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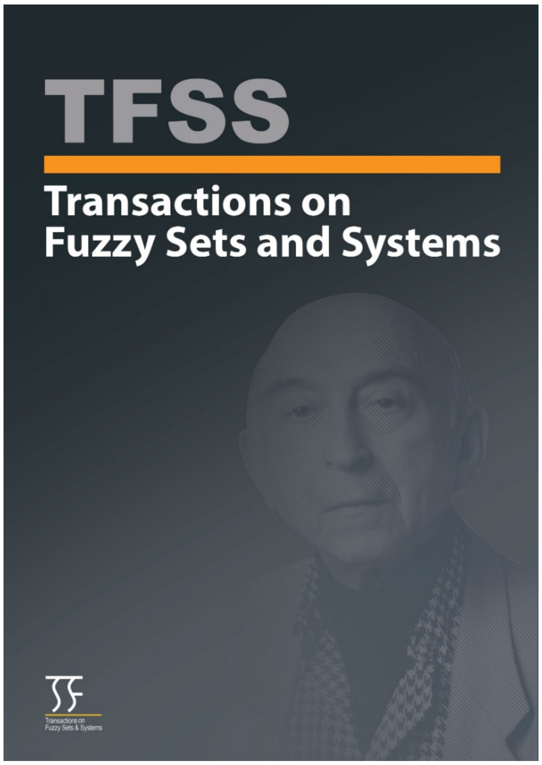

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Enhancing Big Data Governance Framework Implementation Using Novel Fuzzy Frank Operators: An Application to MADM Process

Hamza Iftikhar , **Faisal Mehmood**[*]

Abstract. In today's data-driven landscape, to ensure continuous survival and betterment, the implementation of a robust Big Data Governance Framework (BDGF) is imperative for organizations to effectively manage and harness the potential of their vast data resources. The BDGF serves no purpose when implemented in a random manner. This article delves into the complex decision-making challenges that emerge in the context of implementation of the BDGF under uncertain conditions. Specifically, we aim to analyze and evaluate the BDGF performance using the Multi-Attribute Decision-Making (MADM) techniques aiming to address the intricacies of big data governance uncertainties. To achieve our objectives, we explore the application of Frank operators within the framework of complex picture fuzzy (CPF) sets (CPFs). We introduce complex picture fuzzy Frank weighted averaging (CPFFWA) and complex picture fuzzy Frank ordered weighted averaging (CPFFOWA) operators to enable more accurate implementation of the BDGF. Additionally, we rigorously examine the reliability of these newly proposed fuzzy Frank (FF) operators (FFAOs), taking into consideration essential properties such as idempotency, monotonicity, and boundedness. To illustrate the practical applicability of our approach, we present a case study that highlights the decision-making challenges encountered in the implementation of the BDGF. Subsequently, we conduct a comprehensive numerical example to assess various BDGF implementation options using the MADM technique based on complex picture fuzzy Frank aggregation (CPFFA) operators. Furthermore, we perform a comprehensive comparative assessment of our proposed methodology, emphasizing the significance of the novel insights and results derived. In conclusion, this research article offers a unique and innovative perspective on decision-making within the realm of the BDGF. By employing the CPFFWA and the CPFFOWA operators, organizations can make well-informed decisions to optimize their BDGF implementations, mitigate uncertainties, and harness the full potential of their data assets in an ever-evolving data landscape. This work contributes to the advancement of decision support systems for big data governance (BDG), providing valuable insights for practitioners and scholars alike.

AMS Subject Classification 2020: 03B52; 03B80; 03B50
Keywords and Phrases: Picture fuzzy set, Complex picture fuzzy set, Frank operations, Averaging operators, Geometric operators.

1 Introduction

Although having a BDGF is crucial for companies, a judicious and optimized implementation of the BDGF is a lot more important. The BDGF is like a set of rules and plans to make sure that when a lot of information (Big Data) is collected and used, it's done in a smart and responsible way. It's about making sure the data is accurate, safe, and used in a way that helps rather than causes problems. Just like traffic rules help everyone drive safely on the roads, Big Data Governance rules help manage information in a sensible and secure manner.

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In today's fiercely competitive business environment, companies are constantly challenged to not only attain profitability but also to allocate their funds judiciously. The imperative arises when organizations seek to implement a robust BDGF to manage and harness the potential of their ever-expanding data assets. However, the challenge they face lies in determining how to deploy these financial resources effectively, particularly when confronted with multiple options for their allocation. This problem is further worsened by the inherent complexity of the BDGF, where cost attributes and profit attributes are pivotal components, rendering it a multi-attribute decision-making problem. This study underscores the critical need for a suitable and comprehensive implementation of the BDGF within companies to ensure their continuous survival in competitiveness and achieve long-term profitability. To address this challenge, our research article introduces an innovative approach based on CPFSs and FFA Operators. This novel method offers a powerful tool to tackle the intricacies of multi-attribute decision-making in the context of data governance, ensuring that organizations make well-informed, data-driven choices when allocating resources for optimal outcomes. Through powerful empirical and academic analysis, our research article provides a convincing argument for the implementation of a BDGF in a manner that strictly goes in the companys favor. By leveraging CPFS and FFAOs, organizations can not only navigate the complexities of data governance but also make informed decisions that enhance profitability, reduce risks, and fortify their competitive position in an increasingly data-centric world. This research article serves as a valuable resource for executives, practitioners, and scholars seeking to fortify their organizations' data governance practices in the pursuit of sustainable success.

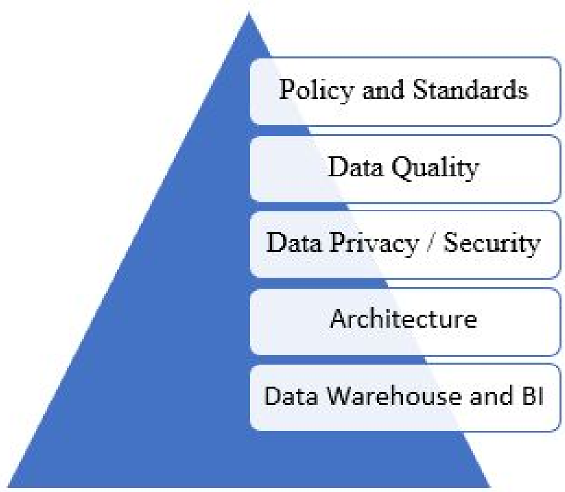

Figure 1: Big data governance framework of a company.

The MADM is a method of organizing and resolving planning and judgmental problems by determining the most appropriate alternative based on an expert's judgment in line with predetermined criteria [1]. This decision-making process is very important and has drawn the attention of many academicians [2]. Typically, professionals or decision-makers resolve the problems to assess the data using fuzzy information [3]. To give just a few examples, numerous researchers have made substantial contributions to the fields of fuzzy sets (FSs) [4], intuitionistic fuzzy sets (IFSs) [3], cubic intuitionistic fuzzy sets [5], linguistic interval-valued IFSs [6], and other generalized sets [7, 8, 9, 10, 11, 12], as well as Pythagorean fuzzy set (Py-FS) [13], which relaxes the IFS limitation $0 \leq \mu + \nu \leq 1$ into $0 \leq \mu^2 + \nu^2 \leq 1$. PFSs are more broadly applicable than IFSs. Using various aggregation operators (AOs), several scholars have presented different sorts of models in these generalized environments [14, 15, 16]. A mathematical function known as an aggregate operator turns a collection of inputs or data into a single datum. In the course of making decisions, aggregation operators are crucial [17, 18] and [19], respectively, creating the ordered weighted averaging (OWA) and ordered weighted geometric (OWG) operators, which, according to their ranking order, applied weights to all values in the collection or data. The weighted averaging operator was developed by [20] for use with IF data, while the geometric aggregation operators were provided by [21]. The FSs and IFSs environments, however, only provide partial information on the data set's components. Picture fuzzy set (PFS) is a novel concept introduced by [22]. It is distinguished by neutral membership, membership, and non-membership features that show data on human decisions like yes, refrain, no, and rejection. [23] Also provided some PFS results. In a PFS environment, [24] proposed WA, OWA, and hybrid averaging operators. The Einstein operations on PFSs were first introduced by [25]. In [26] the most recently used the Hamachar operators in PFSs.

A noteworthy extended form of Lukasiewicz as well as probabilistic t-norm and t-conorm [27] have emerged as Frank t-norm and t-conorm [28]. Moreover, they constitute a sufficiently flexible type of the continuous triangular norm. The Frank models, along with the process of fusion of information, became more adaptable owing to the fact that a certain parameter is used in them, and the literature is replete with numerous works [17, 23, 29, 30, 31, 32] related to these models. Frank operators have gained the attention of researchers in a great number recently [33] Ullah et al used Frank operators to evaluate electric motor cars, and Milosevic et al [34] used those operators for IFS-IBA logical aggregation. Also to improve the MADM process Seikh et al [35, 36] used Frank operators in a very efficient manner. For two types comprising commutative, associative, and growing binary operators, the Alsina and Frank functional equations have skillfully been examined by [37]. Yager [38] developed a paradigm for approximate reasoning using Frank t-norms by examining the additive-generating function of these norms. The scalar cardinality related to Frank t-norms was studied in a novel way by Casasnovas and Torrens [17], who also further established the characteristics of other common t-norms. Sarkoci [39] came to the conclusion that two separate t-norms, Frank and Hamacher t-norms, are actually members of the same family. In order to address the MADM issues, Xing et al. [40] presented aggregation operators pertaining to the PyFs depending on Frank models. Zhou et al. [41] examined a case study of choosing agriculture socialization and looked into several Frank aggregation operations of interval-valued neutron-sophic numbers. Aggregation operators of Frank pertaining to triangular interval type2 FSs were presented by Qin and Liu [42]. Based on Frank t-norm procedures, Qin et al. [43] created more hesitant fuzzy aggregation operators.

It has been determined that the MADM problems addressed in the aforementioned studies in FSs, IFSs, and PFSs environments only handle ambiguity as well as vagueness. All of these models are unable to cope with data insensitivity and periodicity, but a complex data set potentially addresses data periodicity, its continually changing nature and uncertainty at the same time. To deal with these circumstances, Ramot [44] and Ramot et al. [45] proposed the noteworthy notion known as a complex fuzzy set (CFS). In a complex plane, he suggested that a CFS membership degree is expressed as $\mu e^{i\phi\mu}$ the range of which is expanded to a unit disk, where $\mu \in [0, 1]$ and $\phi \in [0, 2]$, Zhang et al. presented certain CFS operating rules and characteristics [46]. Since CFSs first appeared in a variety of real-world sectors, such as biometric procedures, medical investigations, etc., a broad range of applications have been well established. Using the CF data, Bashir and Akram [47] put out the novel idea of ordered weighted quadratic averaging operators. In the realm of the CF, Luqman et al.[48], [49] produced outstanding work. Some notable works can also be found in [50, 51, 52, 53, 54, 55].

The CFS is insufficient to reveal a data set elements inconsistency. Then, Garg and Rani [15], [56] produced the complex intuitionistic fuzzy set (CIFS) and their aggregation operators, which is a generalization of a CFS, in which both an element's membership as well as non-membership were embedded. They provided definitions for the CIFSs intersection, complement, and union. In order to overcome the MADM challenges later, AOs utilizing the CIFSs information were presented by Rani and Garg [57]. Ali et al [17] used complex T-Spherical fuzzy Frank aggregation operators, and yang et al [58] used complex intuitionistic fuzzy Frank (CIFF) aggregation operators for decision making, However, CIFS is unable to cope with data that has been somewhat neglected, such as the fact that it only displays the membership and non-membership degrees of

Table 1: Superiority of CPFS over existing models in literature.

Model	Fuzzy set	Intuitionistic fuzzy set	Picture fuzzy set	Complex fuzzy set	Complex intuitionistic fuzzy set	Complex picture fuzzy set
Uncertainty	✓	✓	✓	✓	✓	✓
Falsity	×	✓	✓	×	✓	✓
Hesitation	×	×	✓	×	×	✓
Periodicity	×	×	×	✓	✓	✓
Handles 2-D data	×	×	×	×	✓	✓
Handles 3-D data	×	×	×	×	×	✓

a data set's elements in a complex plane and is unable to convey whether a choice was made to abstain (neutral value) or reject a piece of information. We introduce the complex picture fuzzy set (CPFS), which is distinguished by membership, neutral membership, and non-membership values in a unit disc of a complex plane, in response to the absence of information in CIFS theory. By criteria $[0, 1]$ and $[0, 2\pi]$, respectively, the amplitude terms and corresponding phase terms of a CPFS are constrained. The set's applicability is increased by complex neutral membership grade, which also makes it simpler for a decision maker to be aware of greater depth as compared to a CIFS. The readers are directed to [59], [60, 61, 49, 48], [22, 2, 43, 44, 57], and [62] for any additional and useful discussions related to AOs as well as the MADM techniques.

The presence of neutral membership gives a CPFS leverage over a CFS and CIFS: The CPFS has a larger range as compared to them. Compared to previous models, CPFSs can address uncertainty and periodicity concurrently and provide significantly more detailed and insightful about an object. The following is a description of the suggested model's motivation:

1. A CPFSs membership degrees have complex values made up of terms for amplitude and phase. The amplitude component of the neutral membership function denotes the abstinence degree. The phase term for this function, however, offers further details, usually regarding periodicity. In essence, the neutral degree has boosted the adaptability of CPFS by providing additional details about an object being evaluated.

