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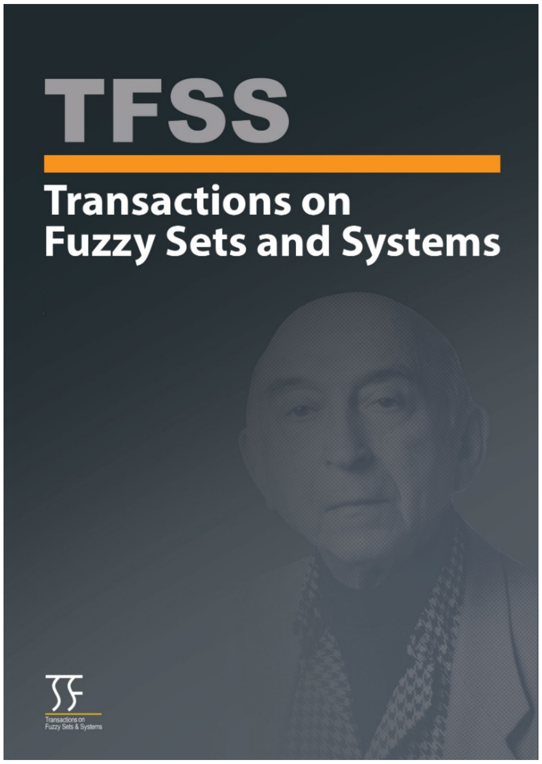

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Fuzzy Implication Operators Applied to Country Health Preparation

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Fuzzy Implication Operators Applied to Country Health Preparation

John N. Mordeson, **Sunil Mathew***, **Aswathi Prabhath**

Abstract. We use a new method to determine a fuzzy similarity measure using fuzzy implication operators. We use this method to determine the fuzzy similarity between the two rankings of countries involving health security and health care. We then find a fuzzy similarity of countries involving the two rankings of countries with respect to national disaster and political disaster.

AMS Subject Classification 2020: 94D05; 03E72
Keywords and Phrases: Fuzzy implication operators, Fuzzy similarity measures, Global health security index, Health care, National disaster, Political stability, Country rankings.

1 Introduction

The Global Health Security Index states that all countries remain dangerously unprepared for future epidemic and pandemic threats, including threats potentially more devastating than COVID-19, [1]. In [2], we ranked the Organization for Economic Cooperation and Development (OECD) countries with respect to their preparation. In [3], countries are ranked with respect to their health care. We find the fuzzy similarity measure between these two rankings. We use implication operators to define a new fuzzy similarity measure to find the fuzzy similarity of these rankings. We also consider the natural disaster risk, the political stability of OECD countries. We provide the rankings as given in [4, 5, 6]. The report in [4] considers a country's vulnerability and exposure to natural hazards to determine a ranking of countries around the world based on their natural disaster risk. The index of Political Stability and Absence of Violence/Terrorism measures perceptions of the likelihood that the government will be destabilized or overthrown by unconstitutional or violent means, including politically motivated violence and terrorism. We used five different fuzzy similarity measures. In three cases, we found the similarities to be medium and in two, we found the similarity to be low.

Let X be a set. Then the **fuzzy power set** of X, denoted $\mathcal{FP}(X)$, is the set of all fuzzy subsets of X. Define the relations $\vee, \wedge$ on the closed interval $[0,1]$ by for all $a, b \in [0,1], a \vee b$ is the maximum of a and b and $a \wedge b$ is the minimum of a and b.

Define $\overline{\wedge} : [0,1] \times [0,1] \to [0,1]$ by $\overline{\wedge}(a,b) = 1$ if $a = b$ and $a \wedge b$ if $a \neq b$. Define $\varnothing : [0,1] \times [0,1] \to [0,1]$ by $\varnothing(a,b) = 1$ if $a = b$ and $\frac{a \wedge b}{a \vee b}$ if $a \neq b$. Note that for all $a, b \in [0,1]$, $\varnothing(a,b) = \frac{a \wedge b}{a \vee b}$.

