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On the results of proving the equivalence of $\mathcal{T}$-contractive mappings

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Abstract. The main goal of this paper is to compare the proof of the existence and uniqueness of fixed points for $\mathcal{T}$-contractive mappings in various metric spaces and different distances regarding some techniques in mathematical analysis. Also, several comparisons to show the efficiency of the obtained result will be considered.

Keywords: $\mathcal{T}$-contraction, fixed point, w-distance, cone metric space, $\mathfrak{c}$-distance, partial order.

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1. Introduction and preliminaries

The concept of contraction is known by Banach contraction principle [4] as follows: Let $(\mathcal{M}, d)$ be a complete metric space and $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ be a contraction, i.e., there exists $\alpha \in [0, 1)$ provided that

$$d(\mathcal{S}a, \mathcal{S}b) \leqslant \alpha d(a, b) \tag{1}$$

for all $a, b \in \mathcal{M}$. Then $\mathcal{S}$ possesses a unique fixed point.

The metric fixed point theory and contractions are an important instrument in nonlinear functional analysis and applications. Hence, since 1922, many mathematicians working in this domain have applied them in nonlinear functions and engineering problems (see [13, 19, 21] and references therein). Let us start with metrics and distances used in the sequel.

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In 1996, a new concept in metric spaces named w-distance is defined by Kada et al. [14].

Definition 1.1 [14] Presume $(\mathcal{M}, d)$ is a metric space. A function $p : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ is called a w-distance if it satisfies the following conditions:

- (w_1) $p(a, c) \leqslant p(a, b) + p(b, c)$ for each $a, b, c \in \mathcal{M}$;
- (w_2) p is lower semi-continuous in its second variable, i.e., if $a \in \mathcal{M}$ and $b_n \to b$ in $\mathcal{M}$, then $p(a, b) \leqslant \liminf_n p(a, b_n)$;
- (w_3) for all $\varepsilon > 0$, there is $\delta > 0$ provided that $p(c, a) \leqslant \delta$ and $p(c, b) \leqslant \delta$ induce that $d(a, b) \leqslant \varepsilon$.

Evidently, each metric is a w-distance but the converse isn't necessarily valid.

Given a cone $P \subset E$ which E is a Banach space, a partial ordering $\preceq$ regarding P is defined by $a \preceq b$ iff $b - a \in P$. We write $a \prec b$ to mean $a \preceq b$ and $a \neq b$, and $a \ll b$ iff $b - a \in intP$ (where $intP$ is the interior of P). The cone P is called solid if $intP \neq \emptyset$ and is called normal if there is $K > 0$ provided that $\theta \preceq a \preceq b$ implies that $\|a\| \leqslant K\|b\|$ for all $a, b \in E$. In 2007, Huang and Zhang defined a new metric, known cone metric, as follows:

Definition 1.2 Let $\mathcal{M}$ be a nonempty set. Suppose that a mapping $d : \mathcal{M} \times \mathcal{M} \to P$ satisfies the followings properties:

- (d_1) $d(a, b) = \theta$ iff $a = b$;
- (d_2) $d(a, b) = d(b, a)$;
- (d_3) $d(a, c) \preceq d(a, b) + d(b, c)$

for each $a, b, c \in \mathcal{M}$. Then, $\mathcal{M}$ is a cone metric and $(\mathcal{M}, d)$ is a cone metric space.

