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Research article

Convergence of collocation Bernoulli wavelet method in solving nonlinear Fredholm integro-differential equations of fractional order

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Abstract

We provide a computer method for solving fractional order nonlinear Fredholm integro-differential equations in this study. This method transforms the core issue into a set of algebraic equations using Bernoulli wavelets. The operational Bernoulli wavelet with fractional integration is obtained and used. It works particularly well for technical applications. The convergence of the suggested strategy is the most crucial aspect to note here. The collocation approach for this issue has a unique approximation since these requirements can be shown using mathematical principles and matrices theory. Finally, some pertinent examples for which the exact solution is known are used in numerical simulation to confirm the effectiveness and relevance. Alternatively, these examples will demonstrate the viability and correctness of the suggested approach.

Keywords: Fractional calculus, Bernoulli wavelets, Fredholm integro-differential equations, collocation method.

1- Introduction

Differential equations (DEs) are a subfield of mathematics having several uses in science and engineering. Based on fractional order integrals and derivatives, fractional differential equations (FDEs) are a relatively recent branch of applied mathematics.

FDEs and fractional integro-differential equations have been used to model a variety of physical and chemical processes in recent years. Actually, the use of fractional calculus provided a more accurate representation of complex natural phenomena [1, 2], such as non-Brownian motion, signal processing[3], system identification [4], control theory [5], viscoelastic materials [6], buckling analysis [7], stress theories [8], and polymers [8, 9].

There has been a great deal of interest in developing numerical techniques for solving the many forms of FDEs and FIDEs that have been proposed for use in standard models. The following are some techniques for solving FDEs that have been proposed: Adomian decomposition method (ADM) [10-12], Laplace decomposition method (LDM) [12], Homotopy perturbation method (HPM) [13-16], Homotopy analysis method (HAM) [17, 18], Iterative method [18, 7], Grunwold-Letnikov method [19], Diethelm algorithm [20], Spectral method [21].

Wavelet theory has been used in several technological disciplines since it was first developed. On the foundation of wavelets, which are localized functions, energy-bounded functions, such as $L^2(\mathbb{R})$are constructed (R). In order to solve fractional differential equations, operational matrices of fractional order integration for the Legendre wavelet [22], Chebyshev wavelet [23-25], Haar wavelet [26], cosine and sine (CAS) wavelet [27, 28] and the second kind Chebyshev wavelet [29, 30] have recently been constructed. For the purpose of producing operational matrices for fractional order integration, all of the wavelet techniques previously discussed use Block-Pulse functions. In order to solve fractional integro-differential equations [28, 31], wavelets have been used in a variety of methods. Legendre wavelets were used in the Meng et al. method's [31] resolution of fractional integro-differential equations. Fractional integro-differential equations with weakly singular kernels were solved using CAS wavelet techniques by Yi and Huang [28].
An efficient method based on Haar wavelets and Block-pulse functions was developed by Saeedi [32] to solve nonlinear Fredholm integro-differential equations of fractional order. Heydari [20] developed a successful Chebyshev wavelets method for solving a class of nonlinear fractional integro-differential equations across a wide interval. In [33] a numerical method for solving a class of nonlinear mixed Fredholm-Volterra integro-differential equations of fractional order is presented.
In this paper, a novel operational method for the solution of the following class of fractional order nonlinear Fredholm integro-differential equations is presented.

$$D^q f(x) - \lambda \int_0^1 k(x,t)[f(t)]^p dt = g(x) \qquad p > 1 \tag{1.1}$$

with these supplementary conditions:

$$f^{(i)}(0) = \delta_i, \qquad i = 0,1,\dots,r-1$$
$$r-1 < q \leq r \; r \in \mathbb{N}$$

where $g \in L^2([0,1)), k \in L^2([0,1)^2)$ are known functions, $f(x)$ is unknown function, D^q is the Caputo fractional differential operator and p is a positive integer.
By extending the solution as Bernoulli wavelets with unknown coefficients, the collocation method reduces the issue to a set of algebraic equations. The primary characteristic of an operational approach is the transformation of a differential problem into an algebraic equation.