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2 Preliminary Results

Definition 2.1. *Let S be a function of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ into $[0,1]$. Then S is called a* ***fuzzy similarity measure*** *on $\mathcal{FP}(X)$ if the following properties hold: $\forall \mu, \nu, \rho \in \mathcal{FP}(X)$:*

(1)$S(\mu,\nu) = S(\nu,\mu)$;

(2)$S(\mu,\nu) = 1$ *if and only if* $\mu = \nu$;

(3) *If* $\mu \subseteq \nu \subseteq \rho$, *then* $S(\mu,\rho) \leq S(\mu,\nu) \wedge S(\nu,\rho)$;

(4) *If* $S(\mu,\nu) = 0$, *then* $\forall x \in X, \mu(x) \wedge \nu(x) = 0$.

Example 2.2. Let μ, ν be fuzzy subsets of a set X. Then M and S are fuzzy similarity measures on $\mathcal{FP}(X)$, where

$$M(\mu,\nu) = \frac{\sum_{x\in X}\mu(x) \wedge \nu(x)}{\sum_{x\in X}\mu(x) \vee \nu(x)},$$

$$S(\mu,\nu) = 1 - \frac{\sum_{x\in X}|\mu(x) - \nu(x)|}{\sum_{x\in X}(\mu(x) + \nu(x))}.$$

Results concerning fuzzy similarity measures can be found in [7, 8].

Definition 2.3. *([9], p. 14) Let I be a function of $[0,1] \times [0,1]$ into $[0,1]$ such that $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$. Then I is called an* ***implication operator****.*

An implication operator I is said to satisfy the **identity principle** if $I(x,x) = 1$ for all $x \in [0,1]$. An implication operator is said to satisfy the **ordering principle** if $x \leq y \Leftrightarrow I(x,y) = 1$, [10]. Clearly, the ordering principle implies the identity principle.

I_1, I_2, and L defined below are implication operators that satisfy the ordering principle.

Example 2.4. Let $x, y \in [0,1]$.

(1) Godel implication operator:$I_1(x,y) = 1$ if $x \leq y, I_1(x,y) = y$ otherwise.

(2) Goguen implication operator: $I_2(x,y) = 1$ if $x \leq y$ and $I_2(x,y) = y/x$ otherwise

(3) Luckasiewicz implication operator: $L(x,y) = (1 - x + y) \wedge 1$.

By ([2], Theorem 3.1), S_L is a fuzzy similarity, where $S_L(\mu_A,\mu_B) = \frac{1}{n}\sum_{x\in X}(1 - \mu_A(x)) \wedge (1 - \mu_B(x)) \wedge (\mu_A(x)) \wedge \mu_B(x)$.

Definition 2.5. *([9], p. 15) Let I be an implication operator. Define the fuzzy subset E_I of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ by for all $\mu, \nu \in \mathcal{FP}(X)$,*

$$E_I(\mu,\nu) = \wedge\{\wedge\{I(\mu(x),\nu(x))|x \in X\}, \wedge\{I(\nu(x),\mu(x))|x \in X\}\}.$$

Then $E_I(\mu,\nu)$ is called the ***degree of sameness*** *of μ and ν.*

In [2], it was decided that the following definition would be more suitable than the previous definition for defining fuzzy similarity measures from implication operators.

Definition 2.6. *Let I be an implication operator. Define $S : \mathcal{FP}(X) \times \mathcal{FP}(X) \to [0,1]$ by for all $(\mu,\nu) \in \mathcal{FP}(X) \times \mathcal{FP}(X), S(\mu,\nu) = \frac{1}{n}\sum_{x\in X} I((\mu(x),\nu(x)) \wedge I((\nu(x),\mu(x)))$. Then S is called a* ***degree of likeness****.*

In ([2], Theorem 2.7), it was shown that the function S of Definition 2.6 is a fuzzy similarity measure.

An implication operator I is called a **hybrid monotonous implication operator if** $I(x,\_)$ is non decreasing for all $x \in [0,1]$ and $i(\_,y)$ is nonincreasing for all $y \in [0,1]$.

Other implication operators can be found in [9].

Let X be a set with n elements, $n > 1$, say $X = \{x_1, ..., x_n)$. Let A be one-to-one function of X onto $\{1, ..., n\}$. Then A is called a **ranking** of X. Define the fuzzy subset μ_A of X by for all $x \in X, \mu_A(x) = \frac{A(x)}{n}$. Then μ_A is called the **fuzzy subset associated with** A.