Regarding Definitions 1.1 and 1.2, Cho et al. [9] defined a c-distance as follows:

Definition 1.3 Presume $(\mathcal{M}, d)$ is a cone metric space. A mapping $q : \mathcal{M} \times \mathcal{M} \to E$ is named a $\mathfrak{c}$-distance when the following are held:

- $(\mathfrak{c}_1)$ $0 \preceq q(a, b)$ for each $a, b \in \mathcal{M}$;
- $(\mathfrak{c}_2)$ $q(a, c) \preceq q(a, b) + q(b, c)$ for all $a, b, c \in \mathcal{M}$;
- $(\mathfrak{c}_3)$ for all $n \geqslant 1$ and $a \in \mathcal{M}$, if $q(a, b_n) \preceq u$ for a $u = u_a$, then $q(a, b) \preceq u$ when $\{b_n\}$ is a sequence in $\mathcal{M}$ converging to some $b \in \mathcal{M}$;
- $(\mathfrak{c}_4)$ for all $c \in intP$, there is $e \in intP$ provided that $q(c, a) \ll e$ and $q(c, b) \ll e$ imply $d(a, b) \ll c$ for all $a, b, c \in \mathcal{M}$.

For all definitions, lemmas and properties with respect to Definitions 1.1-1.3, we refer to [9, 12, 14, 18, 20, 23].

In addition, in 2004, a order version of metric spaces is discussed by Ran and Reurings [20]. Then they proved some fixed point theorems for contractive mappings regarding comparable elements as follows:

Theorem 1.4 [20] Presume $(\mathcal{M}, \sqsubseteq)$ is a partially ordered set, $(\mathcal{M}, d)$ is a complete metric space and $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ is a nondecreasing mapping fulfilling

$$d(\mathcal{S}a, \mathcal{S}b) \leqslant \alpha d(a, b) \tag{2}$$

for any $a, b \in \mathcal{M}$ with $a \sqsubseteq b$, where $\alpha \in [0, 1)$. Also, presume that either

1. $\mathcal{S}$ is continuous; or
2. when a nondecreasing sequence a_n converges to a $a \in \mathcal{M}$, we have $a_n \sqsubseteq a$.

If there is $a_0 \in \mathcal{M}$ satisfying $a_0 \sqsubseteq \mathcal{S}a_0$, $\mathcal{S}$ possesses a fixed point. What's more, if every two fixed points are comparable, then the fixed point is unique.

Regarding one of the newest type of contractions named $\mathcal{T}$-contraction and mentioned in 2008 [15] (also, see [5, 7]), we now review some basic definitions, notations and lemmas.

Definition 1.5 Presume $(\mathcal{M}, d)$ is a metric space and $\mathcal{S}, \mathcal{T} : \mathcal{M} \to \mathcal{M}$ are two mappings. Then $\mathcal{S}$ is named a $\mathcal{T}$-contraction whenever there exists $\alpha \in [0, 1)$ provided that

$$d(\mathcal{T}\mathcal{S}a, \mathcal{T}\mathcal{S}b) \leqslant \alpha d(\mathcal{T}a, \mathcal{T}b) \tag{3}$$

for all $a, b \in \mathcal{M}$.

Evidently, if $\mathcal{T}$ is an identity mapping, then (2) will be changed into (1) and as a result, $\mathcal{T}$-contraction and contraction will be equal in this manner. It's also clear that each $\mathcal{T}$-contraction isn't necessarily a contraction.

***Example* 1.6** Take $\mathcal{M} = [1, \infty)$, the metric $d : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ by $d(a, b) = |a - b|$ for all $a, b \in \mathcal{M}$ and the mappings $\mathcal{S}, \mathcal{T} : \mathcal{M} \to \mathcal{M}$ by $\mathcal{S}a = ka$ and $\mathcal{T}a = \dfrac{1}{a} + 1$ for all $a \in \mathcal{M}$ and $k = 2, 3, \cdots$. Then $\mathcal{S}$ doesn't satisfy in (1) and it isn't a contraction, but

$$d(\mathcal{T}\mathcal{S}a, \mathcal{T}\mathcal{S}b) = |\frac{1}{ka} + 1 - (\frac{1}{kb} + 1)| \leqslant \frac{1}{k}|\frac{1}{a} - \frac{1}{b}| = d(\mathcal{T}a, \mathcal{T}b)$$

for all $a, b \in \mathcal{M}$; that is, $\mathcal{S}$ is a $\mathcal{T}$-contraction with $\alpha = \dfrac{1}{k} \in [0, 1)$.

