q\beta$; a contradiction. Hence, a_i is a non-zero element of $\mathbb{F}_q$ for all $1 \leq i \leq \alpha$, and $b_j = \theta_j u$ where θ_j is a non-zero element of $\mathbb{F}_q$ for $1 \leq j \leq \beta$. Clearly, $(\delta a, \delta b) \in C$ for any $\delta \in \mathbb{F}_q$. Thus, if $a_i \neq \theta_j$ for some $i \neq j$, $|C| > q$; is a contradiction. Hence, $C = \{(\underbrace{\theta, ..., \theta}_{\alpha}, \underbrace{\theta u, ..., \theta u}_{\beta}) : \theta \in \mathbb{F}_q\}$. □

Now, we introduce a class of optimal one-homogenous weight codes. The following lemma gives the well-known Griesmer bound for linear codes.

Lemma 5.4. [15, Theorem 2.7.4] *Let C be a q-ary code of parameters $[n, k, d_H]$, where $k \geq 1$. Then the Griesmer bound is as follows:*

$$n \geq \sum_{i=0}^{k-1} \lceil \frac{d_H}{q^i} \rceil.$$

Definition 5.5. If a linear code C over a finite field $\mathbb{F}_q$ meets the Griesmer bound, then C is called optimal.

Suppose that $\mathbb{F}_q = \{0, f_1 = 1, f_2, ..., f_{q-1}\}$. The following theorem gives a class of optimal codes.

Theorem 5.6. *Let C be an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code of type $(q, q, 1, 0, 1)$ with the following standard form matrix:*

$$G = \left(\begin{array}{c|cc} \mathbf{a} & 0 & \mathbf{b} \\ \mathbf{0} & u & \mathbf{d} \end{array}\right),$$

where $\mathbf{a} = (\underbrace{1, 1, \cdots, 1}_{q \text{ times}})$, $\mathbf{b} = (\underbrace{u, u, \cdots, u}_{(q-1) \text{ times}})$ *and* $\mathbf{d} = (f_1 u, f_2 u, \cdots, f_{(q-1)} u)$. *Then C is a one-homogenous weight code with weight $m = q^2$. Also, $\Phi_{hom}(C)$ is an optimal one-Hamming weight code with parameters $[q^2 + q, 2, q^2]$.*

Proof. Let c be a codeword in C. Then by the definition of $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes and the structure of G, $c = (\delta\mathbf{a} \;\; 0 \;\; \delta\mathbf{b}) + (\mathbf{0} \;\; \delta' u \;\; \delta'\mathbf{d}) = (\delta\mathbf{a} \;\; \delta' u \;\; \delta\mathbf{b} + \delta'\mathbf{d})$ for some $\delta, \delta' \in \mathbb{F}_q$. But the weight of c is equal to

$$\omega(c) = \omega_h(\delta\mathbf{a}) + \omega_{hom}(\delta' u) + \omega_{hom}(\delta\mathbf{b} + \delta'\mathbf{d}) =$$
$$q + q + \omega_{hom}(\delta\mathbf{b} + \delta'\mathbf{d}).$$

Since $\mathbb{F}_q$ is a field, there exists only one element $\theta \in \mathbb{F}_q$ such that $\delta + \theta = 0$. But $\mathbb{F}_q(\delta') = \mathbb{F}_q$. Hence, there exists only one integer $1 \leq i \leq q-1$ such that $\delta + f_i\delta' = 0$. This shows that $\omega_{hom}(\delta\mathbf{b} + \delta'\mathbf{d}) = (q-2)q$. Therefore, $\omega(c) = q^2$. It is easy to see that $\Phi_{hom}(C)$ is a one-Hamming weight code with parameters $[q^2 + q, 2, q^2]$. Now, $q^2 + q = \lceil \frac{q^2}{1} \rceil + \lceil \frac{q^2}{q} \rceil$. □