2. The CPFS is distinct from the conventional ideas of PFS due to the innovative phase term conception. This is caused by the fact that PFS only works with one-dimensional data, which causes data loss. However, when dealing with issues in real-world occurrences, the second dimension must be taken into account. To remedy it, we incorporated the phase term.

3. The T-norm and t-conorm of Frank [24] appear to be fascinating generalizations extracted from the t-norm and t-conorm of probabilistics and Lukasiewicz [6], and they constitute a common as well as a sufficiently compromising branch of these models. Usage of a certain parameter, the robustness of the Frank models and the information fusion process increase manifold.

The structure of this manuscript can be viewed as follows: In Section 2, several essential ideas that are important to comprehending this manuscript have been given. The novel conception of a CPFS with several CPFS attributes and operating rules as well as score and accuracy functions are presented in Section 3. After applying, in the earlier part of section 3, the Frank AOs to the innovative CPFS concept, we introduced the idea of the CPFHWA and the CPFHWG, complex picture fuzzy Frank hybrid averaging (CPFFHA), complex picture fuzzy Frank weighted geometric averaging (CPFFWGA), complex picture fuzzy Frank ordered weighted geometric averaging (CPFFOWG), and complex picture fuzzy Frank hybrid geometric averaging (CPFFHGA) operators in the later parts of this section. For these operators, we demonstrate a few features and outcomes. An MADM technique, along with the method to determining attributes weights, is put forth in section 4 to determine the most preferable alternative using a specific example and the creditworthiness to show the relevance of the proposed work. After that, section 5 deals with a numerical illustration of our proposed work with the help of a real-world problem. To demonstrate our manuscript's superiority and influence over other prevalent approaches, we compare the model we have offered in Section 6 of our paper. Finally, in Section 7, we have concluded our approach and have provided some recommendations for future work as far as this wider and profound area of study is concerned.

2 Preliminaries

This section deals with some of the fundamental definitions and preliminaries.

Definition 2.1. *[37] Consider X as a universal set. The CPFS $\bar{R}$ over X is defined as*

$$\bar{R}=\left\{\left\langle x,\mu_{\bar{R}}(x)e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}(x)e^{i2\pi\eta_{\bar{R}}(x)},\nu_{\bar{R}}(x)e^{i2\pi\nu_{\bar{R}}(x)}\right\rangle \mid x\in X\right\},$$

where $\mu_{\bar{R}}(x)e^{i2\pi\mu_{\bar{R}}(x)}:X\to[0,1],\eta_{\bar{R}}(x)e^{i2\pi\eta_{\bar{R}}(x)}:X\to[0,1]$ and $\nu_{\bar{R}}(x)e^{i2\pi\nu_{\bar{R}}(x)}:X\to[0,1]$ are reffered to as positive, neutral and negative degrees respectively, such that $0\leq\mu_{\bar{R}}(x)e^{i2\pi\mu_{\bar{R}}(x)}+\eta_{\bar{R}}(x)e^{i2\pi\eta_{\bar{R}}(x)}+\nu_{\bar{R}}(x)e^{i2\pi\nu_{\bar{R}}(x)}\leq 1$ for every $x\in X$.

Moreover, $\varkappa_{\bar{R}}(x)e^{i2\pi\mu_{\bar{R}}(x)}=\left(1-\mu_{\bar{R}}(x)-\eta_{\bar{R}}(x)-\nu_{\bar{R}}(x)\right)e^{i2\pi\mu_{\bar{R}}(x)}$ is reffered to as the degree of hesitancy for $\in X$. For our convenience, we denote $\bar{R}=\left(\mu_{\bar{R}}(x)e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$ as the representation of a CPFN.

Definition 2.2. *Let $\bar{R}=\left(\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$ and $\bar{S}=\left(\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\right)$ be CPFNs over a universal set X and $\zeta>0$ belonging to real number, then following are defined some notable operations:*

1. $\bar{R}\leq\bar{S}$, *if* $\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}\leq\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}\leq\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}$ *and* $\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\geq\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}$.

2. $$\bar{R}\bigvee\bar{S}=\left(\begin{array}{c}\max\left\{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}\right\},\min\left\{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{S}(x)}\right\},\\ \min\left\{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\right\}\end{array}\right).$$

3. $$\bar{R}\wedge\bar{S}=\left(\begin{array}{c}\min\left\{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}\right\},\max\left\{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}\right\}\\ ,\max\left\{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\right\}\end{array}\right).$$

4. $\bar{R}^{c}=\left(\nu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\mu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$.

5. $$\bar{R}\wedge\bar{S}=\left(\begin{array}{c}\min\left\{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\mu_{\bar{S}}e^{i2\pi\mu_{\bar{R}}(x)}\right\},\max\left\{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{R}}(x)}\right\},\\ \max\left\{\nu_{\bar{R}}e^{i2\pi v_{\bar{R}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{R}}(x)}\right\}\end{array}\right).$$

6. $$\bar{R}\vee\bar{S}=\left(\begin{array}{c}\max\left\{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}\right\},\min\left\{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}\right\},\\ \min\left\{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\right\}\end{array}\right).$$

7. $$\bar{R}\oplus\bar{S}=\left(\begin{array}{c}\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}+\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}\eta_{\bar{S}}e^{i2\pi\eta_{S}(x)},\\ \nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\end{array}\right).$$

8. $$\bar{R}\otimes\bar{S}=\left(\begin{array}{c}\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}+\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)},\\ \nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}+\nu_{\bar{S}}e^{i2\pi\nu_{S}(x)}-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\nu_{\bar{S}}e^{i2\pi\nu_{S}(x)}\end{array}\right).$$

9. $\zeta \bar{R} = \left(1 - \left(1 - \mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}\right)^{\zeta}, \left(\eta_{\bar{R}}^{\zeta}\right) e^{i2\pi\eta_{\bar{R}}(x)}, \nu_{\bar{R}}^{\zeta} e^{i2\pi\nu_{\bar{R}}(x)}\right)$.

10. $\bar{R}^{\zeta} = \left(\mu_{\bar{R}}^{\zeta} e^{i2\pi\mu_{\bar{R}}(x)}, 1 - \left(1 - \eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}\right)^{\zeta}, 1 - \left(1 - \nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}\right)^{\zeta}\right)$.

Definition 2.3. *For a $CPFN \bar{R} = \left(\mu_{\bar{R}}(x) e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}, \nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}\right)$*

$$\Delta(p) = \frac{2 + \mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)} - \nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}}{4},$$

is defined as score function, where $\Delta(p) \in [0, 1]$.

Definition 2.4. *For a $CPFN \bar{R} = \left(\mu_{\bar{R}}(x) e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}, \nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}\right)$*

$$\nabla(p) = \mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)} + \nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)},$$

is defined as accuracy function, where $\Psi(p) \in [-1, 1]$.

According to Definitions 2.3 and 2.4, if $\bar{R} = \left(\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}, \nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}\right)$ and $\bar{S} = \left(\mu_{\bar{S}} e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{S}} e^{i2\pi\eta_{\bar{R}}(x)}, \nu_{\bar{S}} e^{i2\pi\nu_{\bar{R}}(x)}\right)$ are any two CPFNs then

1. If $\Delta(\bar{R}) > \Delta(\bar{S})$ then $\bar{R} > \bar{S}$.
2. If $\Delta(\bar{R}) < \Delta(\bar{S})$ then $\bar{R} < \bar{S}$.
3. If $\Delta(\bar{R}) = \Delta(\bar{S})$, then
   (a) If $\nabla(\bar{R}) > \nabla(\bar{S})$, then $\bar{R} > \bar{S}$.
   (b) If $\nabla(\bar{R}) = \nabla(\bar{S})$, then $\bar{R} = \bar{S}$.

Definition 2.5. *Let $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right) (k = 1, 2, \ldots, n)$ be CPFNs' collection. Then, by using the CPFWAOs, their aggregated value is also a CPFN and $CPFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \oplus_{k=1}^{n}\left(w_k \bar{R}_k\right)$*

$$= \left(\begin{array}{c} 1 - \prod_{k=1}^{n}\left(1 - \mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}\right)^{w_k}, \left(\prod_{k=1}^{n} \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}\right)^{w_k}, \\ \left(\prod_{k=1}^{n} \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)^{w_k} \end{array}\right),$$

here $w = (w_1, w_2, \ldots, w_n)^t$ denotes the weight-vector of $p_k (k = 1, 2, \ldots, n), w_k \in [0, 1]$ and $\sum_{k=1}^{n} w_k = 1$.

Definition 2.6. *Let $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right) (k = 1, 2, \ldots, n)$ be a collection of CPFNs. A structure $p^n \to p$ such that,*

$$CPFOWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \bigoplus_{k=1}^{n}\left(w_k \bar{R}_{\bar{R}(k)}\right) = \left(1 - \prod_{k=1}^{n}\left(1 - \mu_{\bar{R}_{\bar{R}(k)}}\right)^{w_i}, \prod_{k=1}^{n} \eta_{\bar{R}_{\bar{R}(k)}} w_k, \prod_{k=1}^{n} \nu_{\bar{R}_{\bar{R}(k)}} w_k\right),$$

is known as CPFOWA operator and,$(\rho(1), \rho(2), \ldots, \rho(n))$ denotes the permutation related to $(k = 1, 2, \ldots, n)$, satisfying $p_{\rho(i-1)} \geq p_{\rho(k)}; k = 1, 2, \ldots, n$.

The definition of Frank t-norm and t-conorm is provided as follows:

Definition 2.7. *[28] For a and b as two real numbers, the functions*

$$\mathrm{Fra}(a,b)=\log_r\left(1+\frac{\left(r^a-1\right)\left(r^b-1\right)}{r-1}\right),$$

and

$$\mathrm{Fra}'(a,b)=1-\log_r\left(1+\frac{\left(r^{1-a}-1\right)\left(r^{1-b}-1\right)}{r-1}\right),$$

are defined as Frank t-norm and Frank t-conorm respectively, where $(a,b)\in[0,1]\times[0,1]$ and $r\neq 1$.

Following observations should be considered here [63]:

1. $\mathrm{Fra}'(a,b)\to a+b-ab$, when $r\to 1$, also Fra $(a,b)\to ab$ when $r\to 1$. Therefore, we conclude that sum and product of Frank change into sum and product of probabilistic when $r\to 1$.
2. $\mathrm{Fra}'(a,b)\to \min\{a+b,1\}$ when $r\to\infty$ and Fra $(a,b)\to\max\{0,a+b-1\}$ when $r\to\infty$. Therefore, we conclude that the sum as well as product of Frank change into sum as well as product of Lukasiewicz, when $r\to\infty$.

Example 2.8. Let $a=0.33, b=0.98$ and $r=5$, then,

$$\mathrm{Fra}(0.33,0.98)=\log_4\left(1+\frac{\left(4^{0.33}-1\right)\left(4^{0.98}-1\right)}{5-1}\right)=0.3197.$$

$$\mathrm{Fra}'(0.33,0.98)=1-\log_4\left(1+\frac{\left(4^{1-0.33}-1\right)\left(4^{1-0.98}-1\right)}{5-1}\right)=0.9902.$$

3 Complex Picture Fuzzy Frank Aggregation Operators

By skillfully using the t-norm and t-conorm of Frank, notable operating rules for the CPF environment have been created in this part. We also recommend the CPFFWA, CPFFOWA, CPFFHWA, CPFFWG, CPFFOWG, and CPFFHWA aggregation operators utilizing the operational principles we have defined.

Definition 3.1. *Let $\bar{R}=\left(\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right), \bar{S}=\left(\mu_{\bar{S}}e^{i2\pi\mu_{\bar{J}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{J}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{J}}(x)}\right)$ and $\bar{T}=\left(\mu_{\bar{T}}e^{i2\pi\mu_{\bar{T}}(x)},\eta_{\bar{T}}e^{i2\pi\eta_{\bar{T}}(x)},\nu_{\bar{T}}e^{i2\pi\nu_{\bar{T}}(x)}\right)$ be CPFNs, $r>1$, and $\zeta>0$ is a real number.*

Frank t-norm and t-conorm operations for CPFNs are provided as follows:

1. $\bar{R}\oplus\bar{S}=$

$$=\left(\begin{array}{c}1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)\\\left(r^{1-\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)\\\left(r^{\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),\\\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\\\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right)\end{array}\right).$$

2. $\bar{R}\otimes\bar{S}=$

$$=\left(\begin{array}{c}\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)\\ \left(r^{\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)\\ \left(r^{1-\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),\\ 1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\\ \left(r^{1-\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right)\end{array}\right).$$

3. $\zeta\bar{R}$

$$=\left(\begin{array}{c}1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta-1}}\right),\log_r\left(1+\frac{\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta-1}}\right),\\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta-1}}\right)\end{array}\right).$$

4. $\bar{R}^{\zeta}$

$$=\left(\begin{array}{c}\log_r\left(1+\frac{\left(r^{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta-1}}\right),1-\log_r\left(1+\frac{\left(r^{1-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta-1}}\right),\\ 1-\log_r\left(1+\frac{\left(r^{1-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta-1}}\right)\end{array}\right).$$

Example 3.2. Let $\bar{R}=\left(0.5e^{i2\pi(0.5)},0.3e^{i2\pi(0.1)},0.2e^{i2\pi(0.4)}\right)$ and $\bar{S}=\left(0.1e^{i2\pi(0.4)},0.5e^{i2\pi(0.4)},0.4e^{i2\pi(0.2)}\right)$ be any two CPFNs. Let $r=2$ and $\zeta=3$ in definition 3.1, we get

1. $\bar{R}\oplus\bar{S}=\left(0.5577e^{i2\pi(0.7206)},0.1319e^{i2\pi(0.0327)},0.0669e^{i2\pi(0.06)}\right)$.
2. $\bar{R}\otimes\bar{S}=\left(0.0422e^{i2\pi(0.1793)},0.6680e^{i2\pi(0.4672)},0.0.2243e^{i2\pi(0.3330)}\right)$.
3. $5\bar{R}=\left(0.9348e^{i2\pi(0.9860)},0.0012e^{i2\pi(0.00001)},0.000015e^{i2\pi(0.000015)}\right)$.
4. $\bar{R}^4=\left(0.00003e^{i2\pi(0.0034)},0.0975e^{i2\pi(0.0876)},0.9216e^{i2\pi(0.9216)}\right)$.