Definition 1.7 [5, 7, 15, 16] Assume $(\mathcal{M}, d)$ is a metric space and $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ is a mapping. Then

- S is named sequentially convergent whenever the convergency of $\{\mathcal{S}a_n\}$ for each sequence $\{a_n\}$ implies that $\{a_n\}$ is convergent.
- S is named subsequentially convergent whenever the convergency of $\{\mathcal{S}a_n\}$ for each sequence $\{a_n\}$ implies that $\{a_n\}$ has a convergent subsequence.

Theorem 1.8 Presume $(\mathcal{M}, d)$ is a complete metric space and $\mathcal{S}, \mathcal{T} : \mathcal{M} \to \mathcal{M}$ are two mappings such that $\mathcal{S}$ is a $\mathcal{T}$-contraction and $\mathcal{T}$ is a one-to-one, continuous and sequentially convergent. Then $\mathcal{S}$ possesses a unique fixed point.

It's cleat that if $\mathcal{T} = \mathcal{I}$; that is, $\mathcal{T}$ is an identity mapping, then Theorem 1.8 is the same Banach contraction principle. Since 2009, many authors have defined various $\mathcal{T}$-contractions regarding former contraction and have obtained diverse fixed point theorems in different distances and metrics spaces (see [18, 19] and some references therein).

On the other hand, some authors [3, 6, 23] just showed that the proof of $\mathcal{T}$-contraction theorems for the existence of fixed points can be extra. In the present paper, we discuss the efficiency and equivalence of this type of contractions in some various metric spaces and different distances therein.

2. On the equivalence of former $\mathcal{T}$-contraction results

In this section, regarding two following propositions, we find the equivalence of between every two theorems for finding a fixed point.

Proposition 2.1 [3, 23] Presume $(\mathcal{M}, d)$ is a complete metric space, p is a w-distance on $\mathcal{M}$ and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is a one-to-one, continuous and sequentially convergent. Then $(\mathcal{M}, d^*)$ is a complete metric space and p^* is a w-distance on $\mathcal{M}$ in which

$$d^*(a, b) = d(\mathcal{T}a, \mathcal{T}b) \quad \text{and} \quad p^*(a, b) = p(\mathcal{T}a, \mathcal{T}b) \tag{4}$$

for all $a, b \in \mathcal{M}$, respectively.

In 2012, Aydi et al. [3, Theorem 9] showed that Theorem 1.8 and Banach contraction principle are equivalent. In 2023, Yousefi et al. [23] proved that the w-distance type of Theorem 1.8 and Banach contraction principle are equivalent, too. Here, we find two important equivalent theorems regarding some papers.

Theorem 2.2 Presume $(\mathcal{M}, d)$ is a complete metric space and $\mathcal{S}, \mathcal{T} : \mathcal{M} \to \mathcal{M}$ are two mappings so that $\mathcal{T}$ is one-to-one, continuous and subsequentially convergent and $\mathcal{S}$ is orbitally continuous. Then the following results are equivalent:

(I) (Ćirić's theorem [8], 1976) Let $\mathcal{S}$ satisfy the following condition

$$\begin{aligned} d(\mathcal{S}a, \mathcal{S}b) \leqslant \alpha \max\{&d(a, b), (d(a, b))^{-1} d(a, \mathcal{S}a) d(b, \mathcal{S}b), \\ &\delta(a, b) d(a, \mathcal{S}b) d(b, \mathcal{S}a)\} \end{aligned} \tag{5}$$

for each $a, b \in \mathcal{M}$ with $a \neq b$ and $\alpha \in (0, 1)$, where $\delta(a, b)$ is a real nonnegative function. Also, let each Cauchy sequence $\{\mathcal{S}^{n_i}a\}$ be convergent in $\mathcal{M}$. Then, for each $a \in \mathcal{M}$, $\lim_{n\to\infty} \mathcal{S}^n a = u_a$ and $\mathcal{S}u_a = u_a$. Additionally, if $\delta(a, b) \leqslant (d(a, b))^{-1}$, then $\mathcal{S}$ possesses a unique fixed point.