Example 5.7. Let C be an $\mathbb{F}_5\mathbb{F}_5[u]$-additive code of type $(5, 5, 1, 0, 1)$ with the following standard form matrix:

$$G = \left(\begin{array}{ccccc|ccccc} 1 & 1 & 1 & 1 & 1 & 0 & u & u & u & u \\ 0 & 0 & 0 & 0 & 0 & u & u & 2u & 3u & 4u \end{array}\right).$$

Then C is a one-homogenous weight code with weight $m = 25$. Also, $|C| = 5^{k_0+2k_1+k_2} = 5^2$, and hence $\lambda = 5$. Moreover, $\Phi_{hom}(C)$ is a one-Hamming weight $[30, 2, 25]$ code over $\mathbb{F}_5$ which is an optimal code.

One-Lee weight codes

Now, we study one-Lee weight $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes.

Lemma 5.8. *Let $C \subseteq \mathbb{F}_q^\alpha \times R^\beta$ be an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code. Then*

$$\textstyle\sum_{c\in C} \omega'(c) = \frac{|C|(q-1)}{q}(\alpha + 2\beta).$$

Proof. By the same argument as Lemma 5.1, write the codewords of C as rows of a matrix G. Consider the column j of G, where $\alpha + 1 \leq j \leq \alpha + \beta$. Let J be the ideal of R generated by all elements of the column j. Then $J = \mathfrak{m}$ or $J = R$. Hence, we have the following two cases: Case 1 ($J = \mathfrak{m} = \langle u \rangle$). Clearly, any element of the column j is of the form bu for some $b \in \mathbb{F}_q \setminus \{0\}$. By case 1 of Lemma 5.1, any element of J is an element of the column j. Also, any two elements of J have the same repetition number in the column j. Hence, the sum of the weights of all elements of the column j is equal to

$$\frac{|C|}{|J|}(\sum_{s\in J} \omega_L(s)) = \frac{|C|}{|J|}(2|J \setminus \{0\}|) =$$
$$\frac{|C|}{q}(2(q-1)) = \frac{2|C|}{q}(q-1).$$

Case 2 ($J = R$). By case 2 of Lemma 5.1, any element of $J = R$ is an element of the column j and all elements have the same repetition number. Now, let $a + bu$ be an element of the column j. Then, we have the following cases:

(1) $a = 0, b \neq 0$;
(2) $a \neq 0, b = 0$;
(3) $a, b \neq 0$ and $a = -b$;
(4) $a, b \neq 0$ and $a \neq -b$.

Hence, the sum of the weights of all elements of the column j is equal to

$$\frac{|C|}{q^2}(2(q-1)+q-1+q-1+2(q-1)(q-2)) =$$

$$\frac{|C|(q-1)}{q^2}(4+2(q-2)) = \frac{2|C|}{q}(q-1).$$

If $1 \leq j \leq \alpha$, then by the same argument as Lemma 5.1, the sum of the weights of all elements of the column j is equal to

$$\frac{|C|}{|\mathbb{F}_q|}(\sum_{s\in\mathbb{F}_q}\omega_H(s)) = \frac{|C|}{q}(q-1).$$

Therefore,

$$\sum_{c\in C}\omega'(c) = \frac{|C|(q-1)}{q}(\alpha+2\beta).$$

□

Theorem 5.9. *Let $C \subseteq \mathbb{F}_q^\alpha \times R^\beta$ be a one-Lee weight $\mathbb{F}_q\mathbb{F}_q[u]$-additive code with weight m. Then, there exists a unique positive integer λ such that $m = \lambda\frac{|C|}{q}$ and $\alpha + 2\beta = \lambda(\frac{|C|-1}{q-1})$.*

Proof. It is proved by the same argument as Theorem 5.2. □

Now, we determine the structure of one-Lee weight codes in the case that m is odd.