2- Preliminaries

2-1- Fractional operators

Fractional integration and derivatives have several meanings. The Riemann- Liouville definition and the Caputo definition are the most often used definitions of a fractional integration and derivative, respectively [34].

Definition 2.1. The Riemann-Liouville fractional integral operator of order α is defined as

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \tag{2.1}$$

Where $\alpha > 0$

Definition 2.2. The Caputo definition of fractional differential operator is given by

$$D^\alpha u(t) = \begin{cases} \dfrac{d^r u(t)}{dt^r} & \alpha = r \in \mathbb{N} \\ \dfrac{1}{\Gamma(r-\alpha)} \displaystyle\int_0^t \frac{u^r(s)}{(t-s)^{\alpha-r+1}} ds & 0 \leq r-1 < \alpha < r \end{cases} \tag{2.2}$$

The Caputo fractional derivatives of order α is also defined as $D^\alpha u(t) = I^{r-\alpha} D^r u(t)$ is the usual integer differential operator of order r. The relation between the Riemann-Liouville integral operator I^α and Caputo differential operator D^α is given by the following expressions:

$$D^\alpha I^\alpha u(t) = u(t) \tag{2.3}$$

$$I^\alpha D^\alpha u(t) = u(t) - \sum_{k=0}^{r-1} u^{(k)}(0^+) \frac{t^k}{k!} \tag{2.4}$$

where $t > 0$.

2-2- Bernoulli polynomials

Bernoulli polynomials of order m can be defined by [35]

$$\beta_m(t) = \sum_{i=0}^{m} \binom{m}{i} \alpha_{m-i} t^i \qquad (2.5)$$

where $\alpha_i, i = 0,1,\dots,m$ are Bernoulli numbers. These numbers are a sequence of signed rational numbers which arise in the series expansion of trigonometric functions [35-38] and can be defined by the identity

$$\frac{t}{e^t-1} = \sum_{i=0}^{\infty} \alpha_i \frac{t^i}{i!} \qquad (2.6)$$

Table 1: Bernoulli polynomials and numbers

i	Bernoulli numbers(α_i)	m	Bernoulli polynomials($\beta_m(t)$)
0	1	0	1
1	$-1/2$	1	$t-\frac{1}{2}$
2	$1/6$	2	$t^2-t+1/6$
3	0	3	$t^3-\frac{3}{2}t^2+\frac{1}{2}t$

The first few Bernoulli polynomials and numbers are listed in Table 1. Bernoulli polynomials satisfy the following formula [34]

$$\int_0^1 \beta_n(t)\beta_m(t)dt = (-1)^{n-1}\frac{m!\,n!}{(m+n)!}\alpha_{n+m},\ m,n \ge 1. \qquad (2.7)$$

According to [39], Bernoulli polynomials form a complete basis over the interval $[0,1]$.

2-3- Function approximation by using of Bernoulli polynomials

We can approximate $f(t)$ by using Bernoulli polynomials as

$$f(x) \simeq \sum_{i=0}^{M-1} a_i\beta_i(t) = A^T B(t) \qquad (2.8)$$

where $B(t) = [\beta_0(t), \beta_1(t), \dots, \beta_{M-1}(t)]^T$ and $A = [a_0, a_1, \dots, a_{M-1}]^T$

2-4- Bernoulli wavelets

In recent years, the various basic functions have been used to estimate the solution of integral equations. In this work, we review construction of a basic for $L^2[0,1]$ that this basic consist of orthonormal system Bernoulli wavelets $\psi_{n,m}(t) = \psi(k,\hat{n},m,t)$. These wavelets have four arguments: $\hat{n} = n-1$, $n = 1,2,3,\dots,2^{k-1}$, k can assume any positive integer, m is the order for Bernoulli polynomials and t is the normalized time. We define them on interval [0,1) as follows

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k-1}{2}}\hat{\beta}_m(2^{k-1}t-\hat{n}) & \frac{\hat{n}}{2^{k-1}} \le t < \frac{\hat{n}+1}{2^{k-1}} \\ 0 & \text{otherwise} \end{cases} \qquad (2.9)$$

with

$$\hat{\beta}_m(t) = \begin{cases} 1 & , \quad m=0 \\ \dfrac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}\beta_m(t), & m>0 \end{cases} \qquad (2.10)$$

where $m = 0,1,2,\dots,M-1$ and $n = 1,2,\dots,2^{k-1}$. The coefficient $\frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}$ is for normality.