(II) (Chi's theorem [7], 2009) Let $\mathcal{S}$ satisfy the following condition

$$\begin{aligned} d(\mathcal{T}\mathcal{S}a, \mathcal{T}\mathcal{S}b) \leqslant \alpha \max\{&d(\mathcal{T}a, \mathcal{T}b), (d(\mathcal{T}a, \mathcal{T}b))^{-1} d(\mathcal{T}a, \mathcal{T}\mathcal{S}a) d(\mathcal{T}b, \mathcal{T}\mathcal{S}b), \\ &\delta(a, b) d(\mathcal{T}a, \mathcal{T}\mathcal{S}b) d(\mathcal{T}b, \mathcal{T}\mathcal{S}a)\} \end{aligned} \tag{6}$$

for each $a, b \in \mathcal{M}$ with $a \neq b$ and $\alpha \in (0, 1)$, where $\delta(a, b)$ is a real nonnegative function. Also, let each Cauchy sequence $\{\mathcal{T}\mathcal{S}^{n_i}a\}$ be convergent in $\mathcal{M}$. Then, for each $a \in \mathcal{M}$, $\lim_{n\to\infty} \mathcal{S}^n a = u_a$ and $\mathcal{S}u_a = u_a$. Additionally, if $\delta(a, b) \leqslant (d(\mathcal{T}a, \mathcal{T}b))^{-1}$, then $\mathcal{S}$ possesses a unique fixed point.

Proof. $(II) \to (I)$: It's enough to take $\mathcal{T} = \mathcal{I}$ on $\mathcal{M}$.
$(I) \to (II)$: It's sufficient to apply the left relation in (4). Then (6) will be changed into (5) with the notation d^* and the proof of (II) can be obtained from (I). ■

Note that the mapping $\mathcal{S}$ is orbitally continuous when $\lim_{n\to\infty} \mathcal{S}^{n_i} a = u \in \mathcal{M}$ implies $\lim_{n\to\infty} \mathcal{S}\mathcal{S}^{n_i} a = \mathcal{S}u \in \mathcal{M}$.

Proposition 2.3 Presume $(\mathcal{M}, d)$ is a complete ordered metric space endowed with a partial order "$\sqsubseteq$", p is a w-distance on $\mathcal{M}$ and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is a one-to-one, continuous and sequentially convergent mapping. Then $(\mathcal{M}, d^*)$ is a complete metric space endowed with the partial order $\sqsubseteq$ and p^* is a w-distance on $\mathcal{M}$ in which d^* and p^* are the same d^* and p^* in (4), respectively.

Theorem 2.4 Two following theorems are equivalent:

(I) (Theorem 1.4 introduced by Ran and Reurings, 2004).

(II) (Abbas et al. [1, Theorem 2.1], 2012) Presume $(\mathcal{M}, \sqsubseteq)$ is a partially ordered set and $(\mathcal{M}, d)$ is a complete metric space. Also, assume $\mathcal{S}, \mathcal{T} : \mathcal{M} \to \mathcal{M}$ are two mappings so that $\mathcal{S}$ is nondecreasing and $\mathcal{T}$ is one-to-one, continuous and subsequentially convergence and

$$d(\mathcal{T}\mathcal{S}a, \mathcal{T}\mathcal{S}b) \leqslant \alpha d(\mathcal{T}a, \mathcal{T}b) \tag{7}$$

for any $a, b \in \mathcal{M}$ with $a \sqsubseteq b$, where $\alpha \in [0, 1)$. Also, presume that either

a) $\mathcal{S}$ is continuous or

b) when a nondecreasing sequence a_n converges to a $a \in \mathcal{M}$, we have $a_n \sqsubseteq a$.