Theorem 5.10. *Let C be a one-Lee weight $\mathbb{F}_q\mathbb{F}_q[u]$-additive code of type $(\alpha, \beta, k_0, k_1, k_2)$ with weight m, where m is odd. Then, $k_1 = 0$ and according to Theorem 3.2, the generator matrix of C is given by the following*

$$G = \left(\begin{array}{cc|ccc} I_{k_0} & A_1 & 0 & 0 & uB \\ 0 & 0 & 0 & uI_{k_2} & uD_4 \end{array}\right).$$

Proof. Let $k_1 \neq 0$ and c be a vector in $(0\ \ A_2\ \ I_{k_1}\ \ D_1\ \ D_2+uD_3)$. since C is an R-submodule of $\mathbb{F}_q^\alpha \times R^\beta$, by the Definition 3.1 we have:

$$u.c \in u(0\ \ A_2\ \ I_{k_1}\ \ D_1\ \ D_2+uD_3) =$$
$$(0\ \ 0\ \ uI_{k_1}\ \ uD_1\ \ uD_2).$$

Thus, C contains a non-zero vector of even weight and it is a contradiction. □

The following theorem gives the exact structure of one weight codes with odd distances if q is an even integer.

Theorem 5.11. *Let $C \subseteq \mathbb{F}_q^\alpha \times R^\beta$ be a one-Lee weight $\mathbb{F}_q\mathbb{F}_q[u]$-additive code with weight m. If m is odd, and $q = 2^s$ for some positive integer s, then $C = \{(\underbrace{\theta,...,\theta}_{\alpha},\underbrace{\theta u,...,\theta u}_{\beta}) : \theta \in \mathbb{F}_q\}$.*

Proof. Clearly, $C = \{(\theta,...,\theta,\theta u,...,\theta u) : \theta \in \mathbb{F}_q\}$ is a one-Lee weight code of weight $\alpha + 2\beta$. Now, we show that any one-Lee weight code has this form. By the above theorem, $m = \lambda\frac{|C|}{q}$. Since m is odd and $q = 2^s$, λ should be an odd integer and $\frac{|C|}{q} = 1$. Hence, $m = \lambda = \alpha + 2\beta$. Now, let $(\theta, \delta + \theta u) = (\varepsilon_1, ..., \varepsilon_\alpha, \delta_1 + \theta_1 u, ..., \delta_\beta + \theta_\beta u)$ be a codeword in C. Since $m = \alpha + 2\beta$, $\varepsilon_i \neq 0$ for all i and we have the following two cases:

(1) $\delta_i = 0, \theta_i \neq 0$;
(2) $\delta_i, \theta_i \neq 0$ and $\delta_i \neq -\theta_i$.

But $|C| = q$. Hence, $\delta_i = 0$ for all i, and $\varepsilon_i = \theta_j$ for all $1 \leq i \leq \alpha$ and $1 \leq j \leq \beta$. This completes the proof. □

Now, the following theorem gives a class of one-Lee weight optimal codes.

Theorem 5.12. *Let C be an $\mathbb{F}_q\mathbb{F}_q[u]$-additive code of type $(2, q, 1, 0, 1)$ with the following standard form matrix*

$$G = \left(\begin{array}{c|cc} \mathbf{a} & 0 & \mathbf{b} \\ \mathbf{0} & u & \mathbf{d} \end{array}\right),$$

where $\mathbf{a} = (1, 1)$, $\mathbf{b} = (\underbrace{u, u, \cdots, u}_{(q-1)\ times})$ and $\mathbf{d} = (f_1u, f_2u, \cdots, f_{(q-1)}u)$. Then C is a one-Lee weight code with weight $m = 2q$. Also, $\Phi_{Lee}(C)$ is an optimal one-Hamming weight code with parameters $[2(q+1), 2, 2q]$.