For two rankings A and B of $X, \sum_{x\in X}(A(x) + B(x)) = n(n+1)$ and so $\sum_{x\in X}(\mu_A(x) + \mu_B(x)) = n+1$. Thus for S of Example 2.2,

$$S(\mu_A, \mu_B) = 1 - \frac{\sum_{x\in X} |\mu_A(x) - \nu_B(x)|}{n+1}.$$

3 Main Results

Let S_1 and S_2 be the fuzzy similarity measures defined by I_1 and I_2 under Definition 2.6, respectively. Then

$$\begin{aligned} S_1(\mu, \nu) &= \frac{1}{n} \sum_{x\in X} \mu(x) \barwedge \nu(x), \\ S_2(\mu, \nu) &= \frac{1}{n} \sum_{x\in X} \mu(x) \varnothing \nu(x). \end{aligned}$$

We next consider how small S_1 can be with respect to rankings A and B.

Suppose n is even. Let A be the ranking: $1, 2, ... \frac{n}{2}, \frac{n+2}{2} ..., n-1, n$ and let B be the ranking $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, , , , , 2, 1$. Then

$$\begin{aligned} S_1(\mu_A, \mu_B) &= \frac{1}{n} \sum_{x\in X} \mu_A(x) \barwedge \mu_B(x) = \frac{1}{n}(2(1+2+...+\frac{n}{2}))\frac{1}{n} \\ &= \frac{1}{n}(2([\frac{n}{2}(\frac{n}{2}+1)])/2)\frac{1}{n} = \frac{1}{n^2}(\frac{n^2}{4} + \frac{n}{2}) = \frac{1}{4} + \frac{1}{2n}. \end{aligned}$$

Suppose that n is odd. Let A be the ranking $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ and B be the ranking $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. Then

$$\begin{aligned} S_1(\mu_A, \mu_B) &= \frac{1}{n} \sum_{x\in X} \mu_A(x) \barwedge \mu_B(x) = \frac{1}{n}(1 + 2(1+2+...+\frac{n-1}{2}))\frac{1}{n} \\ &= \frac{1}{n^2}(1 + 2(\frac{n-1}{2})(\frac{n-1}{2}+1)/2) = \frac{1}{n^2}(1 + 2(\frac{n-1}{2}\frac{n+1}{2}\frac{1}{2})) \\ &= \frac{1}{n^2}(1 + \frac{n^2-1}{4}) = \frac{1}{n^2} + \frac{1}{4} - \frac{1}{4n^2} = \frac{1}{4} + \frac{3}{4n^2}. \end{aligned}$$

Example 3.1. Let $n = 6$. Let A be the ranking $1, 2, ..., 5, 6$ and B the ranking $6, ..., 2, 1$. Then $\mu_A(x_i) = \frac{i}{6}$ and $B(x_i) = \frac{6-i+1}{6}, i = 1, 2, ..., 6$. Hence

$$\begin{aligned} \frac{\mu_A(x_1) \wedge \mu_B(x_1)}{\mu_A(x_1) \vee \mu_B(x_1)} &= \frac{\frac{1}{6}}{\frac{6}{6}} = \frac{\mu_A(x_6) \wedge \mu_B(x_6)}{\mu_A(x_6) \vee \mu_B(x_6)}, \\ \frac{\mu_A(x_2) \wedge \mu_B(x_2)}{\mu_A(x_2) \vee \mu_B(x_2)} &= \frac{\frac{2}{6}}{\frac{5}{6}} = \frac{\mu_A(x_5) \wedge \mu_B(x_5)}{\mu_A(x_5) \vee \mu_B(x_5)}, \\ \frac{\mu_A(x_3) \wedge \mu_B(x_3)}{\mu_A(x_3) \vee \mu_B(x_3)} &= \frac{\frac{3}{6}}{\frac{4}{6}} = \frac{\mu_A(x_4) \wedge \mu_B(x_4)}{\mu_A(x_4) \vee \mu_B(x_4)}. \end{aligned}$$