If there is $a_0 \in \mathcal{M}$ satisfying $a_0 \sqsubseteq \mathcal{S}a_0$, $\mathcal{S}$ possesses a fixed point. What's more, if every two fixed points are comparable, then the fixed point is unique.

Proof. $(II) \to (I)$: It's enough to take $\mathcal{T} = \mathcal{I}$ on $\mathcal{M}$.
$(I) \to (II)$: It's sufficient to apply Proposition 2.3. Then (7) will be changed into (2) with the notation d^* and the proof of (II) can be obtained from (I). ■

Remark 1 *Regarding Propositions 2.1 and 2.3, we can obtain all types of $\mathcal{T}$-contractions from their corresponding contractions in both metric spaces and ordered metric spaces and so all theorems for the existence and uniqueness of fixed points in both cases are equivalent.*

Proposition 2.5 [6, 23] Presume $(\mathcal{M}, d)$ is a complete cone metric space and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is one-to-one, continuous and sequentially convergent. Then $(\mathcal{M}, d^*)$ is a cone complete metric space in which

$$d^*(a, b) = d(\mathcal{T}a, \mathcal{T}b) \tag{8}$$

for all $a, b \in \mathcal{M}$.

In 2013, Bekeshie and Naidu [6, Corollary 3.5] showed that the cone metric version of Theorem 1.8 and Banach contraction principle are equivalent. Also, they discussed between the cone-type theorem of some contractions like Kannan, Chatterjea, Edelstein, Zamfirescu etc and their corresponding $\mathcal{T}$-contractions. Now, we can consider some equivalent theorems about the discussed articles of the conic version. For example, we give the following equivalence that can cover all result mentioned by Bekeshie and Naidu.

Theorem 2.6 Presume $(\mathcal{M}, d)$ is a complete cone metric space and P is a solid cone. Also, suppose that $\mathcal{S}, \mathcal{T} : \mathcal{M} \to \mathcal{M}$ are two mappings so that $\mathcal{T}$ is one-to-one, continuous and subsequentially convergent. Then the following results are equivalent:

(I) (Abbas and Rhoades [2, Corollary 2.4], 2009) If $\mathcal{S}$ is a Hardy-Rogers contraction; that is, there exist $\alpha_i \geqslant 0$ for $i = 1, \cdots, 5$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ so that

$$\begin{aligned} d(\mathcal{S}a, \mathcal{S}b) \preceq\ & \alpha_1 d(a, b) + \alpha_2 d(a, \mathcal{S}a) + \alpha_3 d(b, \mathcal{S}b) \\ & + \alpha_4 d(a, \mathcal{S}b) + \alpha_5 d(b, \mathcal{S}a) \end{aligned} \tag{9}$$

for all $a, b \in \mathcal{M}$, then $\mathcal{S}$ possesses a unique fixed point.

(II) (Filipović et al. [11, Theorem 2.1], 2011) If $\mathcal{S}$ is a $\mathcal{T}$-Hardy-Rogers contraction; that is, there exist $\alpha_i \geqslant 0$ for $i = 1, \cdots, 5$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ so that

$$d(\mathcal{T}\mathcal{S}a, \mathcal{T}\mathcal{S}b) \preceq \alpha_1 d(\mathcal{T}a, \mathcal{T}b) + \alpha_2 d(\mathcal{T}a, \mathcal{T}\mathcal{S}a) + \alpha_3 d(\mathcal{T}b, \mathcal{T}\mathcal{S}b) + \alpha_4 d(\mathcal{T}a, \mathcal{T}\mathcal{S}b) + \alpha_5 d(\mathcal{T}b, \mathcal{T}\mathcal{S}a) \tag{10}$$

for all $a, b \in \mathcal{M}$, then $\mathcal{S}$ possesses a unique fixed point.