Proof. It is proved by the same argument as Theorem 5.6. ☐

Example 5.13. Let C be an $\mathbb{F}_3\mathbb{F}_3[u]$-additive code of type $(2,3,1,0,1)$ with the following standard form matrix

$$G = \left(\begin{array}{cc|ccc} 1 & 1 & 0 & u & u \\ 0 & 0 & u & 2u & u \end{array}\right).$$

Then C is a one-Lee weight code with weight $m = 6$. Also, $|C| = 3^{k_0+2k_1+k_2} = 3^2$, and hence $\lambda = 2$. Moreover, $\Phi_{Lee}(C)$ is a one-Hamming weight $[8,2,6]$ linear code over $\mathbb{F}_3$ which is an optimal code.

6. Constacyclic $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes

Recently, Qian et al.[21] have studied the cyclic codes over $\mathbb{Z}_p\mathbb{Z}_p[u]$-additive codes. In this section, we introduce constacyclic $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes and obtain the structure of these codes.

Constacyclic codes over R

To obtain the structure of constacyclic additive codes over $\mathbb{F}_q \times R$, we need the structure of linear constacyclic codes over $\mathbb{F}_q$ and R. There are many researches about linear constacyclic codes over $\mathbb{F}_q$ and the structure of these codes for arbitrary length has been given; see [8, 23]. In this section, we remind some results about linear constacyclic codes over R.

Since R is a chain ring, all elements of $R \setminus \mathfrak{m}$ are unit. Hence, R has precisely $q(q-1)$ units, which are of the form $\delta + \theta u$, where $\delta \in \mathbb{F}_q \setminus \{0\}$ and $\theta \in \mathbb{F}_q$. Let $\lambda = \delta + \theta u$ be a unit of R. Consider the following correspondence:

$$\begin{aligned} p_\lambda : R^n &\longrightarrow R[x]/\langle x^n - \lambda\rangle, \\ (a_0, a_1, ..., a_{n-1}) &\longmapsto a_0 + a_1x + ... + a_{n-1}x^{n-1} + \langle x^n - \lambda\rangle. \end{aligned}$$

Clearly p_λ is an R-module isomorphism. Also, it is easy to see that C is a λ-constacyclic code if and only if $p_\lambda(C)$ is an ideal of $R[x]/\langle x^n - \lambda\rangle$. We will identify $(R)^n$ with $R[x]/\langle x^n - \lambda\rangle$ under p_λ and for simplicity, we write the polynomial $a_0 + a_1x + ... + a_{n-1}x^{n-1}$ instead of the residue class $a_0 + a_1x + ... + a_{n-1}x^{n-1} + \langle x^n - \lambda\rangle$. By this correspondence, to obtain the structure of λ-constacyclic codes over R, we determine the ideals of $R[x]/\langle x^n - \lambda\rangle$.

In this paper, we denote the residue ring $R[x]/\langle x^n - \lambda\rangle$ by $R_{n,\lambda}$. Also, for the non-zero elements $\delta, \theta \in \mathbb{F}_q$, the residue rings $\mathbb{F}_q[x]/\langle x^n - \delta\rangle$ and $(\mathbb{F}_q)^2[x]/\langle x^n - (\delta,\theta)\rangle$ are denoted by $(\mathbb{F}_q)_{n,\delta}$ and $(\mathbb{F}_q)^2_{n,(\delta,\theta)}$ respectively.

The following theorem determines a class of constacyclic codes over R.

Theorem 6.1. [13, Theorem 4.13] *Let C be a λ-constacyclic code over R of length n such that $\gcd(n,p) = 1$. Then $C = \langle g_0(x), ug_1(x)\rangle \subseteq R_{n,\lambda}$, where $g_1(x)|g_0(x)|(x^n - \lambda)$.*

There are many studies about linear constacyclic codes over R of length n which is not coprime to p; see [7, 9, 10, 11]. We recalled the structure of these codes of length p^s from [9].