Let $n = 5$. Let A be the ranking $1, 2, ..., 5$,and B the ranking $5, ..., 2, 1$. Then $\mu_A(x_i) = \frac{i}{5}$ and $B(x_i) = \frac{5-i+1}{5}, i = 1, 2, ..., 5$. Hence

$$\begin{aligned}
\frac{\mu_A(x_1) \wedge \mu_B(x_1)}{\mu_A(x_1) \vee \mu_B(x_1)} &= \frac{\frac{1}{5}}{\frac{5}{5}} = \frac{\mu_A(x_5) \wedge \mu_B(x_5)}{\mu_A(x_5) \vee \mu_B(x_5)}, \\
\frac{\mu_A(x_2) \wedge \mu_B(x_2)}{\mu_A(x_2) \vee \mu_B(x_2)} &= \frac{\frac{2}{5}}{\frac{4}{5}} = \frac{\mu_A(x_4) \wedge \mu_B(x_4)}{\mu_A(x_4) \vee \mu_B(x_4)}, \\
\frac{\mu_A(x_3) \wedge \mu_B(x_3)}{\mu_A(x_3) \vee \mu_B(x_3)} &= \frac{\frac{3}{5}}{\frac{3}{5}} = \frac{\mu_A(x_3) \wedge \mu_B(x_3)}{\mu_A(x_3) \vee \mu_B(x_3)}.
\end{aligned}$$

We see that for n odd, the middle term will yield the value 1.

The following discussion is to determine the smallest value a fuzzy similarity measure can be with respect to rankings. Let S be any fuzzy similarity measure with respect to some rankings A and B. We determine the smallest value S can be for the following reason: Say, the smallest value S can be is S^*. Then the ratio $\frac{S-S^*}{1-S^*}$ ranges from 0 to 1. A clearer picture of the similarity is thus provided.

Lemma 3.2. *(1) Suppose n is even. Let A be the ranking: $1, 2, ...\frac{n}{2}, \frac{n+2}{2}..., n-1, n$ and let B be the ranking $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, , , , , 2, 1$. Then $\frac{1}{n}(\frac{\frac{n}{2}}{\frac{n}{2}+1} + ... + \frac{2}{n-1} + \frac{1}{n}) = (n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}$.*

(2) Suppose n is odd. Let A be the ranking $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ and B be the ranking $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. Then $\frac{\frac{n-1}{2}}{\frac{n+3}{2}} + ... + \frac{2}{n-1} + \frac{1}{n} = (n+1)\sum_{j=\frac{n+1}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}$.

Proof. (1) $\frac{\frac{n}{2}}{\frac{n}{2}+1} + ... + \frac{2}{n-1} + \frac{1}{n} =$

$$\begin{aligned}
\sum_{i=1}^{\frac{n}{2}} \frac{i}{n-i+1} &= \sum_{j=\frac{n}{2}+1}^{n} \frac{n-j+1}{j} = \sum_{j=\frac{n}{2}+1}^{n} (\frac{n}{j} - 1 + \frac{1}{j}) \\
&= (n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}.
\end{aligned}$$

(2) $\frac{\frac{n-1}{2}}{\frac{n+3}{2}} + ... + \frac{2}{n-1} + \frac{1}{n} = \sum_{i=1}^{\frac{n-1}{2}} \frac{i}{n-i+1}$.

Let $j = n-i+1$. Then $i = n-j+1$ and $j = n, n-1, ..., \frac{n}{2} + \frac{3}{2}$. Now

$$\begin{aligned}
\sum_{i=1}^{\frac{n-1}{2}} \frac{i}{n-i+1} &= \sum_{j=\frac{n+3}{2}}^{n} \frac{n-j+1}{j} = \sum_{j=\frac{n+3}{2}}^{n} (\frac{n}{j} - 1 + \frac{1}{j}) \\
&= (n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}.
\end{aligned}$$

□

Theorem 3.3. *(1) Suppose n is even. Let A be the ranking: $1, 2, ...\frac{n}{2}, \frac{n+2}{2}..., n-1, n$ and let B be the ranking $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, , , , , 2, 1$. Then $S_2(\mu_A, \mu_B) = \frac{2}{n}[(n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}]$.*

(2) Suppose n is odd. Let A be the ranking $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ and B be the ranking $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. Then $S_2(\mu_A, \mu_B) = \frac{1}{n}[[(n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}]2 + 1]$.