Proof. $(II) \to (I)$: It's enough to take $\mathcal{T} = \mathcal{I}$ on $\mathcal{M}$.
$(I) \to (II)$: It's sufficient to apply Proposition 2.5. Then (10) will be changed into (9) with the notation d^* and the proof of (II) can be obtained from (I). ■

Regarding Proposition 2.5, all fixed point results for a $\mathcal{T}$-contraction mapping in cone metric spaces mentioned by Rahimi et al. [18] can be obtained from their usual contractions and conversely. With this interpretation, it seems that the proof of all results mentioned by Wangwe and Kumar in 2022 are extra and they can be obtained from their corresponding results.

Remark 2 *In [3, 6], the authors claimed that fixed point theorems for both usual contractive mapping and $\mathcal{T}$-contractive mapping are same and as a result, $\mathcal{T}$-contractions aren't real generalization, but it's not true. For instance, considering Example 1.6, we aren't able to show the existence of fixed point by taking Banach contraction principle but we can prove the existence of such points for a mapping $\mathcal{S}$ by applying Theorem 1.8. Hence, we can just conclude that the direct proofs are extra. Indeed, we can disregard long proofs mentioned in many papers on $\mathcal{T}$-contraction theorems but these theorems and corollaries are real and interesting!*

3. Generalized results

Combining Propositions 2.1 and 2.5, we first introduce next elementary proposition.

Proposition 3.1 Presume $(\mathcal{M}, d)$ is a complete cone metric space with a $\mathfrak{c}$-distance q therein and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is one-to-one, continuous and sequentially convergent. Then q^* is a $\mathfrak{c}$-distance in which

$$q^*(a, b) = q(\mathcal{T}a, \mathcal{T}b) \tag{11}$$

Proof. For all $a, b, c \in \mathbb{R}$, we have

- $q^*(a, c) = q(Ta, Tc) \preceq q(Ta, Tb) + q(Tb, Tc) = q^*(a, b) + q^*(b, c)$. Hence, $q^*(a, c) \preceq q^*(a, b) + q^*(b, c)$;
- for all $n \geqslant 1$ and $a \in \mathcal{M}$ and by definition $\mathfrak{c}$-distance, if $q^*(a, b_n) \preceq u$ for a $u = u_a$, then $q^*(a, b) \preceq u$ when $\{b_n\}$ is a sequence in $\mathcal{M}$ converging to a $b \in \mathcal{M}$;
- Let $c \in intP$ be arbitrary. Also, assume that there is $e \in intP$ provided that $q^*(c, a) \ll e$ and $q^*(c, b) \ll e$. Then, by the definition of q^*, $q(\mathcal{T}c, \mathcal{T}a) \ll e$ and $q(\mathcal{T}c, \mathcal{T}b) \ll e$. As q is a $\mathfrak{c}$-distance, $d(\mathcal{T}a, \mathcal{T}b) \ll c$. Hence, by the definition of d^*, we have $d^*(a, b) \ll c$. In conclusion, $q^*(c, a) \ll e$ and $q^*(c, b) \ll e$ induce that $d^*(a, b) \ll c$.

Therefore, q^* is a $\mathfrak{c}$-distance on $\mathcal{M} \times \mathcal{M}$. ■

Moreover, by using the definition of the $\mathfrak{c}$-distance q^*, Lemma 2.12 of [9] is also valid for this distance. Now, we are ready to establish some results for $\mathcal{T}$-contractions that are equivalent to usual contractions.

Theorem 3.2 Presume $(\mathcal{M}, d)$ is a complete cone metric space and P is a solid cone. Also, suppose that q is a $\mathfrak{c}$-distance and $\mathcal{S}, \mathcal{T} : \mathcal{M} \to \mathcal{M}$ are two mappings so that $\mathcal{T}$ is one-to-one, continuous and subsequentially convergent. Then the following results are equivalent:

(I) (Sintunavarat et al. [22, non-ordered Banach version of Theorem 3.1], 2011) If there exists $\alpha \in [0, 1)$ so that

$$q(\mathcal{S}a, \mathcal{S}b) \preceq \alpha q(a, b) \tag{12}$$

for all $a, b \in \mathcal{M}$, then $\mathcal{S}$ possesses a unique fixed point. Moreover, if $\mathcal{S}x^* = x^*$, then $q(x^*, x^*) = \theta$.