Theorem 6.2. [9, Theorem 4.4] *Let C be a $(\delta + \theta u)$-constacyclic code over R of length p^s such that $\delta, \theta \in \mathbb{F}_q \setminus \{0\}$ and $s = am + r$ for non-negative integers a, r with $0 \leq r \leq m-1$. Then $C = \langle(\delta_0 x - 1)^i\rangle \subseteq R_{p^s,\delta+\theta u}$ for some $i \in \{0, 1, \cdots, 2p^s\}$, where $\delta_0 = \delta^{-p^{m-r}}$.*

The above theorem gives the structure of $(\delta + \theta u)$-constacyclic codes of length p^s in the case that δ and θ are non-zero elements in $\mathbb{F}_q$. In the following theorems, we have the structure of these codes in the case that $\theta = 0$. First let $(\delta + \theta u) = 1$. Hence, we have cyclic codes.

Theorem 6.3. [9, Theorem 5.4] *Cyclic codes of length p^s over R, i.e., ideals of the ring $R_{p^s,1}$, are*

1) *Type 1 (trivial ideals): $\langle 0\rangle$, $\langle 1\rangle$.*

2) *Type 2 (principal ideals with non-monic polynomial generators): $\langle u(x-1)^i\rangle$, where $0 \leq i \leq p^s - 1$.*

3) *Type 3 (principal ideals with monic polynomial generators): $\langle (x-1)^i + u(x-1)^t h(x)\rangle$, where $1 \leq i \leq p^s - 1$, $0 \leq t \leq i$, and either $h(x)$ is 0 or $h(x)$ is a unit, where it can be represented as $h(x) = \sum_j h_j(x-1)^j$, with $h_j \in \mathbb{F}_q$, and $h_0 \neq 0$.*

4) *Type 4 (nonprincipal ideals): $\langle (x-1)^i + u\sum_{j=0}^{\omega-1} c_j(x-1)^j, u(x-1)^\omega\rangle$, where $1 \leq i \leq p^s - 1$, $c_j \in \mathbb{F}_q$, and $\omega < T$, where T is the smallest integer such that $u(x-1)^T \in \langle (x-1)^i + u\sum_{j=0}^{i-1} c_j(x-1)^j$; or equivalently, $\langle (x-1)^i + u(x-1)^t h(x), u(x-1)^\omega\rangle$, with $h(x)$ as in Type 3, and $\deg(h) \leq \omega - t - 1$.*

Now, by the structure of cyclic codes in the Theorem 6.3 and the ring isomorphism in the following theorem, we have the structure of δ-constacyclic codes.

Theorem 6.4. [9, Proposition 6.1] *Let δ_0 be defined such as Theorem 6.2. Then the map $\phi : R_{p^s,1} \longrightarrow R_{p^s,\delta}$*

given by $f(x) \longmapsto f(\delta_0 x)$ is a ring isomorphism. In particular, for $A \subseteq R_{p^s,1}$, $B \subseteq R_{p^s,\delta}$ if $\phi(A) = B$, then A is an ideal of $R_{p^s,1}$ if and only if B is an ideal of $R_{p^s,\delta}$. Equivalently, A is a cyclic code of length p^s over R if and only if B is a δ-constacyclic code of length p^s over R.

Constacyclic $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes

Now, by the definition of constacyclic codes in the above subsection, we define constacyclic $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes.

Definition 6.5. Let α and β be two positive integers and $(\lambda_1, \lambda_2) \in \mathbb{F}_q \times R$. An $\mathbb{F}_q\mathbb{F}_q[u]$-additive code $C \subseteq \mathbb{F}_q^\alpha \times R^\beta$ is called (λ_1, λ_2)-constacyclic if

$$(\lambda_1 s_{\alpha-1}, s_0, ..., s_{\alpha-2}, \lambda_2 r_{\beta-1}, r_0, ..., r_{\beta-2}) \in C,$$

whenever $(s_0, s_1, ..., s_{\alpha-1}, r_0, r_1, ..., r_{\beta-1}) \in C$.