Proof. (1) $S_2(\mu_A, \mu_B) = \frac{1}{n}(\frac{\frac{n}{2}}{\frac{n+2}{2}} + ... + \frac{2}{n-1} + \frac{1}{n})2 = \frac{1}{n}(\frac{\frac{n}{2}}{\frac{n}{2}+1} + ... + \frac{2}{n-1} + \frac{1}{n})2 = \frac{2}{n}[(n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}]$ by Lemma 3.2 (1).

(2) $S_2(\mu_A, \mu_B) = \frac{1}{n}((\frac{\frac{n-1}{2}}{\frac{n+3}{2}} + ... + \frac{2}{n-1} + \frac{1}{n})2 + 1) = \frac{1}{n}[[(n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}]2 + 1]$ by Lemma 3.2 (2). □

We next determine approximate values for $\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}$ when n is even and $\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}$ when n is odd. Recall that $H_n = \sum_{j=1}^{n} \frac{1}{j}$ is a harmonic sum which sums approximately to $\gamma + \ln 2$, where γ is the Euler-Mascheroni constant, $\gamma \approx 0.5772$ and where $\approx$ denotes approximately equal to.

Let n be even. Consider $\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}$. We have $\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j} = \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{\frac{n}{2}} \frac{1}{j} \approx \gamma + \ln n - (\gamma + \ln \frac{n}{2}) = \ln n - \ln \frac{n}{2} = \ln 2$.

Let n be odd. Consider $\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}$. We have $\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j} = \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{\frac{n+1}{2}} \frac{1}{j} \approx \gamma + \ln n - (\gamma + \ln \frac{n+1}{2}) = \ln n - \ln \frac{n+1}{2} = \ln \frac{2n}{n+1}$.

Theorem 3.4. *(1) Suppose n is even. Let A be the ranking: $1, 2, ... \frac{n}{2}, \frac{n+2}{2} ..., n-1, n$ and let B be the ranking $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, , , , , 2, 1$. Then $S_2(\mu_A, \mu_B) \approx 0.386 + \frac{2}{n} \ln 2$.*

(2) Suppose n is odd. Let A be the ranking $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ and B be the ranking $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. Then $S_2(\mu_A, \mu_B) \approx 2 \ln \frac{2n}{n+1} + \frac{2}{n} \ln \frac{2n}{n+1} - 1 + \frac{2}{n}$.

Proof. Theorem 3.3 is used in the following arguments.

(1) We have

$$\begin{aligned} S_2(\mu_A, \mu_B) &= \frac{1}{n}(\sum_{j=1}^{\frac{n}{2}} \frac{j}{n-j+1})2 \\ &= \frac{2}{n}[(n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}] \\ &\approx \frac{2}{n}[(n+1)\ln 2 - \frac{n}{2}] \\ &= (2 + \frac{2}{n})\ln 2 - 1 \\ &= 2\ln 2 + \frac{2}{n}\ln 2 - 1 \\ &\approx 0.386 + \frac{2}{n}\ln 2. \end{aligned}$$

(2) We have

$$\begin{aligned} S_2(\mu_A, \mu_B) &= \frac{1}{n}[[(n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}]2 + 1] \\ &= \frac{2}{n}[(n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}] + \frac{1}{n} \\ &\approx \frac{2}{n}[(n+1)\ln \frac{2n}{n+1} - \frac{n-1}{2}] + \frac{1}{n} \\ &= (2 + \frac{2}{n})\ln \frac{2n}{n+1} - (1 - \frac{1}{n}) + \frac{1}{n} \\ &= 2\ln \frac{2n}{n+1} + \frac{2}{n}\ln \frac{2n}{n+1} - 1 + \frac{2}{n}. \end{aligned}$$

□

Proposition 3.5. *Let $S_1, ..., S_n$ be fuzzy similarity measures on $\mathcal{FP}(X)$. Let $w_i \in [0,1]$ be such that $\sum_{i=1}^n w_i = 1$. Then $\sum_{i=1}^n w_i S_i$ is a fuzzy similarity measure on $\mathcal{FP}(X)$.*