Theorem 3.3. *Let* $\bar{R}=\left(\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$, *and* $\bar{S}=\left(\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\right)$ *and* $\bar{T}=\left(\mu_{\bar{T}}e^{i2\pi\mu_{\bar{T}}(x)},\eta_{\bar{T}}e^{i2\pi\eta_{\bar{T}}(x)},\nu_{\bar{T}}e^{i2\pi\nu_{\bar{T}}(x)}\right)$ *be any three CPFNs. Let* $r>1$ *and* ζ,ζ_1,ζ_2 *are positive real numbers, then*

1. $\bar{R}\oplus\bar{S}=\bar{S}\oplus\bar{R}$.
2. $\bar{R}\otimes\bar{S}=\bar{S}\otimes\bar{R}$.
3. $\zeta(\bar{R}\oplus\bar{S})=\zeta\bar{R}\oplus\zeta\bar{S}$.

4. $\zeta_1\bar{R}\oplus\zeta_2\bar{R}=(\zeta_1+\zeta_2)\,\bar{R}$.

5. $(\bar{R}\otimes\bar{S})^{\zeta}=\bar{R}^{\zeta}\otimes\bar{S}^{\zeta}$.

6. $\bar{R}^{\zeta_1}\otimes\bar{R}^{\zeta_2}=\bar{R}^{\zeta_1+\zeta_2}$.

Proof. For three CPFNs $\bar{R}=\left(\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right),\bar{S}=\left(\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\right)$ and $\bar{T}=\left(\mu_{\bar{T}}e^{i2\pi\mu_{\bar{T}}(x)},\eta_{\bar{T}}e^{i2\pi\eta_{\bar{T}}(x)},\nu_{\bar{T}}e^{i2\pi\nu_{\bar{T}}(x)}\right)$ with $\zeta,\zeta_1,\zeta_2>0$. According to Definition 3.1, we can obtain

1. $$\bar{R}\oplus\bar{S}=\left(\begin{array}{c}1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)\\\left(r^{1-\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)\\\left(r^{\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),\\\log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)}{r-1}\right)\end{array}\right)$$

$$=\left(\begin{array}{c}1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)\\\left(r^{1-\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)\\\left(r^{\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),\\\log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)}{r-1}\right)\end{array}\right)=\bar{S}\oplus\bar{R}.$$

2. $$\bar{R}\otimes\bar{S}=\left(\begin{array}{c}\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)\\\left(r^{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)\end{array}\right)}{r-1}\right),1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)\\\left(r^{1-\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right)\\1-\log_r\left(1+\frac{\left(r^{1-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\left(r^{1-\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)}{r-1}\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)\\\left(r^{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)\end{array}\right)}{r-1}\right),1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)\\\left(r^{1-\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right)\\1-\log_r\left(1+\frac{\left(r^{1-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\left(r^{1-\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)}{r-1}\right)\end{array}\right)=\bar{S}\otimes\bar{R}.$$

3. $$\zeta(\bar{R}\oplus\bar{S})=\zeta\left(\begin{array}{c} 1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)\\ \left(r^{1-\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right), \log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)\\ \left(r^{\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),\\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)}{r-1}\right)\end{array}\right)$$

$$=\left(\begin{array}{c} 1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta}\\ \left(r^{1-\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)^{\zeta}\end{array}\right)}{(r-1)^{2\zeta-1}}\right), \log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)^{\zeta}\\ \left(r^{\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)^{\zeta}\end{array}\right)}{(r-1)^{2\zeta-1}}\right),\\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta}\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)^{\zeta}}{(r-1)^{2\zeta-1}}\right)\end{array}\right).$$

Now

$$\zeta\bar{R}\oplus\zeta\bar{S}=\left(\begin{array}{c} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right), \log_r\left(1+\frac{\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right),\\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right)\end{array}\right)$$

$$\oplus\left(\begin{array}{c} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right), \log_r\left(1+\frac{\left(r^{\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right),\\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right)\end{array}\right)$$

$$=\left(\begin{array}{c} 1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta}\\ \left(r^{1-\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)^{\zeta}\end{array}\right)}{(r-1)^{2\zeta-1}}\right), \log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)^{\zeta}\\ \left(r^{\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)^{\zeta}\end{array}\right)}{(r-1)^{2\zeta-1}}\right),\\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta}\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)^{\zeta}}{(r-1)^{2\zeta-1}}\right)\end{array}\right).$$

Therefore,

$$\zeta(\bar{R} \oplus \bar{S}) = \zeta\bar{R} \oplus \zeta\bar{S}.$$

4. $$\zeta_1\bar{R} \oplus \zeta_2\bar{R} = \begin{pmatrix} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta_1}}{(r-1)^{\zeta_1}}\right), \log_r\left(1+\frac{\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)^{\zeta_1}}{(r-1)^{\zeta_1}}\right) \\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta_1}}{(r-1)^{\zeta_1}}\right) \end{pmatrix}$$

$$\oplus \begin{pmatrix} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta_2}}{(r-1)^{\zeta_2}}\right), \log_r\left(1+\frac{\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)^{\zeta_2}}{(r-1)^{\zeta_2}}\right) \\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta_2}}{(r-1)^{\zeta_2}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta_1+\zeta_2}}{(r-1)^{\zeta_1+\zeta_2}}\right), \log_r\left(1+\frac{\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}\right)^{\zeta_1+\zeta_2}}{(r-1)^{\zeta_1+\zeta_2}}\right) \\ , \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta_1+\zeta_2}}{(r-1)^{\zeta_1+\zeta_2}}\right) \end{pmatrix}.$$

$$= (\zeta_1 + \zeta_2)\,\bar{R}.$$

5. $$\left(\bar{R}_1 \otimes \bar{R}_2\right)^{\zeta} = \begin{pmatrix} \log_r\left(1+\frac{\left(\begin{pmatrix} r^{\mu_{\bar{R}_1}e^{i2\pi\mu_{\bar{R}_1}(x)}}-1 \end{pmatrix}\begin{pmatrix} r^{\mu_{\bar{R}_2}e^{i2\pi\mu_{\bar{R}_2}(x)}}-1 \end{pmatrix}\right)}{r-1}\right), 1-\log_r\left(1+\frac{\left(\begin{pmatrix} r^{1-\eta_{\bar{R}_1}e^{i2\pi\eta_{\bar{R}_1}(x)}}-1 \end{pmatrix}\begin{pmatrix} r^{1-\eta_{\bar{R}_2}e^{i2\pi\eta_{\bar{R}_2}(x)}}-1 \end{pmatrix}\right)}{r-1}\right) \\ 1-\log_r\left(1+\frac{\left(r^{1-\nu_{\bar{R}_1}e^{i2\pi\nu_{\bar{R}_1}(x)}}-1\right)\left(r^{1-\nu_{\bar{R}_2}e^{i2\pi\nu_{\bar{R}_2}(x)}}-1\right)}{r-1}\right) \end{pmatrix}^{\zeta}$$

$$= \left(\begin{array}{c} \log_r \left(1 + \frac{\left(\left(r^{\mu_{\bar{R}_1} e^{i2\pi\mu_{\bar{R}_1}(x)}} - 1 \right) \left(r^{\mu_{\bar{R}_2} e^{i2\pi\mu_{\bar{R}_2}(x)}} - 1 \right) \right)^{\zeta}}{(r-1)^{2\zeta-1}} \right), 1 - \log_r \left(1 + \frac{\left(\left(r^{1-\eta_{\bar{R}_1} e^{i2\pi\eta_{\bar{R}_1}(x)}} - 1 \right) \left(r^{1-\eta_{\bar{R}_2} e^{i2\pi\eta_{\bar{R}_2}(x)}} - 1 \right) \right)^{\zeta}}{(r-1)^{2\zeta-1}} \right) \\ 1 - \log_r \left(1 + \frac{\left(\left(r^{1-\nu_{\bar{R}_1} e^{i2\pi\nu_{\bar{R}_1}(x)}} - 1 \right) \left(r^{1-\nu_{\bar{R}_2} e^{i2\pi\nu_{\bar{R}_2}(x)}} - 1 \right) \right)^{\zeta}}{(r-1)^{2\zeta-1}} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \log_r \left(1 + \frac{\left(r^{\mu_{\bar{R}_1} e^{i2\pi\mu_{\bar{R}_1}(x)}} - 1 \right)^{\zeta}}{(r-1)^{\zeta}} \right), 1 - \log_r \left(1 + \frac{\left(r^{1-\eta_{\bar{R}_1} e^{i2\pi\eta_{\bar{R}_1}(x)}} - 1 \right)^{\zeta}}{(r-1)^{\zeta}} \right), \\ 1 - \log_r \left(1 + \frac{\left(r^{1-\nu_{\bar{R}_1} e^{i2\pi\nu_{\bar{R}_1}(x)}} - 1 \right)}{(r-1)^{\zeta}} \right) \end{array} \right).$$

6. $$\bar{R}^{\zeta_1} \otimes \bar{R}^{\zeta_2} = \left(\begin{array}{c} \log_r \left(1 + \frac{\left(r^{\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}} - 1 \right)^{\zeta_1}}{(r-1)^{\zeta_1-1}} \right), 1 - \log_r \left(1 + \frac{\left(r^{1-\eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}} - 1 \right)^{\zeta_1}}{(r-1)^{\zeta_1-1}} \right), \\ 1 - \log_r \left(1 + \frac{\left(r^{1-\nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}} - 1 \right)^{\zeta_1}}{(r-1)^{\zeta_1-1}} \right) \end{array} \right)$$

$$\otimes \left(\begin{array}{c} \log_r \left(1 + \frac{\left(r^{\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}} - 1 \right)^{\zeta_2}}{(r-1)^{\zeta_2-1}} \right), 1 - \log_r \left(1 + \frac{\left(r^{1-\eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}} - 1 \right)^{\zeta_2}}{(r-1)^{\zeta_2-1}} \right), \\ 1 - \log_r \left(1 + \frac{\left(r^{1-\nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}} - 1 \right)^{\zeta_2}}{(r-1)^{\zeta_2-1}} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \log_r \left(1 + \frac{\left(r^{\mu} e^{i2\pi\mu_{\bar{R}}(x)} - 1 \right)^{\zeta_1+\zeta_2}}{(r-1)^{\zeta_1+\zeta_2-1}} \right), 1 - \log_r \left(1 + \frac{\left(r^{1-\eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}} \right)^{\zeta_1+\zeta_2}}{(r-1)^{\zeta_1+\zeta_2-1}} \right), \\ 1 - \log_r \left(1 + \frac{\left(r^{1-\nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}} - 1 \right)^{\zeta_1+\zeta_2}}{(r-1)^{\zeta_1+\zeta_2-1}} \right) \end{array} \right) = \bar{R}^{\zeta_1+\zeta_2}.$$

□

3.1 Complex Picture Fuzzy Frank Arithmetic Aggregation Operators

Definition 3.4. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *be CPFNs' collection. Then, a function* $p^n \to p$

$$CPFFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \bigoplus_{k=1}^{n}\left(w_k \bar{R}_k\right),$$

is known as the CPFFWA operator with $w = (w_1, w_2, \ldots, w_n)^t$ *as the weight vector of* $\bar{R}_k(k = 1, 2, \ldots, n), w_k \in [0, 1]$ *and* $\sum_{k=1}^{n} w_k = 1$.

As a result, the following consequential theorem is obtained.