Consider the map $\pi_{(\lambda_1,\lambda_2)} : \mathbb{F}_q^\alpha \times R^\beta \longrightarrow (\mathbb{F}_q)_{\alpha,\lambda_1} \times R_{\beta,\lambda_2}$ with the following definition $(s_0, s_1, ..., s_{\alpha-1}, r_0, r_1, ..., r_{\beta-1}) \longmapsto (s_0 + s_1 x + ... + s_{\alpha-1}x^{\alpha-1} + \langle x^\alpha - \lambda_1\rangle, r_{0+} r_1 x + ... + r_{\beta-1}x^{\beta-1} + \langle x^\beta - \lambda_2\rangle)$. It is easy to see that $\pi_{(\lambda_1,\lambda_2)}$ is an R-module isomorphism. We will identify $\mathbb{F}_q^\alpha \times R^\beta$ with $(\mathbb{F}_q)_{\alpha,\lambda_1} \times R_{\beta,\lambda_2}$ under $\pi_{(\lambda_1,\lambda_2)}$ and for simplicity, we write $(s_0 + s_1 x + ... + s_{\alpha-1}x^{\alpha-1}, r_{0+} r_1 x + ... + r_{\beta-1}x^{\beta-1})$ for above residue class.

Lemma 6.6. *A subset C of $\mathbb{F}_q^\alpha \times R^\beta$ is a (λ_1, λ_2)-constacyclic code if and only if $\pi_{(\lambda_1,\lambda_2)}(C)$ is an $R[x]$-submodule of $(\mathbb{F}_q)_{\alpha,\lambda_1} \times R_{\beta,\lambda_1}$.*

Proof. Clearly, $(\mathbb{F}_q)_{\alpha,\lambda_1} \times R_{\beta,\lambda_2}$ is an $R[x]$-module. Since $\pi_{(\lambda_1,\lambda_2)}$ is an R-module isomorphism, C is an R-submodule if and only if $\pi_{(\lambda_1,\lambda_2)}(C)$ is an R-submodule. Now, for an element $(s_\alpha, r_\beta) = (s_0, s_1, ..., s_{\alpha-1}, r_0, r_1, ..., r_{\beta-1}) \in C$, let $\sigma(s_\alpha, r_\beta) = (\lambda_1 s_{\alpha-1}, s_0, ..., s_{\alpha-2}, \lambda_2 r_{\beta-1}, r_0, ..., r_{\beta-2})$. Thus

$$\sigma(s_\alpha, r_\beta) \in C \Leftrightarrow x\pi_{(\lambda_1,\lambda_2)}(s_\alpha, r_\beta) =$$

$$\pi_{(\lambda_1,\lambda_2)}(\sigma(s_\alpha, r_\beta)) \in \pi_{(\lambda_1,\lambda_2)}(C).$$

This completes the proof. □

We identify C with $\pi_{(\lambda_1,\lambda_2)}(C)$. Now, we find the generator polynomials of C.

Theorem 6.7. *A subset C of $(\mathbb{F}_q)_{\alpha,\lambda_1} \times R_{\beta,\lambda_2}$ is a (λ_1, λ_2)-constacyclic code if and only if $C = \langle (g, 0), (h_1, f_1), ..., (h_r, f_r)\rangle_{R[x]}$ such that*

(1) *$C_1 = \langle g\rangle$ is a λ_1-constacyclic code over $\mathbb{F}_q$ of length α,*

(2) *$C_2 = \langle f_1, ..., f_r\rangle_{R[x]}$ is a λ_2-constacyclic code over R of length β,*