Proof. Let $S = \sum_{i=1}^n w_i S_i$ and $\mu, \nu, \rho \in \mathcal{FP}(X)$. Then $S(\mu,\nu) = \sum_{i=1}^n w_i S_i(\mu,\nu) = \sum_{i=1}^n w_i S_i(\nu,\mu) = S(\nu,\mu)$. Now $S(\mu,\nu) = 1 \Leftrightarrow \sum_{i=1}^n w_i S_i(\mu,\nu) = 1 \Leftrightarrow S_i(\mu,\nu) = 1$ for $i = 1,...,n \Leftrightarrow \mu = \nu$. Suppose that $\mu \subseteq \nu \subseteq \rho$. Then $S_i(\mu,\rho) \leq S_i(\mu,\nu) \wedge S_i(\nu,\rho), i = 1,...,n$. Hence

$$\begin{aligned}\sum_{i=1}^n w_i S_i(\mu,\rho) &\leq \sum_{i=1}^n w_i[S_i(\mu,\nu) \wedge S_i(\nu,\rho)] = \sum_{i=1}^n w_i S_i(\mu,\nu) \wedge w_i S_i(\nu,\rho) \\ &\leq \sum_{i=1}^n w_i S_i(\mu,\nu) \wedge \sum_{i=1}^n w_i S_i(\nu,\rho) = S(\mu,\nu) \wedge S(\nu,\rho).\end{aligned}$$

Suppose $S(\mu,\nu) = 0$. Then $\sum_{i=1}^n w_i S_i(\mu,\nu) = 0$. Thus $S_i(\mu,\nu) = 0$ for all i such that $w_i > 0$. Thus for all $x \in X, \mu(x) \wedge \nu(x) = 0$. □

Proposition 3.6. *Let $S_1, ..., S_n$ be fuzzy similarity measures on $\mathcal{FP}(X)$. Let $w_i \in [0,1]$ be such that $\sum_{i=1}^n w_i = 1$. Let a_i be the smallest value S_i can be, $i = 1,...,n$. Then $\sum_{i=1}^n w_i a_i$ is the smallest value $\sum_{i=1}^n w_i S_i$ can be.*

Proof. Suppose $(\sum_{i=1}^n w_i S_i)(\mu,\nu) = b$. Then $\sum_{i=1}^n (w_i S_i)(\mu,\nu) = b$. Let $S_i(\mu,\nu) = b_i, i = 1,...,n$. Then $b_i \geq a_i, i = 1,...,n$. Now $b = \sum_{i=1}^n w_i b_i$ and so $b \geq \sum_{i=1}^n w_i a_i$. □

Converting a fuzzy similarity measures to a measure using the smallest value it can be, converts the measure to the interval $[0,1]$. We can say if this converted value lies between 0 and 0.2, the similarity is very low, from 0.2 to 0.4 the similarity is low, from 0.4 to 0.6 the similarity is medium, from 0.6 to 0.8 high, and from 0.8 to 1 very high.

4 Country Health

The 2021 Global Health Security Index measures the capacities of 195 countries to prepare for epidemics and pandemics. All countries remain dangerously unprepared for future epidemics and pandemic threats, including threats potentially more devastating than Covid-19, [3]. In [1], a ranking of countries with respect to health care is provided. We provide the ranking with respect to OECD countries.

Table 1: OECD health security and health care rankings

Country	Health Security	Health Care	Country	Health Security	Health Care
Australia	2	9	Korea, Rep.	8	1
Austria	22	5	Latvia	14	
Belgium	19	4	Lithuania	18	26
Canada	4	19	Luxembourg	35	
Chile	23	30	Mexico	21	23
Czech Rep.	30	12	Netherlands	10	3
Denmark	11	8	New Zealand	12	16
Estonia	25	18	Norway	17	13
Finland	3	11	Poland	24	29
France	13	6	Portugal	27	22
Germany	7	10	Slovak Rep.	29	28
Greece	32	31	Slovenia	5	27
Hungary	28	33	Spain	15	7
Iceland	34		Sweden	9	20
Ireland	26	32	Switzerland	20	17
Israel	36	15	Turkey	33	21
Italy	31	25	United Kingdom	6	14
Japan	16	2	United States	1	24