Theorem 3.5. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *be CPFNs' collection, then the aggregated value is also a CPFN, and*

$$CPFFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \bigoplus_{k=1}^{n}\left(w_k \bar{R}_k\right)$$

$$= \begin{pmatrix} 1 - \log_r\left(1 + \prod_{k=1}^{n}\left(r^{1-\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}}(x)}} - 1\right)^{w_k}\right), \log_r\left(1 + \prod_{k=1}^{n}\left(r^{\eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right)^{w_k}\right), \\ \log_r\left(1 + \prod_{k=1}^{n}\left(r^{\nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}}(x)}}\right)^{w_k}\right) \end{pmatrix}.$$

Proof. Method of mathematical induction would be used for proving this theorem. We take $n = 2$, and by using Frank operations for CPFNs, we get

$$CPFFWA\left(\bar{R}_1, \bar{R}_2\right) = \bigoplus_{k=1}^{2} w_k = w_1\bar{R}_1 \bigoplus w_2\bar{R}_2$$

$$= \begin{pmatrix} 1 - \log_r\left(1 + \frac{\left(r^{1-\mu_{\bar{R}_1} e^{i2\pi\mu_{\bar{R}_1}(x)}} - 1\right)^{w_1}}{(r-1)^{w_1-1}}\right), \log_r\left(1 + \frac{\left(r^{\eta_{\bar{R}_1} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right)^{w_1}}{(r-1)^{w_1-1}}\right), \\ \log_r\left(1 + \frac{\left(r^{\nu_{\bar{R}_1} e^{i2\pi\nu_{\bar{R}_1}(x)}}\right)^{w_1}}{(r-1)^{w_1-1}}\right) \end{pmatrix}$$

$$\oplus \begin{pmatrix} 1 - \log_r\left(1 + \frac{\left(r^{1-\mu_{\bar{R}_2} e^{i2\pi\mu_{\bar{R}_2}(x)}}\right)^{w_2}}{(r-1)^{w_2-1}}\right), \log_r\left(1 + \frac{\left(r^{\eta_{\bar{R}_2} e^{i2\pi\eta_{\bar{R}_2}(x)}}\right)^{w_2} - 1}{(r-1)^{w_2-1}}\right), \\ \log_r\left(1 + \frac{\left(r^{\nu_{\bar{R}_2} e^{i2\pi\nu_{\bar{R}_2}(x)}} - 1\right)^{w_2}}{(r-1)^{w_2-1}}\right) \end{pmatrix}$$

$$= \left(\begin{array}{c} 1-\log_r\left(1+\prod_{k=1}^{2}\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right), \log_r\left(1+\prod_{k=1}^{2}\left(r^{\eta_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{k=1}^{2}\left(r^{\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right) \end{array} \right).$$

$$\left[\because \sum_{k=1}^{2} w_k = 1\right]$$

Therefore, for $n=2$, the result is true.
By considering the given result as true for $n=s$, we have,

$$CPFFWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right) = \bigoplus_{k=1}^{s} w_k\bar{R}_k$$

$$= \left(\begin{array}{c} 1-\log_r\left(1+\prod_{k=1}^{s}\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right), \log_r\left(1+\prod_{k=1}^{s}\left(r^{\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{k=1}^{s}\left(r^{\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right) \end{array} \right).$$

Now, for $n=s+1$, we have,

$$CPFFWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_{n+1}\right) = \bigoplus_{k=1}^{s+1} w_k\bar{R}_k = \bigoplus_{i=1}^{s} w_k\bar{R}_k \bigoplus w_{s+1}\bar{R}_{s+1}$$

$$= \left(\begin{array}{c} 1-\log_r\left(1+\frac{\prod_{k=1}^{s}\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}}{(r-1)^{\sum_{i=1}^{s}w_i-1}}\right), \\ \log_r\left(1+\frac{\prod_{k=1}^{s}\left(r^{\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}}-1\right)^{w_k}}{(r-1)^{\sum_{i=1}^{s}w_i-1}}\right), \\ \log_r\left(1+\frac{\prod_{k=1}^{s}\left(r^{\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}}-1\right)^{w_k}}{(r-1)^{\sum_{i=1}^{s}w_i-1}}\right) \end{array} \right)$$

$$\oplus\left(\begin{array}{c} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{R}_{S+1}}e^{i2\pi\mu_{\bar{R}_{S+1}}(x)}}-1\right)^{w_{S+1}}}{(r-1)^{w_{S+1}-1}}\right) \\ \log_r\left(1+\frac{\left(r^{\eta_{\bar{R}_{S+1}}e^{i2\pi\eta_{\bar{R}_{S+1}}(x)}}-1\right)^{w_{S+1}}}{(r-1)^{w_{S+1}-1}}\right) \\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}_{S+1}}e^{i2\pi\nu_{\bar{R}_{S+1}}(x)}}-1\right)^{w_{S+1}}}{(r-1)^{w_{S+1}-1}}\right) \end{array}\right)$$

$$=\left(\begin{array}{c} 1-\log_r\left(1+\prod_{i=1}^{s+1}\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{i=1}^{s+1}\left(r^{\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{i=1}^{s+1}\left(r^{\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right) \end{array}\right).$$

$$\text{as } \sum_{k=1}^{s+1} w_k = 1.$$

Which shows the result is valid for $n = s + 1$, if it is valid for $n = s$. Hence, method of induction shows the validity of our result, no matter what natural number n is. □

Example 3.6. For $\bar{R}_1 = \left(0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.4)}\right), \bar{R}_2 = \left(0.1e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.2)}\right), \bar{R}_3 = \left(0.1e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.2)}, 0.7e^{i2\pi(0.5)}\right), \bar{R}_4 = \left(0.6e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.3)}\right)$ with weights $= (0.2, 0.3, 0.1, 0.4)$ and $r = 2$, step by step working of the operator is given as follows:

$$CPFFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \left(\begin{array}{c} 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu_{\bar{R}_k}}e^{i2\pi\mu_{\bar{R}}(x)}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{\eta_{\bar{R}_k}}e^{i2\pi\eta_{\bar{R}}(x)}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{\nu_{\bar{R}_k}}e^{i2\pi\nu_{\bar{R}}(x)}-1\right)^{w_k}\right) \end{array}\right)$$

$$=\left(\begin{array}{c} 1-\log_2\left(\begin{array}{c} 1+\left(2^{1-(0.5)e^{i2\pi(0.5)}}-1\right)^{0.2}+\left(2^{1-(0.1)e^{i2\pi(0.4)}}-1\right)^{0.3} \\ +\left(2^{1-(0.1)e^{i2\pi(0.1)}}-1\right)^{0.1}+\left(2^{1-(0.6)e^{i2\pi(0.2)}}-1\right)^{0.4} \end{array}\right), \\ \log_2\left(\begin{array}{c} 1+\left(2^{(0.3)e^{i2\pi(0.1)}}-1\right)^{0.2}+\left(2^{(0.5)e^{i2\pi(0.4)}}-1\right)^{0.3} \\ +\left(2^{(0.1)e^{i2\pi(0.1)}}-1\right)^{0.1}+\left(2^{(0.1)e^{i2\pi(0.2)}}-1\right)^{0.4} \end{array}\right), \\ \log_2\left(\begin{array}{c} 1+\left(2^{(0.1)e^{i2\pi(0.4)}}-1\right)^{0.2}+\left(2^{(0.4)e^{i2\pi(0.2)}}-1\right)^{0.3} \\ +\left(2^{(0.7)e^{i2\pi(0.5)}}-1\right)^{0.1}+\left(2^{(0.3)e^{i2\pi(0.3)}}-1\right)^{0.4} \end{array}\right) \end{array}\right)$$

$$=\left(0.321e^{i2\pi(0.321)}, 0.205e^{i2\pi(0.216)}, 0.330e^{i2\pi(0.297)}\right).$$

Theorem 3.7. *(Idempotent). Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)$ $(k = 1, 2, \ldots, n)$ *be a collection of CPFNs which are all identical, i.e.,* $\bar{R}_k = \bar{R}$ *for all* k*, where* $R = \left(\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}, v_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}\right)$*, then* $CPFFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \bar{R}$.

Proof. As for every $\bar{R}_k = \bar{R}$, therefore,

$$CPFFWA\left(\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n\right) =$$

$$= \left(\begin{array}{c} 1 - \log_r\left(1 + \prod_{k=1}^n \left(r^{1-\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}} - 1\right)^{w_i}\right), \log_r\left(1 + \prod_{k=1}^n \left(r^{\eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right)^{w_k}\right), \\ \log_r\left(1 + \prod_{k=1}^n \left(r^{\nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}} - 1\right)^{w_k}\right) \end{array} \right)$$

$$= \left(\begin{array}{c} 1 - \log_r\left(1 + \prod_{k=1}^n \left(r^{1-\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}} - 1\right)^{w_k}\right), \log_r\left(1 + \prod_{k=1}^n \left(r^{\eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right)^{w_k}\right), \\ \log_r\left(1 + \prod_{k=1}^n \left(r^{\nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}} - 1\right)^{w_k}\right) \end{array} \right)$$

$$= \left(\begin{array}{c} 1 - \log_r\left(1 + \prod_{k=1}^n \left(r^{1-\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}} - 1\right)^{\sum_{k=1}^n w_k}\right), \log_r\left(1 + \prod_{k=1}^n \left(r^{1-\eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right)^{\sum_{k=1}^n w_k}\right), \\ \log_r\left(1 + \prod_{k=1}^n \left(r^{1-\nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}} - 1\right)^{\sum_{k=1}^n w_k}\right) \end{array} \right)$$

$$= \left(\begin{array}{c} 1 - \log_r\left(1 + \prod_{k=1}^n \left(r^{1-\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}} - 1\right)\right), \log_r\left(1 + \prod_{k=1}^n \left(r^{1-\eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right)\right), \\ \log_r\left(1 + \prod_{k=1}^n \left(r^{1-\nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}} - 1\right)\right) \end{array} \right)$$

$$= \left(\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}, v_{\bar{R}} e^{i2\pi v_{\bar{R}}(x)}\right) = \bar{R}.$$

Hence, it completes the proof. □

Theorem 3.8. *(Boundedness). Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)$ $(k = 1, 2, \ldots, n)$ *be CPFNs'collection. Let* $\bar{R}^- = \min\left\{\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right\}$*, and* $\bar{R}^+ = \max\left\{\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right\}$*. Then,*

$$\bar{R}^- \leq CPFFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) \leq \bar{R}^+.$$

Proof. Let $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)$ ($k = 1, 2, \ldots, n$ is a collection of CPFN. Let

$$\bar{R}^- = \min\left\{\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right\} = \left(\mu^-{}_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta^-{}_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu^-{}_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right),$$

and

$$\bar{R}^+ = \max\left\{\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right\} = \left(\mu^+{}_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta^+{}_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu^+{}_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right).$$

Then, we have,

$$\begin{aligned} \mu^- \bar{R}_k e^{i2\pi\mu_{\bar{R}_k}(x)} &= \min_k \left\{\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}\right\}, \eta^-{}_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)} \\ &= \max_k \left\{\eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}\right\}, \nu^-_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)} = \max_k \left\{\nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right\} \\ \mu^+{}_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)} &= \max_k \left\{\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}\right\}, \eta^+{}_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)} \\ &= \min_k \left\{\eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}\right\}, \nu^+{}_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)} = \min_k \left\{\nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right\}. \end{aligned}$$

Now,

$$1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\left(\mu^-{}_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}\right)}-1\right)^{w_k}\right)$$
$$\leq 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right)$$
$$\leq 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\left(\mu^+{}_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}\right)}-1\right)^{w_k}\right).$$
$$\log_r\left(1+\prod_{k=1}^{n}\left(r^{\left(\eta^+{}_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}\right)}-1\right)^{w_k}\right)\Bigg)$$
$$\leq \log_r\left(1+\prod_{k=1}^{n}\left(r^{\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}-1}\right)^{w_k}\right)$$
$$\leq \log_r\left(1+\prod_{k=1}^{n}\left(r^{\left(\eta^-_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}-1\right)}\right)^{w_k}\right).$$

and

$$\log_r\left(1+\prod_{k=1}^{n}\left(r^{\left(\nu^+{}_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)}-1\right)^{w_k}\right)\Bigg)$$
$$\leq \log_r\left(1+\prod_{k=1}^{n}\left(r^{\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right)\Bigg)$$
$$\leq \log_r\left(1+\prod_{k=1}^{n}\left(r^{\left(\nu^-{}_{\bar{R}_k}^{i2\pi\nu_{\bar{R}_k}(x)}\right)}-1\right)^{k}\right)\Bigg).$$

Therefore,

$$\bar{R}^-\leq CPFFWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)\leq \bar{R}^+.$$

□

Theorem 3.9. *(Monotonicity) Let the two sets $\bar{R}_k$ and $\bar{R}'_k(k=1,2,\ldots,n)$ of CPFNs, if for all k $\bar{R}_k\leq \bar{R}'_k$, then $CPFFWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)\leq$ CPFFWA $\left(\bar{R}'_1,\bar{R}'_2,\ldots,\bar{R}'_n\right)$.*

Proof. Since $\bar{R}_k\leq \bar{R}'_k$ for all $k=1,2,\ldots,n$, then, $\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}\leq \mu'_{\bar{R}_k}e^{i2\pi\mu'\bar{R}_k(x)}, r^{\eta_{\bar{R}_k}}e^{i2\pi\eta_{\bar{R}_k}(x)}\leq \eta'_{\bar{R}_k}e^{i2\pi\eta'_{\bar{R}_k}(x)}$ and $\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\geq \nu'_{\bar{R}_k}e^{i2\pi\nu'\bar{R}_k(x)}$ for all $k=1,2,\ldots,n$. Now

$$\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k} \geq \left(r^{1-\mu'_{\bar{R}_k}e^{i2\pi\mu'_{\bar{R}_k}(x)}}-1\right)^{w_k}$$
$$\Rightarrow \log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right)$$
$$\geq \log_r\left(1+\prod_{i=\mathrm{k}}^{n}\left(r^{1-\mu'\bar{R}_k e^{i2\pi\mu'\bar{R}_k(x)}}-1\right)^{w'_k}\right)$$
$$\Rightarrow 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right)$$
$$\leq 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu'_{\bar{R}_k}e^{i2\pi\mu'\bar{R}_k(x)}}-1\right)^{w_k}\right).$$

Similarly, it can be shown that

$$\log_r\left(1+\prod_{k=1}^{n}\left(r^{\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}}-1\right)^{w_k}\right)$$
$$\leq \log_r\left(1+\prod_{k=1}^{n}\left(r^{\eta'_{\bar{R}_k}e^{i2\pi\eta'_{\bar{R}_k}(x)}}-1\right)^{w_k}\right).$$

And

$$\geq \log_r\left(1+\prod_{k=1}^{n}\left(r^{\nu'\bar{R}_k e^{i2\pi\nu'_{\bar{R}_k}(x)}}-1\right)^{w_i}\right).$$

Therefore,

$$CPFFWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right) \leq CPFFWA\left(\bar{R}'_1,\bar{R}'_2,\ldots,\bar{R}'_n\right).$$

Now, the $CPFFOWA$ operator will be introduced.