Definition 2.3. A function $\psi \in L^2$ is called an orthonormal wavelet if the collection of functions $\psi_{n,m}(t), n,m \in \mathbb{Z}$, as defined in

$$\psi_{n,m}(t) := 2^{n/2}\psi(2^j t - k) \qquad (2.11)$$

is an orthonormal basis of L^2 [28].

According to Theorem 4.2. in [29] and Definition 3.1 Bernoulli wavelets is a basis.

2-5- Expanding Bernoulli wavelet via the Bernoulli polynomials

The Bernoulli wavelets may be expanded into an M-term Bernoulli polynomials as

$$\Psi(t) = \Phi B(t) \qquad (2.12)$$

where Φ is the transformation matrix of the Bernoulli wavelet to the Bernoulli polynomials. For example, E. Keshavarz et al. in [40] obtain in case M = 3 and k = 2:

$$\Phi = \begin{cases} \varphi_1 = [a_{ij}]_{6\times 3} & 0 \le t < \frac{1}{2} \\ \varphi_2 = [b_{ij}]_{6\times 3} & \frac{1}{2} \le t < 1 \end{cases} \tag{2.13}$$

They also obtain Φ in more general case for arbitrary M and k = 2. for more details, see in [40].

2-6- Expanding Bernoulli polynomials via BPFs

First, let us expand i-th Bernoulli polynomials by using of BPFs:

$$\beta_i(t) \simeq \sum_{j=0}^{m'-1} e_j b_j(t) \tag{2.14}$$

where $e_j = m' \int_0^1 \beta_i(t) b_j(t) dt$. By using of BPFs properties, we have

$$e_j = m' \int_0^1 \beta_i(t) b_j(t) dt = m' \int_{\frac{j}{m'}}^{\frac{j+1}{m'}} \beta_i(t) dt$$

$$= m' \int_{\frac{j}{m'}}^{\frac{j+1}{m'}} \sum_{n=0}^{i} \binom{i}{n} \alpha_{i-n} t^n dt \tag{2.15}$$

$$= m' \sum_{n=0}^{i} \left[\binom{i}{n} \alpha_{i-n} \int_{\frac{j}{m'}}^{\frac{j+1}{m'}} t^n \right] dt \tag{2.16}$$

$$= m' \sum_{n=0}^{i} \left[\binom{i}{n} \alpha_{i-n} \left[\frac{t^{n+1}}{n+1}\right]_{\frac{j}{m'}}^{\frac{j+1}{m'}} \right] \tag{2.17}$$

$$= m' \sum_{n=0}^{i} \left[\binom{i}{n} \frac{\alpha_{i-n}}{(m')^{n+1}(n+1)} [((j+1)^{n+1} - j^{n+1})] \right]$$

$$= m' \sum_{n=0}^{i} \left[\binom{i}{n} \frac{\alpha_{i-n}}{(m')^{n+1}(n+1)} \left[\sum_{r=0}^{n} \binom{n+1}{r} j^r \right] \right]$$

Now, we can obtain E transform matrix that as follows:

$$B(t) = E B_{m'}(t) \tag{2.18}$$

where $E = [e_j]$.