Let M and S be the fuzzy similarity measures of Example 2.2. We deleted the countries in the Health Security ranking that were not in the Health Care ranking and then reranked the Health Security countries. We found that $S(\mu_A, \mu_B) = 1 - \frac{223}{1122} = 1 - 0.199 = 0.801$. By ([10], Theorem 2.10), $S(\mu_A, \mu_B) = \frac{2M(\mu_A,\mu_B)}{1+M(\mu_A,\mu_B)}$. Hence $M(\mu_A, \mu_B) = \frac{S(\mu_A,\mu_B)}{2-S(\mu_A,\mu_B)} = \frac{0.801}{1.199} = 0.668$. With the countries deleted, $n = 33$. Thus the smallest M can be is $\frac{n+1}{3n-1} = \frac{34}{98} = 0.347$. The smallest S can be is $\frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{66} = 0.515$. Therefore,

$$\frac{0.668 - 0.347}{1 - 0.347} = \frac{0.321}{0.653} = 0.492$$

and

$$\frac{0.801 - 0.515}{1 - 0.515} = \frac{0.286}{0.485} = 0.590.$$

We see that in both cases the similarity is medium.

A fuzzy similarity measure using implication operators was defined in [2]: $S_L(\mu_A, \mu_B) = \frac{1}{n}\sum_{x\in X}[(1 - \mu_A(x)) \wedge (1 - \mu_B(x)) + \mu_A(x) \wedge \mu_B(x)]$. We have by ([2], Proposition 3.5) that $S_L = S + \frac{1}{n}(S - 1)$. Thus $S_L(\mu_A, \mu_B) = 0.801 + \frac{1}{33}(0.801 - 1) = 0.801 - 0.006 = 0.795$. The smallest $S_L(\mu_A, \mu_B)$ is $\frac{1}{2} + \frac{1}{2n^2} = \frac{1}{2} + \frac{1}{2178} = 0.5 + 0.000459$ which we round off to 0.5. Thus$\frac{0.795-0.5}{1-0.5} = 0.59$. The similarity is thus medium.

We have that

$$\begin{aligned} S_1(\mu_A, \mu_B) &= \frac{1}{n}\sum_{x\in X} \mu_A \overline{\wedge} \mu_B(x) \\ &= \frac{1}{33}(\frac{428}{33}) = 0.393. \end{aligned}$$

The smallest S_1 can be is $\approx \frac{1}{4} + \frac{3}{4n^2} = 0.25 + 0.003 = 0.253$. Thus $\frac{0.393-0.253}{1-0.253} = \frac{0.140}{0.747} = 0.187$ and so the similarity is very low.

We find that $S_2(\mu_A, \mu_B) = \frac{17.921}{33} = 0.543$. The smallest S_2 can be is $\approx 2\ln\frac{66}{34} + \frac{2}{34} - 1 + \frac{3}{33} = 0.426$. Thus we have $\frac{0.543-0.426}{1-0.426} = \frac{0.117}{0.514} = 0.228$.

Hence the similarity is low.

We have that $\frac{1}{3}S_1 + \frac{1}{3}S_2 + \frac{1}{3}S_L \approx \frac{1}{3}(0.393+0.543+0.795) = \frac{1}{3}(1.649) = 0.577$. The smallest $\frac{1}{3}S_1 + \frac{1}{3}S_2 + \frac{1}{3}S_L$ can be is $\approx \frac{1}{3}(0.276 + 0.426 + 0.500) = \frac{1}{3}(1.202) = 0.401$.

Now $\frac{0.577-0.401}{1-0.401} = \frac{0.176}{0.499} = 0.353$. Here the similarity is low.

5 Natural Disaster, Political Stability, and Political Risk

We next consider the natural disaster risk, [4], the political stability, [6], and the political risk, [5], of OECD countries. We provide the rankings as given in [4, 5, 6]. The report in [4] systematically considers a country's vulnerability and exposure to natural hazards to determine a ranking of countries around the world based on their natural disaster risk. The index of Political Stability and Absence of Violence/Terrorism measures perceptions of the likelihood that the government will be destabilized or overthrown by unconstitutional or violent means, including politically motivated violence and terrorism. The index is an average of several other indexes from the Economist Intelligence Unit, the Economic Forum, and the Political Risk Services, among others, [6]. The Political Risk Index is the overall measure of risk for a given country, calculated by using all 17 risk components from the PRS Methodology including turmoil, financial transfer, direct investment, and export markets. The Index provides a basic convenient way to compare countries directly as well as demonstrating changes over the last five years, [5].

The rankings in the following tables are from low to high.