□

Definition 3.10. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)$ $(k=1,2,\ldots,n)$ *be a collection of CPFNs. The function* $\bar{R}^n \to \bar{R}$ *such that,*

$$CPFFOWA\left(\bar{R}_1,\bar{R}_2,\ldots\bar{R}_n\right) = \bigoplus_{k=1}^{n} w_k \bar{R}_{\bar{R}(k)},$$

is defined as the $CPFFOWA$ *operator of dimension* n *with weight vector* $w=(w_1,w_2,\ldots,w_n)^t$ *of* $\bar{R}_k(k=1,2,\ldots,n); w_k \in [0,1]$ *and* $\sum_{k=1}^{n} w_k = 1, (\bar{R}(1),\bar{R}(2),\ldots,\bar{R}(n))$ *represents permutation of* $(k=1,2,\ldots,n)$*, such that for every* $k=1,2,\ldots,n, \bar{R}_{\bar{R}(k-1)} \geq \bar{R}_{\bar{R}(k)}$.

By using the above definition, we get the following theorem.

Theorem 3.11. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)$ $(k=1,2,\ldots,n)$ *be a collection of CPFNs. Then, a function* $\bar{R}^n \to \bar{R}$ *containing a weight vector* $w=(w_1,w_2,\ldots,w_n)^t; w_k \in [0,1]$ *and* $\sum_{k=1}^{n} w_k = 1$.

So,

$$CPFFOWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)=\oplus_{\mathrm{w}_{k=1}}^{n}w_k\bar{R}_{\bar{R}(k)}$$

$$=\begin{pmatrix} 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu_{\bar{R}_{\bar{R}(k)}}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{\eta_{\bar{R}(k)}e^{i2\pi\eta_{\bar{R}_k}(x)}}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right)\Big) \end{pmatrix},$$

is defined as the CPFFOWA operator of dimension n with $(\bar{R}(1),\bar{R}(2),\ldots,\bar{R}(n))$ represents permutation of $(k=1,2,\ldots,n)$, such that for every $k=1,2,\ldots,n$, $\bar{R}_{\bar{R}(k-1)}\geq\bar{R}_{\bar{R}(k)}$.

Proof. By using the $CPFFOWA$ operators, the following properties can be easily proved. □

Theorem 3.12. *(Idempotent). Let $\bar{R}_k=\left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)},\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)},\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k=1,2,\ldots,n)$ be a collection of identical CPFNs, i.e., $\bar{R}_i=\bar{R}$ for all k. Then, $CPFFOWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)=\bar{R}$.*

Theorem 3.13. *(Boundedness). Let $\bar{R}_k=\left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)},\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)},\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k=1,2,\ldots,n)$ be a collection of CPFNs. Let $\bar{R}^-=\min\left\{\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right\}$ and $\bar{R}^+=\max\left\{\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right\}$. Then, $\bar{R}^-\leq CPFFOWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)\leq\bar{R}^+$.*

Theorem 3.14. *(Monotonicity). Let $\bar{R}_k$ and $\bar{R}'_k(k=1,2,\ldots,n)$ be any two sets of CPFNs, if $\bar{R}_k\leq\bar{R}'_k$ for all k. Then, $CPFFOWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)\leq CPFFOWA\left(\bar{R}'_1,\bar{R}'_2,\ldots,\bar{R}'_n\right)$.*

Theorem 3.15. *(Commutativity). Let $\bar{R}_k$ and $\bar{R}'_k(k=1,2,\ldots,n)$ be any two sets of CPFNs, then CPFFOWA $\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)=CPFFOWA\left(\bar{R}'_1,\bar{R}'_2,\ldots,\bar{R}'_n\right)$, where $\bar{R}'_k$ denotes the permutation of $\bar{R}_k(k=1,2,\ldots,n)$.*

The weights associated with the CPFFWA operator in Definition 3.4 are in the most basic form of a CPF value, but the weights associated with the CPFFOWG operator in Definition 3.10 are not so. Which tells the weights associated with the CPFFWAO as well as the CPFFOWAO convey different viewpoints that are conflicting with one another. However, both viewpoints are intended to be similar in a broad sense. Only to overcome such a shortcoming, we are now introducing the CPFFHA operator.

Definition 3.16. *Let $\bar{R}_k=\left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)},\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)},\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k=1,2,\ldots,n)$ is a collection of CPFNs. Then, a function $\bar{R}^n\to\bar{R}$ such that,*

$$CPFFHA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)=\oplus_{k=1}^{n}\bar{w}_k\dot{\bar{R}}_{\bar{R}(k)}$$

$$\begin{pmatrix} 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\dot{\mu}_{\bar{R}_{\bar{R}(k)}}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{\bar{w}_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{\dot{\eta}_{\bar{R}_{\bar{R}(k)}}e^{i2\pi\eta_{\bar{R}_k}(x)}}-1\right)^{\bar{w}_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{\dot{\nu}_{\bar{R}_{\bar{R}(k)}}e^{i2\pi\nu_{\bar{R}_k}(x)}}-1\right)^{\bar{w}_k}\right)\Big) \end{pmatrix},$$

is defined as the CPFFHA operator of dimension n with $\bar{w}=(\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_n)^t$ as aggregation associated weight vector, $\sum_{k=1}^{n}\bar{w}_k=1, w=(w_1,w_2,\ldots,w_n)^t$ as weight vector of $\bar{R}_k(k=1,2,\ldots,n), w_k\in[0,1]$ and

$\sum_{k=1}^{n} w_k = 1$. $\bar{R}_{\bar{R}(k)}$ *represents* k^{th} *weighted greatest CPF value for* $\dot{p}_k\,(\dot{p}_k = nw_kp_k, k = 1, 2, \ldots, n)$*, and* n *being the balancing coefficient.*

Remark 3.17. *When* $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^t$*, then* $\bar{R}_k = n \times \frac{1}{n} \times \bar{R}_k = \bar{R}_k$ *for* $k = 1, 2, \ldots, n$*. When this happens the CPFFHA operator becomes the CPFFOWA operator. CPFFHA operator becomes the CPFFWA operator, if* $\bar{w} = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^t$*. As a result, the CPFFWA and the CPFFOWA operators are particular varieties of the CPFFHA operators. Therefore, the CPFFHA operator, which indicates the magnitude of the stated disagreements and their structured situations, appears to be a generalization of both the CPFFWA and the CPFFOWA operators.*

3.2 Complex Picture Fuzzy Frank Geometric Aggregation Operators

Definition 3.18. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right) (k = 1, 2, \ldots, n)$ *be a collection of CPFNs. Then a function* $p^n \to p$ *such that*

$$CPFFWG\,(R_1, R_2, \ldots, R_n) = \bigotimes_{k=1}^{n} (R_k)^{w_k},$$

is defined as the CPFFWG operator with $w = (w_k, w_k, \ldots, w_k)^t$ *as the weight vector of* $R_k (k = 1, 2, \ldots, n)$*,* $w_k \in [0, 1]$ *and* $\sum_{k=1}^{n} w_k = 1$*.*

As a result, the following consequential theorem is obtained.

Theorem 3.19. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right) (k = 1, 2, \ldots, n)$ *be a collection of CPFNs, then by using CPFFWG operator, their aggregated value is also a CPFN, and*

$$CPFFWG\,(R_1, R_2, \ldots, R_n) = \bigotimes_{k=1}^{n} (R_k)^{w_k}$$

$$= \begin{pmatrix} \log_r \left(1 + \prod_{k=1}^{n} \left(r^{\mu_{R_k} e^{i2\pi\mu_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right), \\ 1 - \log_r \left(1 + \prod_{k=1}^{n} \left(r^{1-\eta_{R_k} e^{i2\pi\eta_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right), \\ 1 - \log_r \left(1 + \prod_{k=1}^{n} \left(r^{1-\nu_{R_k} e^{i2\pi\nu_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right) \end{pmatrix}.$$

Proof. This theorem can be proved by using the method of proof of Theorem 3.5. □

Example 3.20. For $\bar{R}_1 = \left(0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.4)}\right), \bar{R}_2 = \left(0.1e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.2)}\right), \bar{R}_3 = \left(0.1e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.2)}, 0.7e^{i2\pi(0.5)}\right), \bar{R}_4 = \left(0.6e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.3)}\right)$ with weights $= (0.2, 0.3, 0.1, 0.4)$ and $r = 2$, step by step working of the CPFFWG operator is given as follows:

$$CPFFWG\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \begin{pmatrix} \log_r \left(1 + \prod_{k=1}^{n} \left(r^{\mu_{R_k} e^{i2\pi\mu_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right), \\ 1 - \log_r \left(1 + \prod_{k=1}^{n} \left(r^{1-\eta_{R_k} e^{i2\pi\eta_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right) \\ 1 - \log_r \left(1 + \prod_{k=1}^{n} \left(r^{1-\upsilon_{R_k} e^{i2\pi\upsilon_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right) \end{pmatrix}$$

$$\begin{pmatrix} \log_2 \begin{pmatrix} 1 + \begin{pmatrix} 2^{(0.5)e^{i2\pi(0.5)}} \\ -1 \end{pmatrix}^{0.2} + \begin{pmatrix} 2^{(0.1)e^{i2\pi(0.4)}} \\ -1 \end{pmatrix}^{0.3} \\ \begin{pmatrix} 2^{(0.1)e^{i2\pi(0.1)}} \\ -1 \end{pmatrix}^{0.1} + \begin{pmatrix} 2^{(0.6)e^{i2\pi(0.2)}} \\ -1 \end{pmatrix}^{0.4} \end{pmatrix} \\ 1-\log_2 \begin{pmatrix} 1 + \begin{pmatrix} 2^{(0.3)e^{i2\pi(0.1)}} \\ -1 \end{pmatrix}^{0.2} + \begin{pmatrix} 2^{(0.5)e^{i2\pi(0.4)}} \\ -1 \end{pmatrix}^{0.3} \\ \begin{pmatrix} 2^{(0.1)e^{i2\pi(0.2)}} \\ -1 \end{pmatrix}^{0.1} + \begin{pmatrix} 2^{(0.1)e^{i2\pi(0.2)}} \\ -1 \end{pmatrix}^{0.4} \end{pmatrix} \\ 1-\log_2 \begin{pmatrix} 1 + \begin{pmatrix} 2^{(0.1)e^{i2\pi(0.4)}} \\ -1 \end{pmatrix}^{0.2} + \begin{pmatrix} 2^{(0.4)e^{i2\pi(0.2)}} \\ -1 \end{pmatrix}^{0.3} \\ \begin{pmatrix} 2^{(0.7)e^{i2\pi(0.5)}} \\ -1 \end{pmatrix}^{0.1} + \begin{pmatrix} 2^{(0.3)e^{i2\pi(0.3)}} \\ -1 \end{pmatrix}^{0.4} \end{pmatrix} \end{pmatrix}$$

$$= \left(0.239e^{i2\pi(0.263)}, 0.272e^{i2\pi(0.221)}, 0.425e^{i2\pi(0.357)}\right).$$

The CPFFWG operator makes it simple to prove the following properties:

Theorem 3.21. *(Idempotent). Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *be identical CPFNs' collection, i.e., for every* $k, R_k = R$. *So,* $CPFFWG(R_1, R_2, \ldots, R_n) = R$.

Theorem 3.22. *(Boundedness). Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *be CPFNs' collection. Take* $R^- = \min\{R_1, R_2, \ldots, R_n\}$ *an* $R^+ = \max\{R_1, R_2, \ldots, R_n\}$. *Then* $R^- \leq CPFFWG(R_1, R_2, \ldots, R_n) \leq R^+$.

Theorem 3.23. *(Monotonicity Property). Let the sets* R_i *and* $R'_k(k = 1, 2, \ldots, n)$ *of CPFNs, if for every* $k, R_i \leq R'_k$, *we have* $CPFFWG(R_1, R_2, \ldots, R_n) \leq CPFFWG(R'_1, R'_2, \ldots, R'_n)$.

At this point, CPFFOWG operator has been introduced.