(II) (Fadail and Bin Ahmad [10, Theorem 1], 2014) If there exists $\alpha \in [0, 1)$ so that

$$q(\mathcal{TS}a, \mathcal{TS}b) \preceq \alpha q(\mathcal{T}a, \mathcal{T}b) \tag{13}$$

for all $a, b \in \mathcal{M}$, then $\mathcal{S}$ possesses a unique fixed point. Moreover, if $\mathcal{S}x^* = x^*$, then $q(\mathcal{T}x^*, \mathcal{T}x^*) = \theta$

Proof. $(II) \to (I)$: It's enough to take $\mathcal{T} = \mathcal{I}$ on $\mathcal{M}$.
$(I) \to (II)$: It's sufficient to apply Proposition 3.1. Then (13) will be changed into (12) with the notation q^* and the proof of (II) can be obtained from (I). ■

Similarly, we can obtain a theorem mentioned by Pal et al. [17] without a direct proof as follows:

Theorem 3.3 [17, Theorem 3.1, 2019] Let $(\mathcal{M}, d)$ be a complete cone metric space and P be a solid cone. Also, let q be a $\mathfrak{c}$-distance and $\mathcal{S}, \mathcal{T} : \mathcal{M} \to \mathcal{M}$ be two mappings so that $\mathcal{T}$ is one-to-one, continuous and subsequentially convergent. If there exist nonnegative functions $u(a, b)$, $r(a, b)$, $s(a, b)$ and $t(a, b)$ provided that

$$\sup_{a,b\in\mathcal{M}} \{u(a, b) + r(a, b) + s(a, b) + 2t(a, b)\} \leqslant \alpha < 1 \tag{14}$$

and

$$\begin{aligned} q(\mathcal{TS}a, \mathcal{TS}b) \preceq\ & u(a, b)q(\mathcal{T}a, \mathcal{T}b) + r(a, b)q(\mathcal{T}a, \mathcal{TS}a) \\ & + s(a, b)q(\mathcal{T}b, \mathcal{TS}b) + 2t(a, b)[q(\mathcal{T}a, \mathcal{TS}b) + q(\mathcal{T}b, \mathcal{TS}a)] \end{aligned} \tag{15}$$

holds for all $a, b \in \mathcal{M}$, then $\mathcal{S}$ has a unique fixed point.

Proof. Using (11), we have

$$\begin{aligned} q^*(\mathcal{S}a, \mathcal{S}b) \preceq\ & u(a, b)q^*(a, b) + r(a, b)q^*(a, \mathcal{S}a) + s(a, b)q^*(b, \mathcal{S}b) \\ & + 2t(a, b)[q^*(a, \mathcal{S}b) + q^*(b, \mathcal{S}a)] \end{aligned}$$

for all $a, b \in \mathcal{M}$. The residual of the proof can be done for the $\mathfrak{c}$-distance q^* as it's proved by other authors before, so the proof ends. ■

As you can see, we obtained Pal et al.'s result by a small proof and using former theorems, which is why we proved the constructive Proposition 3.1. If we take $q = d$ in Theorem 3.3, we will have Theorem 2.1 of Rahimi et al. [18].

4. Conclusion

In this work, we discussed the equivalence of the proof between fixed point theorems for $\mathcal{T}$-contractions and contractions in both metric spaces and cone metric spaces. Also, we talked about our results in their w-distances and $\mathfrak{c}$-distances. Beside, we showed the equivalence between results in the framework of ordered version of metric spaces. To continue of this paper, one can discuss the equivalence of the proof of theorems in another spaces like G-metric spaces, graphical metric spaces [13], $\mathcal{F}$-metric spaces etc.

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