(3) *$g | x^\alpha - \lambda_1$ over $\mathbb{F}_q$,*

(4) *$h_1, ..., h_r$ are elements of $(\mathbb{F}_q)_{\lambda_1,\alpha}$,*

(5) *$|C| = |C_1||C_2|$.*

Proof. Let $C \subseteq (\mathbb{F}_q)_{\alpha,\lambda_1} \times R_{\beta,\lambda_2}$ be a (λ_1, λ_2)-constacyclic code. Clearly, the projection map $\phi : C \longrightarrow R_{\beta,\lambda_2}$ is an $R[x]$-homomorphism. Hence, $Im(\phi)$ is an $R[x]$-submodule of R_{β,λ_2}. As $\langle x^\beta - \lambda_2\rangle.Im(\phi) \subseteq \langle x^\beta - \lambda_2\rangle.R_{\beta,\lambda_2} = 0$, $Im(\phi)$ is an ideal of R_{β,λ_2}. In other words, $Im(\phi)$ is a linear λ_2-constacyclic code over R of length β, say C_2. Let $C_2 = \langle f_1, ..., f_r\rangle_{R[x]} = \langle \phi(h_1, f_1), ..., \phi(h_r, f_r)\rangle_{R[x]}$. Now, $\ker\phi$ is an $R[x]$-submodule of C. Let $C_1 = \{g \in (\mathbb{F}_q)_{\alpha,\lambda_1} : (g, 0) \in \ker\phi\}$, then clearly C_1 is an $R[x]$-submodule of $(\mathbb{F}_q)_{\alpha,\lambda_1}$. But the map $\tau : R \longrightarrow \mathbb{F}_q$, in the R-module structure of $\mathbb{F}_q$, is surjective. Hence, C_1 is an $\mathbb{F}_q[x]$-submodule of $(\mathbb{F}_q)_{\alpha,\lambda_1}$. Since $\langle x^\alpha - \lambda_1\rangle.C_1 \subseteq \langle x^\alpha - \lambda_1\rangle.(\mathbb{F}_q)_{\alpha,\lambda_1} = 0$, C_1 is an ideal of $(\mathbb{F}_q)_{\alpha,\lambda_1}$. In other words, C_1 is a λ_1-constacyclic code over $\mathbb{F}_q$ of length α. If $C_1 = \langle g\rangle$, then $\ker\phi = \langle (g, 0)\rangle_{R[x]}$. Therefore, $C = \langle (g, 0), (h_1, f_1), ..., (h_r, f_r)\rangle_{R[x]}$. Since ϕ is an $R[x]$-homomorphism, $\frac{C}{\ker\phi} \cong C_2$, hence $|C| = |\ker\phi||C_2| = |C_1||C_2|$. □

Proposition 6.8. *With the assumptions of Theorem 6.7, let*

$$C = \langle (g, 0), (h_1, f_1), ..., (h_r, f_r)\rangle_{R[x]}$$

be a (λ_1, λ_2)-constacyclic code. Then, we can assume that $\deg h_i < \deg g$ for all i; $1 \leq i \leq r$.

Proof. Since the coefficients of g are invertible, we assume that g is monic. Let $\deg h_i \geq \deg g$ for some i; $\deg h_i - \deg g_j = \ell \geq 0$. Also, let $a \in \mathbb{F}_q$ be the leading coefficient of h_i. Then $(h_i, f_i) = (h_i - ax^\ell g, f_i) + ax^\ell(g, 0)$. Thus, $\langle (h_i, f_i), (g, 0)\rangle = \langle (h_i - ax^\ell g, f_i), (g, 0)\rangle$. Hence, we can use $h_i - ax^\ell g$ instead of h_i. By this method we can reduce $\deg h_i$. □

Proposition 6.9. *Let*

$$C = \langle (g, 0), (h_1, f_1), ..., (h_r, f_r)\rangle_{R[x]}$$

be a (λ_1, λ_2)-constacyclic code. Then $g \mid (x^\beta - \lambda_2)h_i$, for all $i = 1, 2, ..., r$.