Definition 3.24. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *be CPFNs' collection. The n-dimensional CPFFOWG operator takes the form of a function* $R^n \to R$ *such that,*

$$CPFFOWG(R_1, R_2, \ldots R_n) = \bigotimes_{k=1}^{n} \left(R_{\rho(k)}\right)^{w_k},$$

where $w = (w_1, w_2, \ldots, w_n)^t$ *is a representation of weight vector for* $R_k(k = 1, 2, \ldots, n), w_k \in [0, 1]$; $\sum_{k=1}^{n} w_k = 1$. *Moreover,* $(\rho(1), \rho(2), \ldots, \rho(n))$ *appear to be representation of permutation of* $(k = 1, 2, \ldots, n)$, *such that for every* $k = 1, 2, \ldots, n$, $R_{\rho(k-1)} \geq R_{\rho(k)}$.

On the basis of Frank product operation on CPFN utilizing CPFFOWG operators, the following theorem is constructed.

Theorem 3.25. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *is CPFNs' collection. The n-dimensional CPFFOWG operator takes the form of a function* $R^n \to R$. *So,*

$$CPFFOWG(R_1, R_2, \ldots R_n) = \otimes_{k=1}^{n} \left(R_{\rho(k)}\right)^{w_k}$$

$$= \begin{pmatrix} \log_r \left(1+\prod_{k=1}^{n}\left(r^{\mu_{R_{\rho(k)}} e^{i2\pi\mu_{R_k}(x)}}-1\right)^{w_k}\right), \\ 1-\log_r \left(1+\prod_{k=1}^{n}\left(r^{1-\eta_{R_{\rho(k)}}}-1\right)^{w_k}\right), \\ 1-\log_r \left(1+\prod_{k=1}^{n}\left(r^{1-\nu_{R_{\rho(k)}}}-1\right)^{w_k}\right) \end{pmatrix}.$$

Here $w=(w_1,w_2,\ldots,w_n)^t$ is weight vector satisfying $w_k\in[0,1]; \sum_{k=1}^{n} w_k=1$. Moreover, $(\rho(1),\rho(2),\ldots,\rho(n))$ appear to be representation of permutation of $(k=1,2,\ldots,n)$, such that for every $k=1,2,\ldots,n, R_{\rho(k-1)}\geq R_{\rho(k)}$.

The CPFFOWG operator can be used to investigate the properties provided below.

Theorem 3.26. *(Idempotent). Let $\bar{R}_k=\left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)},\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)},\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k=1,2,\ldots,n)$ be a collection of identical CPFNs, i.e., $\bar{R}_k=\bar{R}$ for all k. Then, $CPFFOWG\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)=\bar{R}$.*

Theorem 3.27. *(Boundedness). Let $\bar{R}_k=\left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)},\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)},\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k=1,2,\ldots,n)$ be a collection of CPFN. Let $\bar{R}^-=\min\{\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\}$ an $\bar{R}^+=\max\{\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\}$. Then $\bar{R}^-\leq CPFFOWG(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n)\leq\bar{R}^+$.*

Theorem 3.28. *(Monotonicity). Let the sets $\bar{R}_k$ and $\bar{R}'_k(k=1,2,\ldots,n)$ of CPFNs, if for every $k,\bar{R}_k\leq\bar{R}'_k$, then $CFFFOWG\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)\leq CPFFOWAG\left(\bar{R}'_1,\bar{R}'_2,\ldots,\bar{R}'_n\right)$.*

Theorem 3.29. *(Commutativity). Let the sets $\bar{R}_k$ and $\bar{R}'_k(k=1,2,\ldots,n)$ of CPFNs, then CPFFOWG $\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)=CPFFOWG\left(\bar{R}'_1,\bar{R}'_2,\ldots,\bar{R}'_n\right)$, where $\bar{R}'_k$ denotes the permutation of $\bar{R}_k(k=1,2,\ldots,n)$.*

The weights associated with the CPFFWG operator in Definition 3.18 are in the most basic form of a PF value, but the weights associated with CPFFOWG operator in Definition 3.24 are in the real form of the ordered locations of CPF values. The weights given in CPFFWG and CPFFOWG operators convey different viewpoints that are conflicting with one another in this way. However, both viewpoints are intended to be similar in a broad sense. Only to overcome such a shortcoming, we are now introducing CPFFHG operator.

Definition 3.30. *Let $\bar{R}_k=\left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)},\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)},\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)$ (where k varies from 1 to n) be the CPFNs' collection. Then, a function $\bar{R}^n\to\bar{R}$ such that,*

$$CPFFHG\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)=\bigotimes_{k=1}^{n}\left(\bar{R}_{\rho(k)}\right)^{\bar{w}_k}$$

$$\begin{pmatrix} \log_r \left(1+\prod_{k=1}^{n}\left(r^{1-\dot{\mu}_{\bar{R}_{\rho(k)}} e^{i2\pi\mu_{\rho_k}(x)}}-1\right)^{\bar{w}_k}\right), \\ 1-\log_r \left(1+\prod_{k=1}^{n}\left(r^{1-\dot{\eta}_{\bar{R}_{\rho(k)}} e^{i2\pi\eta_{\rho_k}(x)}}-1\right)^{\bar{w}_k}\right), \\ 1-\log_r \left(1+\prod_{k=1}^{n}\left(r^{1-\nu_{\bar{R}_{\rho(k)}}} e^{i2\pi\nu_{\rho_k}(x)}-1\right)^{\bar{w}_k}\right) \end{pmatrix},$$

is defined as the CPFFHG operator of dimension n with $\bar{w}=(\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_n)^t$ as aggregation associated weight vector, $\sum_{k=1}^{n}\bar{w}_k=1, w=(w_1,w_2,\ldots,w_n)^t$ as weight vector of $\bar{R}_k(k=1,2,\ldots,n), w_k\in[0,1]$ and $\sum_{k=1}^{n}w_k=1$. $\bar{R}_{\bar{R}(k)}$ represents k^{th} weighted greatest CPF value for $\dot{p}_k\left(\dot{p}_k=nw_kp_k,k=1,2,\ldots,n\right)$, and n being the balancing coefficient.

Remark 3.31. *When* $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^t$*, then* $\bar{R}_k = n \times \frac{1}{n} \times \bar{R}_k = \bar{R}_k$ *for* $k = 1, 2, \ldots, n$*. When this happens, the CPFFHG operator becomes the CPFFOWG operator. The CPFFHG operator becomes the CPFFWG operator, if* $\bar{w} = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^t$*. As a result, the CPFFWG and the CPFFOWG operators are particular varieties of the CPFFHG operators. Therefore, the CPFFHG operator, which indicates the magnitude of the stated disagreements and their structured situations, appears to be a generalization of both the CPFFWG and the CPFFOWG operators.*

4 Model for the MADM Using Complex Picture Fuzzy Data

The purpose of this part is to discuss an effective strategy for solving the MADM process, as well as a method that may be used to identify the attribute weights that are necessary.

4.1 An Overview of the DM Issue

An innovative method to MADM problems has been proposed in this part, in which we will use CPF information along with manipulation of the CPFFWA and CPFFWG operators. For this purpose, let $P = \{P_1, P_2, \ldots, P_m\}$ represent a discrete collection of m alternatives to be chosen and $Q = \{Q_1, Q_2, \ldots, Q_n\}$ represent an order of attributes to be evaluated. Also, the weight vector is $w = \{w_1, w_2, \ldots, w_n\}$ related to attributes $H_j (j = 1, 2, \ldots, n)$ where $w_k (k = 1, 2, 3, \ldots, n) \in \mathrm{R}$ such that $w_k > 0$; $\sum_{k=1}^{n} w_k = 1$. We let $P = (\pi_{ij})_{m \times n} = ((\mu_{ij}, \eta_{ij}, v_{ij}))_{m \times n}$ as the CPF decision matrix, where π_{ij} is the possible value for which the alternative F_i satisfies the attribute H_j with the condition $\mu_{ij} + \eta_{ij} + v_{ij} \leq 1$ and $\mu_{ij}, \eta_{ij}, v_{ij} \in [0, 1]$. The illustration of this algorithm has been given following:

4.2 Determination of the Attribute Weights

During the decision-making process, DM challenges inevitably include numerous attributes. They don't have to give each other the same amount of weight. Take, for example, a decision where, in one case, the product's price is more important than its functionality; in another, the product's functionality may be more important than its price, reliability, or other considerations. This means that while solving a problem, various attributes play a role, each with its relevance. For DM to be very effective, selecting the appropriate attribute weights is crucial. Following is the method that can be taken into account for computing the attributes' weights accurately.

For a CPFN, $\bar{R} = \left(\mu_{\bar{R}}(x) e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}, v_{\bar{R}} e^{i2\pi v_{\bar{R}}(x)}\right)$, its hesitation degree has been given as:

$$\tau(p) = 2 - \left(\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)} + \nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}\right).$$

Cases involving DM always require an algorithm which has a lesser hesitancy degree due to the fact that attribute plays an important role during making process. Simply, we can say that an algorithm with lesser hesitancy degree would be more accurate than the one having a greater hesitancy degree. As a result the object would be more important when the hesitancy degree is lesser as compared to when hesitancy is greater. Keeping this in consideration, the following hesitancy matrix $\mathcal{R}$ has been constructed for the given alternatives

$$\mathcal{R} = \begin{pmatrix} \tau_{\bar{R}_{11}} & \tau_{\bar{R}_{12}} & \cdots & \tau_{\bar{R}_{1n}} \\ \tau_{\bar{R}_{21}} & \tau_{\bar{R}_{22}} & \cdots & \tau_{\bar{R}_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{\bar{R}_{m1}} & \tau_{\bar{R}_{m2}} & \cdots & \tau_{\bar{R}_{mn}} \end{pmatrix}.$$

Each $\tau_{\bar{R}_{pq}}$ has been calculated by using hesitation function. Hence, the weight vector $\hat{W}_j$ is determined as

$$\hat{\mathrm{W}}_j = \frac{2-\left(\frac{1}{m}\sum_{i=1}^{m}\tau_{\bar{R}_{ij}}\right)}{\sum_{j=1}^{n}\left(2-\left(\frac{1}{m}\sum_{i=1}^{m}\tau_{\bar{R}_{ij}}\right)\right)}.$$

Algorithm

Following is a presentation of the proposed MADM problem using CPF data related to the proposed CPFFWA and CPFFWG operators:

Step I: Construction of the CPF decision matrix $P=(\pi_{ij})_{m\times n}=((\mu_{ij},\eta_{ij},v_{ij}))_{m\times n}$.

Step II: Transformation of the matrix $P=(\pi_{ij})_{m\times n}=((\mu_{ij},\eta_{ij},v_{ij}))_{m\times n}$ into a normalize PF matrix $P'=\left(\pi'_{ij}\right)_{m\times n}=\left(\left(\mu'_{ij},\eta'_{ij},v'_{ij}\right)\right)_{m\times n}$ by Equation (1).

$$\pi'_{ij}=\begin{cases}(\mu_{ij},\eta_{ij},v_{ij}), & \text{if } H_j \text{ a benefit attribute}\\ (v_{ij},\eta_{ij},\mu_{ij}), & \text{if } H_j \text{ a cost attribute}\end{cases} \tag{1}$$

Step III: Determination of attribute weights of alternatives by using hesitation function, and the following

$$\hat{\mathrm{W}}_j = \frac{2-\left(\frac{1}{m}\sum_{i=1}^{m}\tau_{\bar{R}_{ij}}\right)}{\sum_{j=1}^{n}\left(2-\left(\frac{1}{m}\sum_{i=1}^{m}\tau_{\bar{R}_{ij}}\right)\right)}.$$

Step IV: Calculation of the information σ_k, which is collective, for the alternative A_k with the aid of following equation:

$$\sigma_f = CPFFWA\left(\pi'_{f'}{}', \pi'_{f2_f},\ldots,J'_{fn}\right)=\bigoplus_{k=1}^{n}(w_k\pi_{fk})$$

$$=\begin{pmatrix} 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu' p_{fk}}-1\right)^{w_k}\right), \log_r\left(1+\prod_{k=1}^{n}\left(r^{\eta' p_{fk}}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{v' p_{fk}}-1\right)^{w_k}\right)\end{pmatrix}. \tag{2}$$

And

$$\sigma_f = \mathrm{CPFFWG}\left(\pi_{f1}{}', \pi_{f2}{}',\ldots,\pi'_{fn}\right)=\bigotimes_{k=1}^{n}(\gamma_{fk})^{w_k}$$

$$=\begin{pmatrix} \log_r\left(1+\prod_{k=1}^{n}\left(r^{\mu' p_{fk}}-1\right)^{w_k}\right), 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\eta' p_{fk}}-1\right)^{w_k}\right), \\ 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-v' p_{fk}}-1\right)^{w_k}\right)\end{pmatrix}. \tag{3}$$

Step V: Usage of definition 2.3 to calculate the score value for each alternative.

Step VI: The optimal decision is to select F_k if $\Delta(\sigma_f)=\max_l\{\Delta(\sigma_l)\}$.