Table 2: OECD natural disaster and political stability rankings

Country	Natural Disaster	Political Stability	Country	Natural Disaster	Political Stability
Australia	34	17	Korea, Rep.	28	23
Austria	7	14	Latvia	13	22
Belgium	19	24	Lithuania	14	18
Canada	33	12	Luxembourg	1	3
Chile	30	32	Mexico	36	34
Czech Rep.	3	9	Netherlands	18	13
Denmark	5	10	New Zealand	29	1
Estonia	11	19	Norway	16	5
Finland	8	8	Poland	20	29
France	24	30	Portugal	22	11
Germany	17	20	Slovak Rep.	4	27
Greece	25	31	Slovenia	9	21
Hungary	2	15	Spain	27	26
Iceland	10	2	Sweden	12	7
Ireland	15	16	Switzerland	6	4
Israel	21	35	Turkey	31	36
Italy	26	25	United Kingdom	23	28
Japan	32	6	United States	35	33

Let M and S be the fuzzy similarity measures of Example 2.2. Here $n = 36$. We have that $S(\mu_A, \mu_B) = 1 - \frac{290}{1332} = 0.782$. Thus $M(\mu_A, \mu_B) = \frac{S(\mu_A,\mu_B)}{2-S(\mu_A,\mu_B)} = \frac{0.782}{1.218} = 0.642$. The smallest M can be is $\frac{n+2}{3n+2} = \frac{38}{110} = 0.345$.

Hence $\frac{0.642-0.345}{1-0.345} = 0.453$. Therefore, the similarity is medium. The smallest S can be is $\frac{n/2+1}{n+1} = \frac{19}{37} = 0.514$. Thus $\frac{0.782-0.514}{1-0.514} = 0.551$. Hence the similarity is medium.

$S_L(\mu_A, \mu_B) = 1 - \frac{1}{36^2}(149+141) = 1 - \frac{1}{1296}(290) = 0.7762$. The smallest $S_L(\mu_A, \mu_B)$ can be is 0.5. Thus $\frac{0.776-0.5}{1-0.5} = \frac{0.276}{0.5} = 0.552$. Hence the fuzzy similarity measure is medium.

$S_1(\mu_A, \mu_B) = \frac{\sum_{x \in X} \mu_A(x) \overline{\wedge} \mu_B(x)}{n} = \frac{549/36}{36} = 0.424$ and $S_2 \approx S_1(\mu_A, \mu_B) + \ln 2 - \frac{5}{8} = 0.424 + 0.068 = 0.492$. The smallest S_1 can be is $\frac{1}{4} + \frac{1}{n} = 0.25 + 0.028 = 0.278$ since $n = 36$ is even. Thus $\frac{0.492-0.278}{1-0.278} = \frac{0.214}{0.722} = 0.296$. Hence the similarity is low.

We find that $S_2(\mu_A, \mu_B) = \frac{22}{36} = 0.611$. The smallest S_2 can be is $\approx 0.386 + \frac{2}{36}(0.693) = 0.442$. Thus $\frac{0.611-0.442}{1-0.442} = \frac{0.169}{0.558} = 0.303$. Once again the similarity is low.

We have that $\frac{1}{3}S_1 + \frac{1}{3}S_2 + \frac{1}{3}S_L \approx \frac{1}{3}(0.424+0.611+0.776) = \frac{1}{3}(1.811) = 0.604$. The smallest $\frac{1}{3}S_1 + \frac{1}{3}S_2 + \frac{1}{3}S_L$ can be is $\approx \frac{1}{3}(0.278 + 0.442 + 0.500) = \frac{1}{3}(1.220) = 0.407$.

Now $\frac{0.604-0.407}{1-0.402} = \frac{0.197}{0.598} = 0.329$. The average similarity is low.

6 Conclusion

We used fuzzy implication operators to define the fuzzy similarity between the two rankings of countries involving health security and health care. We then found a fuzzy similarity involving the rankings of countries with respect to national disaster and political disaster. In each case, we found the similarity measures to be medium for S_L, M, and S and low for S_1 and S_2. Future research could involve other regions in the world other than the OECD countries. It was shown in ([2], Theorem 3.6) that $M \subseteq S_L \subseteq S$. It is clear that $S_1 \subseteq S_2$. Another potential project is to determine the relationship between S_2 and M. Further reading on implication operators can be found in [11].

Conflict of Interest: The authors declare no conflict of interest.

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