Proof. Consider the projection map $\phi : C \longrightarrow R_{\beta,\lambda_2}$ in the proof of Theorem 6.7. Then $C_1 = \langle g\rangle = \{f \in$

$(\mathbb{F}_q)_{\alpha,\lambda_1} : (f,0) \in \ker\phi\}$. Now, $(x^\beta - \lambda_2)(h_i, f_i) = ((x^\beta - \lambda_2)h_i, 0) \in \ker\phi$. Hence, $(x^\beta - \lambda_2)h_i \in \langle g\rangle$. This completes the proof. □

Remark 6.10. If $(\lambda_1, \lambda_2) = (1, 1)$ as in Definition 6.5, then we have cyclic $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes. Hence, by above results, we can obtain the structure of cyclic $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes.

Now, we give the exact structure of constacyclic $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes for some special lengths.

Corollary 6.11. *Let $C \subseteq (\mathbb{F}_q)_{\alpha,\lambda_1} \times R_{\beta,\lambda_2}$ be a (λ_1, λ_2)-constacyclic code. If α and β are coprime to p, then $C = \langle (g, 0), (h_1, f_1), (h_2, uf_2)\rangle_{R[x]}$ such that $g|(x^\alpha - \lambda_1)$ and $f_2 \mid f_1|(x^\beta - \lambda_2)$*

Proof. It follows from Theorems 6.1 and 6.7. □

Let $b \in \mathbb{F}_q$ and s be a positive integer. If $s = am + r$ for non-negative integers a, r with $0 \le r \le m - 1$, then define $b_0 = a^{-p^{m-r}}$. The following theorem determines all constacyclic $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes in the case that α and β are powers of p.

Theorem 6.12. *Let $C \subseteq (\mathbb{F}_q)_{\alpha,\lambda_1} \times R_{\beta,\lambda_2}$ be a (λ_1, λ_2)-constacyclic code. Assume that $\alpha = p^{s_1}, \beta = p^{s_2}, \lambda_1 = \gamma$ and $\lambda_2 = \delta + \theta u$. Then $C = \langle ((\gamma_0 x + 1)^i, 0), (h, (\delta_0 x - 1)^j)\rangle$ for some $i \in \{0, 1, \cdots, p^{s_1}\}$ and $j \in \{0, 1, \cdots, 2p^{s_2}\}$.*

Proof. By [9, Theorem 3.4], $C_1 = \langle (\gamma_0 x + 1)^i\rangle$, for some $i \in \{0, 1, \cdots, p^{s_1}\}$. Also, by Theorem 6.2, $C_2 = \langle (\delta_0 x - 1)^j\rangle$, for some $j \in \{0, 1, \cdots, 2p^{s_2}\}$. Now, by Theorem 6.7, we have the result. □

Theorem 6.13. *With the assumptions of the Theorem 6.12, let $\theta = 0$ and $C \subseteq (\mathbb{F}_q)_{\alpha,\lambda_1} \times R_{\beta,\lambda_2}$ be a (γ, δ)-constacyclic code. Then,*

$$C = \langle (g, 0), (h_1, f_1), ..., (h_r, f_r)\rangle_{R[x]},$$

such that

(1) $C_1 = \langle g\rangle = \langle (\gamma_0 x + 1)^i\rangle$ for some $i \in \{0, 1, \cdots, p^{s_1}\}$,

(2) $C_2 = \langle f_1, ..., f_r\rangle_{R[x]} = \phi(I)$ where $\phi : R_{p^s,1} \longrightarrow R_{p^s,\delta}$ is given by $f(x) \longmapsto f(\delta_0 x)$ and I is an ideal of $R_{p^{s_2},1}$ defined in Theorem 6.3.

Proof. It follows from Theorems 6.3, 6.4 and 6.7. □

7. Conclusion

In this paper, we studied the structure of $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes. We obtained the generator matrix of these codes and described their dual codes. We defined Lee weight and homogenous weight over $\mathbb{F}_q\mathbb{F}_q[u]$-additive codes and studied one-weight codes with respect to these two weight functions. Finally, by the Gray image of these codes, we obtained $[q^2 + q, 2, q^2]$ and $[2(q + 1), 2, 2q]$ one-weight optimal codes over $\mathbb{F}_q$.

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