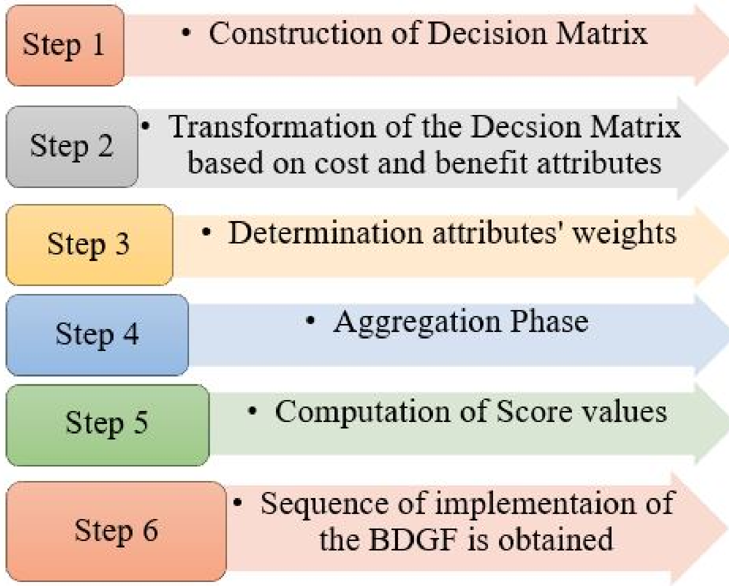

Figure 2: Sequential representation of the algorithm.

5 Numerical Illustration

In order to show the potential assessment of commercialization with the aid of PF data, we are prepared to draw a numerical problem in this part.

A BDGF is of great importance for companies since it regulates the rules under which data flows through different streams and appropriate access is granted to the users. Most of the companies work very much on improving their data assets but fail to understand that a robust data governance framework is needed in which segregation of users' access to sensitive data, access to the data within the organization among stakeholders in hierarchical order is of primary importance, and the responsibilities of the employees are well organized. Failing to have this type of robust framework can lead to uncertain results for the company, hence making the company's survival vulnerable.

Suppose a renowned international organization has come up with the idea of utilizing a handsome amount from its net annual profit in order to improve the company's reputation. Due to increasing risks and malfunctioning in their data assets, the company decides to use the amount in an optimized implementation of its BDGF.

BDGF covers more than one area to be focused on, and also cost attributes and benefit attributes spread uncertainty among decision makers. This brings them to think: In what order the BDGF should be implemented? What area of the BDGF should be focused on very first, followed by the next and so on. Hence an MADM problem arises to be solved for true implementation of the framework so that the company could get the most benefit and make its data assets more profitable. Following are the alternative choices/focus areas of the BDGF which have gained managing board's attention:

1. A_1 : Policy and Standards
2. A_2 : Data Quality
3. A_3 : Data Privacy and Security
4. A_4 : Architecture

Table 2: Decision matrix containing information about alternatives and attributes.

	B_1		$\boldsymbol{B}_2$
A_1	$\left(0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.1)}0.2e^{i2\pi(0.4)}\right)$	A_1	$\left(0.1e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.4)}0.4e^{i2\pi(0.2)}\right)$
A_2	$\left(0.6e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.3)}0.1e^{i2\pi(0.4)}\right)$	A_2	$\left(0.2e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.2)}0.3e^{i2\pi(0.1)}\right)$
A_3	$\left(0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.1)}0.2e^{i2\pi(0.4)}\right)$	A_3	$\left(0.3e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.4)}0.2e^{i2\pi(0.1)}\right)$
A_4	$\left(0.3e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.1)}0.3e^{i2\pi(0.2)}\right)$	A_4	$\left(0.4e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.4)}0.3e^{i2\pi(0.4)}\right)$
A_5	$\left(0.2e^{i2\pi(0.7)}, 0.4e^{i2\pi(0.1)}0.4e^{i2\pi(0.1)}\right)$	A_5	$\left(0.5e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.1)}0.1e^{i2\pi(0.6)}\right)$
	B_3		B_4
A_1	$\left(0.7e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.2)}0.1e^{i2\pi(0.1)}\right)$	A_1	$\left(0.6e^{i2\pi0.2}, 0.1e^{i2\pi(0.2)}0.3e^{i2\pi(0.3)}\right)$
A_2	$\left(0.4e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.1)}0.1e^{i2\pi(0.7)}\right)$	A_2	$\left(0.2e^{i2\pi0.3}, 0.6e^{i2\pi(0.3)}0.1e^{i2\pi(0.3)}\right)$
A_3	$\left(0.6e^{i2\pi(0.4)}, 0.1e^{i2\pi(0.5)}0.1e^{i2\pi(0.1)}\right)$	A_3	$0.5e^{i2\pi(0.1)}, 0.2e^{i2\pi0.1}\left(0.2e^{i2\pi(0.1)}\right)$
A_4	$\left(0.5e^{i2\pi(0.5)}, 0.2e^{i2\pi(0.1)}0.2e^{i2\pi(0.2)}\right)$	A_4	$\left(0.4e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.4)}0.2e^{i2\pi(0.1)}\right)$
A_5	$\left(0.8e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.2)}0.1e^{i2\pi(0.2)}\right)$	A_5	$\left(0.5e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.2)}0.1e^{i2\pi(0.1)}\right)$

5. A_5 : Data Warehouse and Business Intelligence (BI)

As it is difficult to choose amongst the options because they each meet distinct criteria, the problem of making a decision arises. Keeping this in view, the governing board has therefore established the following noteworthy attributes:

1. B_1 : Profit enhancement.

2. B_2 : Benefits of the clients.

3. B_3 : Maintenance cost.

4. B_4 : Mangement support.

Given that each alternative claims to maximize a distinct attribute, making a decision in this situation is challenging. A_1, A_2, A_3, A_4 and A_5 are focus areas of the BDGF.Moreover, B_1, B_2, B_4 are related to benefit attributes, and B_3 is related to cost attributes. Let $P = (\pi_{ij})_{m\times n} = ((\mu_{ij}, \eta_{ij}, v_{ij}))_{m\times n}$, a CPF matrix, be the representation of the alternative A_i with respect to the attributes B_i. Table 2 shows the assessment of the alternatives.

We use the CPFFWA and the CPFFWG operators to create an MADM theory with CPF data for the sake of choosing the best alternative $A_i (i = 1, 2, 3, 4, 5)$ by the following way:

Step I: The CPF decision matrix $P = (\pi_{ij})_{m\times n} = ((\mu_{ij}, \eta_{ij}, v_{ij}))_{m\times n}$ has been created as follows:

Step II: By a careful exploitation of Equation (1), the CPF matrix of table 1 has been normalized as $P' = \left({\pi'_{ij}}'\right)_{m\times n} = \left(\left(\mu'_{ij}, \eta'_{ij}, v'_{ij}\right)\right)_{m\times n}$ given as follows:

Step III: Now we determine the weights of attributes by using the hesitancy function and

$$\hat{\mathrm{W}}_j = \frac{2 - \left(\frac{1}{m}\sum_{i=1}^{m} \tau_{\bar{R}_{ij}}\right)}{\sum_{j=1}^{n}\left(2 - \left(\frac{1}{m}\sum_{i=1}^{m} \tau_{\bar{R}_{ij}}\right)\right)}.$$

The resultant attributes weights are obtained:

$$\hat{W} = (0.268012, 0.259366, 0.242075, 0.230548).$$

Table 3: Transformed decision matrix.

	B_1		B_1
A_1	$\left(0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.4)}\right)$	A_1	$\left(0.1e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.2)}\right)$
A_2	$\left(0.6e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.4)}\right)$	A_2	$\left(0.2e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.1)}\right)$
A_3	$\left(0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.4)}\right)$	A_3	$\left(0.3e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.1)}\right)$
A_4	$\left(0.3e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.1)}, 0.3e^{i2\pi(0.2)}\right)$	A_4	$\left(0.4e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.4)}\right)$
A_5	$\left(0.2e^{i2\pi(0.7)}, 0.4e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.1)}\right)$	A_5	$\left(0.5e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.6)}\right)$
	B_3		B_4
A_1	$\left(0.1e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.2)}, 0.7e^{i2\pi(0.5)}\right)$	A_1	$\left(0.6e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.3)}\right)$
A_2	$\left(0.1e^{i2\pi(0.7)}, 0.1e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.1)}\right)$	A_2	$\left(0.2e^{i2\pi(0.3)}, 0.6e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.3)}\right)$
A_3	$\left(0.1e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.4)}\right)$	A_3	$\left(0.5e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.1)}\right)$
A_4	$\left(0.2e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.1)}, 0.5e^{i2\pi(0.5)}\right)$	A_4	$\left(0.4e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.1)}\right)$
A_5	$\left(0.1e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.2)}, 0.8e^{i2\pi(0.2)}\right)$	A_5	$\left(0.5e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.1)}\right)$

Table 4: Aggregated vales.

	CPFFWA		CPFFWG
σ_1	$\left(0.35e^{i2\pi(0.32)}, 0.20e^{i2\pi(0.20)}, 0.36e^{i2\pi(0.33)}\right)$	σ_1	$\left(0.23e^{i2\pi(0.26)}, 0.27e^{i2\pi(0.23)}, 0.42e^{i2\pi(0.35)}\right)$
σ_2	$\left(0.30e^{i2\pi(0.48)}, 0.26e^{i2\pi(0.20)}, 0.18e^{i2\pi(0.18)}\right)$	σ_2	$\left(0.23e^{i2\pi(0.40)}, 0.34e^{i2\pi(0.22)}, 0.23e^{i2\pi(0.23)}\right)$
σ_3	$\left(0.36e^{i2\pi(0.27)}, 0.24e^{i2\pi(0.21)}, 0.26e^{i2\pi(0.20)}\right)$	σ_3	$\left(0.30e^{i2\pi(0.20)}, 0.25e^{i2\pi(0.29)}, 0.31e^{i2\pi(0.26)}\right)$
σ_4	$\left(0.32e^{i2\pi(0.37)}, 0.18e^{i2\pi(0.20)}, 0.31e^{i2\pi(0.25)}\right)$	σ_4	$\left(0.31e^{i2\pi(0.31)}, 0.19e^{i2\pi(0.25)}, 0.33e^{i2\pi(0.31)}\right)$
σ_5	$\left(0.34e^{i2\pi(0.39)}, 0.20e^{i2\pi(0.13)}0.24e^{i2\pi(0.19)}\right)$	σ_5	$\left(0.27e^{i2\pi(0.31)}, 0.26e^{i2\pi(0.14)}0.42e^{i2\pi(0.28)}\right)$

It is worth noting that $\sum_{j=1}^{n} \hat{W}_j = 1$

Step IV: By taking $r = 2$, and using the CPFFWA and the CPFFWG operators, the collective values $\sigma_f(f = 1, 2, 3, 4, 5)$ alternatives A_i 's have been obtained as follows:

Step V: The definition 2.3 has been used to compute score values $\Delta(\sigma_i)\,(i = 1, 2, 3, 4, 5)$ of the overall CPFN $\sigma_i(1, 2, 3, 4, 5)$ as follows:

Note 3: There may occur many instances, although the probability is somewhat very low, that the score values of two or more alternatives becomes equal. In that case the order of the alternatives is decided by their accuracy values.

Step VI: Finally, we can say that the company should select A_2 as the most preferable alternative in both the cases. So the enterprise should: first consider focusing on Data Quality, then on Data Warehouse and Business Intelligence (BI), followed by Data Privacy and Security, Architecture, and Policy and Standards

5.1 Analysis of Changing the Parameter r on the Outcome of Decision-making:

Various values, for the sake of ranking our considered alternatives, of operational parameter r could be applied in our proposed method. Keeping this in consideration, we set numerous r values for classifying the innovative

Table 5: Order of alternatives for implementation.

Operators	$\Delta(\sigma_1)$	$\Delta(\sigma_2)$	$\Delta(\sigma_3)$	$\Delta(\sigma_4)$	$\Delta(\sigma_5)$	Alternatives' Ranking/Order
CPFFWA	0.5096	0.6018	0.5540	0.5445	0.5851	$A_2 > A_5 > A_3 > A_4 > A_1$
CPFFWG	0.4300	0.5413	0.4820	0.4963	0.4701	$A_2 > A_5 > A_3 > A_4 > A_1$

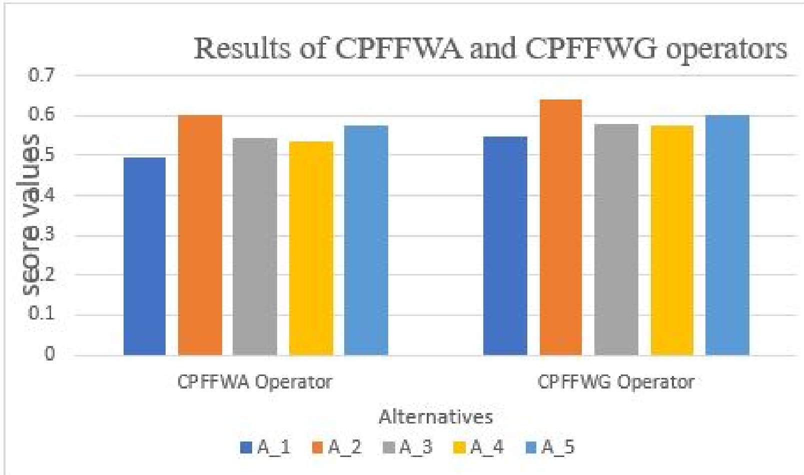

Figure 3: Score values obtained by CPFFWA and CPFFWG operators.

Table 6: Behavior of alternative with changing values of r in the CPFFWA operator.

r	$\Delta(\sigma_1)$	$\Delta(\sigma_2)$	$\Delta(\sigma_3)$	$\Delta(\sigma_4)$	$\Delta(\sigma_5)$	Order of ranking	Optimal alternative
2	0.4958	0.6033	0.5428	0.5330	0.5741	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
3	06819	0.7497	0.7115	0.7053	0.7313	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
4	0.7495	0.8016	0.7714	0.7665	0.7870	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
10	0.8482	0.8805	0.8623	0.8594	0.8718	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
30	0.8972	0.9191	0.9068	0.9048	0.9132	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2

numerical MADM example in order to investigate the adaptability and sensitivity of the parameter r.

From table 6 and figure 4, it is evident that if $2 \leq r \leq 30$ although the obtained aggregated outcomes of the alternatives $A_i(i = 1, 2, 3, 4, 5)$ are different yet the order of ranking does not change. The order of ranking of the alternatives is in this case is depicted to be $A_2 > A_5 > A_3 > A_4 > A_1$. Moreover, the figure 3 depicts that the value of alternatives keeps on becoming refined as the value of parameter r increases. For example, the value of alterative 2 starts from 0.6033 and reaches 0.9191 as the value of r reaches 30, hence showing a refined behavior. The similar trend can also be observed for the remaining alternative from the figure 3.

From table 7 and figure 5, it is evident that if $\leq r \leq 30$, altought the obtained aggregated outcomes of the alternatives $A_i(i = 1, 2, 3, 4, 5)$ are different yet the order of ranking does not change. The order of raking

Table 7: Behavior of alternative with changing values of r in the CPFFWG operator.

r	$\Delta(\sigma_1)$	$\Delta(\sigma_2)$	$\Delta(\sigma_3)$	$\Delta(\sigma_4)$	$\Delta(\sigma_5)$	Order of ranking	Optimal alternative
2	0.4300	0.5413	0.4820	0.4963	0.4701	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
3	0.2713	0.3415	0.3041	0.3131	0.3257	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
4	0.2150	0.2706	0.2410	0.2481	0.2350	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
10	0.1294	0.1629	0.1451	0.1494	0.1415	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
30	0.0876	0.1103	0.0982	0.1011	0.0958	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2

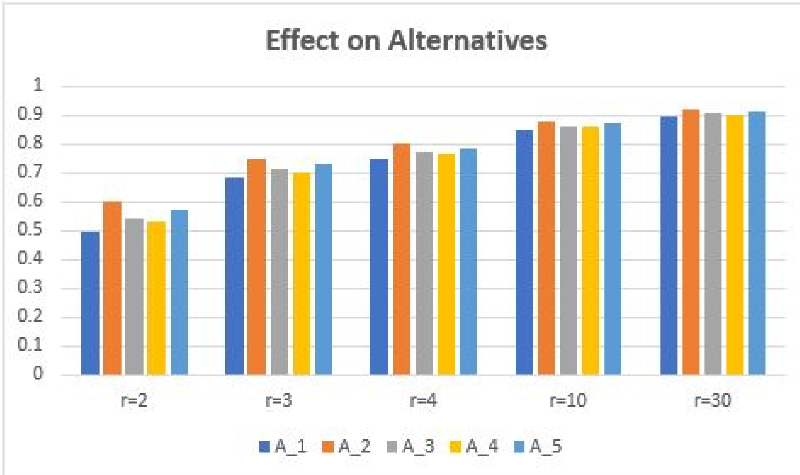

Figure 4: Departure of value of alternatives from their initial values to the finest value as r increases.

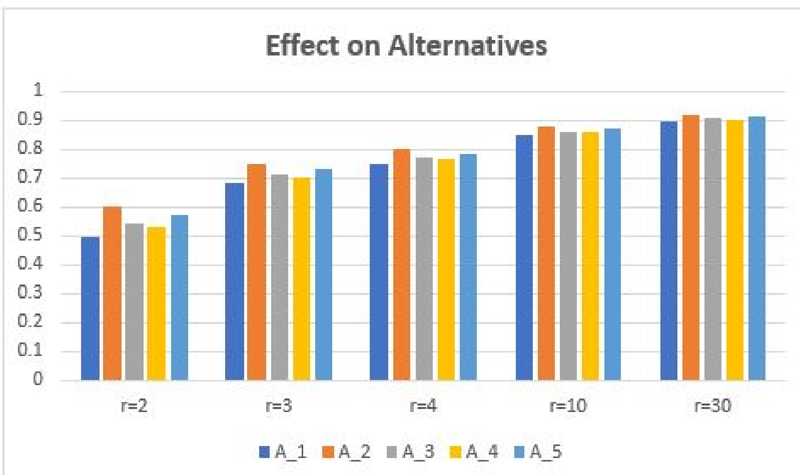

Figure 5: The value of alternatives keeps on becoming refined as the value of parameter r increases.

of the alternatives is in this case is depicted to be $A_4 > A_3 > A_5 > A_1 > A_2$. It is to be noted that although order of ranking of the alternatives are different, yet the best alternative is the same i.e., A_2. Moreover, the figure 5 depicts that the value of alternatives keeps on becoming refined as the value of parameter r increases. For example, the value of alterative 2 starts from 0.5413 and reaches 0.1103 - depicting the maximum refining- as the value of r reaches 50, hence showing a refined behavior. The similar trend can also be observed for the remaining alternative from the figure 5.

Generally, we can say that our proposed method has enough flexibility and accessibility which clearly allows decision makers to take the value of parameters based upon their choice.

It should be noted that CPFFWG operator has shown more flexible behavior than CPFFWA operator in our proposed MADM method by giving more refinement in the score values of the alternatives. Contrary to CPFFWG, the CPFFWA does not show that much flexibility. Therefore, it can be concluded that CPFFWG operator has responded more to variation in values of r than CPFFWA in this MADM problem and becomes more important for smoothly solving this type of MADM problem while keeping variations in values of r according to the decision maker's choice.

Table 8: Comparison of our suggested operators.

Aggregation Operators	Score values					Ranking/Order
	A_1	A_2	A_3	A_4	A_5	
Current work	0.4958	0.6033	0.5428	05330	0.5741	$A_2 > A_5 > A_3 > A_4 > A_1$
Current work	0.5494	0.6396	0.5791	0.5744	0.6018	$A_2 > A_5 > A_3 > A_4 > A_1$
CPFHWA[64]	-0.4151	-0.4934	-0.3868	-0.3376	-04312	$A_2 > A_5 > A_3 > A_4 > A_1$
CPFHWG[64]	-0.4969	-0.3737	-0.3545	-0.3174	-0.479	$A_2 > A_5 > A_3 > A_4 > A_1$
PFHWA[62]	0.5029	0.5640	0.5546	0.5085	0.5519	$A_2 > A_5 > A_3 > A_4 > A_1$
PFHWG[62]	0.4002	0.4956	0.4865	0.4892	0.4192	$A_2 > A_5 > A_3 > A_4 > A_1$
CIFWA[15]	0.36284	0.4516	0.4371	0.4216	0.4501	$A_2 > A_5 > A_3 > A_4 > A_1$

6 Comparative Studies

We contrast our suggested Frank aggregation operators with other current, well-known aggregation operators in the CPF context to ensure their usefulness and to explore their merits. Table 3 presents the comparison outcomes.

6.1 Comparison with Picture Fuzzy (PF) Operators:

We contrast our suggested approach with PFSs operators. When compared to the PFHWG and PFHWA operators given by Wei [62], we can see that in the presence of parameter r they appear to be mere particular incidences of our suggested operators. Moreover, these operators also suffer from the absence of a periodicity function due to which they just become very particular cases of our suggested operators when periodicity functions are taken to be zero. Furthermore owing to the fact the order of alternatives (sequence of implementation of the BDGF) remains unaltered, our approach becomes more consistent. These observations incline us to state that our newly established techniques are therefore more broadly applicable.

6.2 Comparison with CIF Operators:

We compare the operators of CIFSs with our proposed method. In contrast to the CIFWA operators [15], we can observe that they seem to be just specific instances of our proposed operators when parameter r is present. Moreover, these operators lack a hesitation function, which makes them special instances of our proposed operators when the hesitancy functions are assumed to be zero. Furthermore, our method becomes more consistent because the alternatives' order (sequence of implementation of the BDGF doesn't change. These findings lead us to the following conclusion: When compared to CIF operators, our recently developed methods are therefore more widely applicable.

6.3 Comparison with CPF Operators:

We contrast our suggested approach with CPFSs operators. When compared to the CPFHWA and CPFHWG operators [64] we can see that the ranking of the alternatives does not change. The score values obtained by these operators are compared with our suggested operators and are shown in Table 8, the ranking order (sequence of implementation of the BDGF has been shown in Table 9.

Additionally, our observation is also that some AOs [65] and [56] are not capable of dealing with our decision matrix. Our suggested operators, which are based on the t-norm and t-conorm of Frank, are more sophisticated and may take into account the link between different arguments. Our suggested operators also

Table 9: Comparison of our suggested operators.

Aggregation Operators	Framework	Ranking/Order
CPFFWA (our proposed)	CPFSs	$A_2 > A_5 > A_3 > A_4 > A_1$
CPFFWG (our proposed)	CPFSs	$A_2 > A_5 > A_3 > A_4 > A_1$
CPFHWA[64]	CPFSs	$A_2 > A_5 > A_3 > A_4 > A_1$
CPFHWF[64]	CPFSs	$A_2 > A_5 > A_3 > A_4 > A_1$
CIFWA[15]	CIFSS	$A_2 > A_5 > A_3 > A_4 > A_1$
CPyFWA[65]	CPyFs	Failed
CIVFWA[56]	CIVFSs	Failed
PFHWA[62]	PFSs	$A_2 > A_5 > A_3 > A_4 > A_1$
PFHWG[62]	PFSs	$A_2 > A_5 > A_3 > A_4 > A_1$

display Lukasiewicz sum and product as the parameter gets closer to infinity. In light of the various values of r, we have come to the conclusion that practically all of the arithmetic and geometric aggregation operations related to CPFNs are, in fact, a part of CPF Frank aggregating operator.

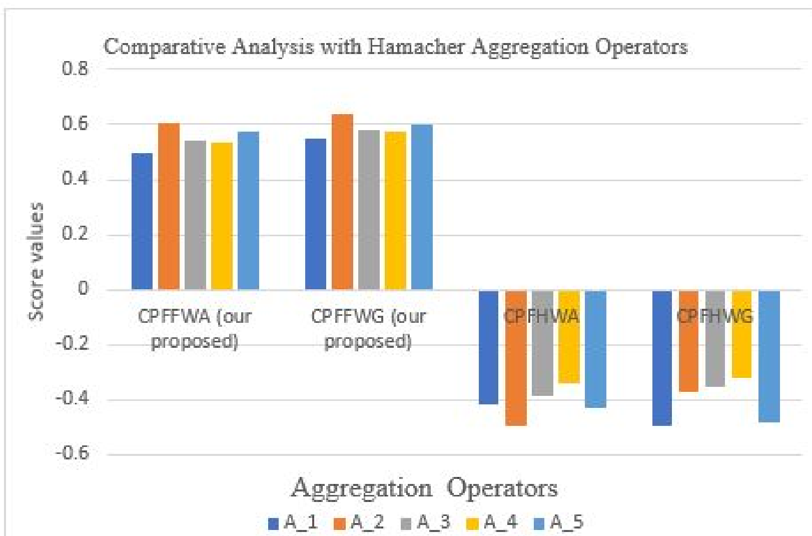

Figure 6: comparison with CPFHWA and CPHWG operators.

7 Conclusion

In conclusion, our study has developed a novel approach, CPFSs, which extends standard PFSs and IFSs. The study concentrated on the MADM problems, demonstrating the versatility and usefulness of CPFSs through the use of multiple aggregation approaches. Our research revealed compelling findings, demonstrating the stability and superiority of complicated CPFFA operators. Specifically, the use of the CPFFWA, CPFFWG, CPFFHA, and CPFFOWA operators has enhanced the applicability of CPFSs for MADM problems. In aggregating complex judgment criteria, Frank weighted geometric performed admirably. The methodological approach, which included Frank techniques for aggregation, gave a methodical framework for addressing MADM problems more rigorously than in the frameworks of PFSs, IFSs, CIFS etc. This research article has enabled decision makers to make more sound and appealing decisions for their maximum benefit. Despite these encouraging outcomes, it is critical to recognize some limits. The study concentrated on a specific set of aggregation approaches and may not cover the complete range of options. The use of CPFSs in real-world applications necessitates additional validation and testing. To address the aforementioned limitations and extend the impact of our findings, future research directions should explore the following: Investigate extensions of CPFSs, such as complex q-rung picture fuzzy sets, to enhance the scope and applicability of the proposed model. Extend the application of CPFSs to various decision-making techniques beyond MADM, including but not limited to COPRAS [66], and VIKOR[67]. Explore and develop additional complex picture fuzzy aggregation operators utilizing different t-norms and t-conorms to enhance the flexibility and robustness of the proposed model. Conduct extensive empirical studies to validate the proposed CPFS-based model in diverse real-world decision-making scenarios, ensuring their practical relevance and reliability. Our method also finds its applications in some notable problems such as plastic waste management [68], electric vehicle charging station site selection problems[69], and bio-medical waste management[70] etc. By focusing on the results obtained and providing a detailed methodological overview, we aim to offer a comprehensive understanding of the capabilities and limitations of CPFSs, facilitating their practical adoption in decision-making processes.

Conflict of Interest: The authors declare no conflict of interest.

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