3- Function approximation by using Bernoulli wavelets

So, suppose that f be an arbitrary function in $L^2([0,1])$, there exist the unique coefficients $c_{1,0}, c_{1,1}, \dots, c_{2^{k-1},M-1}$ such that

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) = C^T \Psi(t) \tag{3.2}$$

where T indicates transposition, C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c_{1,0}, c_{1,1}, \dots, c_{1,M-1}, c_{2,0}, c_{2,1}, \dots, c_{2,M-1}, \dots, c_{2^{k-1},0}, \dots, c_{2^{k-1},M}]^T \tag{3.3}$$

$$\Psi(t) = [\psi_{1,0}(t), \psi_{1,1}(t), \dots, \psi_{1,M-1}(t), \dots, \psi_{2^{k-1},0}(t), \dots, \psi_{2^{k-1},M}(t)]^T \tag{3.4}$$

To compute C, we must evalute two matrices as follows:

- F matrix is

$$F = [f_{1,0}, f_{1,1}, \dots, f_{1,M-1}, f_{2,0}, f_{2,1}, \dots, f_{2,M-1}, \dots, f_{2^{k-1},0}, \dots, f_{2^{k-1},M}]^T \tag{3.5}$$

where $f_{i,j} = \int_0^1 f(t) \psi_{i,j}(t) dt$.

- D is a matrix of order $2^{k-1}M \times 2^{k-1}M$ and $D = [d_{n,m}^{i,j}]$, where

$$d_{n,m}^{i,j} = \int_0^1 \psi_{i,j}(t) \psi_{n,m}(t) dt \tag{3.6}$$

Using Eq. (3.2) we obtain

$$f_{i,j} = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \int_0^1 \psi_{n,m}(t) \psi_{i,j}(t) dt = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} d_{n,m}^{i,j}$$

Therefore

$$F^T = C^T D \Rightarrow C = F^T D^{-1} \tag{3.7}$$

4- Operational matrix

4-1- Bernoulli operational matrix of the fractional integration

The Riemann-Liouville fractional integration of the vector $B(t)$ can be expressed by

$$I^q B(t) = F^{(q)} B(t) \tag{4.1}$$

where $F^{(q)}$ is the $M \times M$ Riemann-Liouville operational matrix of integration. we have

$$I^q \beta_i(t) = I^q \left(\sum_{r=0}^{i} \binom{i}{r} \alpha_{i-r} t^r\right) t^{r+q} = \sum_{r=0}^{i} \binom{i}{r} \alpha_{i-r} I^q t^r = \sum_{r=0}^{i} \frac{i!\alpha_{i-r}}{(i-r)!\Gamma(r+1+q)} = \sum_{r=0}^{i} b_{i,r}^{(q)} t^{r+q} \tag{4.2}$$

where

$$b_{i,r}^{(q)} = \frac{i!\alpha_{i-r}}{(i-r)!\Gamma(r+1+q)} \tag{4.3}$$

Assume t^{q+r} can be expanded in terms of Bernoulli polynomials

$$t^{q+r} \simeq \sum_{j=0}^{M-1} c_{r,j} \beta_j(t) \tag{4.4}$$

by Substituting Eq. (5.3), we have

$$I^q \beta_i(t) = \sum_{r=0}^{i} b_{i,r}^{(q)} t^{r+q} = \sum_{r=0}^{i} b_{i,r}^{(q)} \sum_{j=0}^{M-1} c_{r,j} \beta_j(t) = \sum_{j=0}^{M-1} \left(\sum_{r=0}^{i} \theta_{i,j,r}\right) \beta_j(t)$$

where

$$\theta_{i,j,r} = b_{i,r}^{(q)} c_{r,j} \tag{4.5}$$

Eq. (5.4) can be rewritten as

$$I^q \beta_i(t) \simeq \left[\sum_{r=0}^{i} \theta_{i,0,r}, \sum_{r=0}^{i} \theta_{i,1,r}, \dots, \sum_{r=0}^{i} \theta_{i,M-1,r}\right] B(t)$$

where $i = 0,1,\dots,M-1$. Therefore, we have

$$F^q = \begin{pmatrix} \theta_{0,0,0} & \theta_{0,1,0} & \cdots & \theta_{0,M-1,0} \\ \sum_{r=0}^{1} \theta_{1,0,r} & \sum_{r=0}^{1} \theta_{1,1,r} & \cdots & \sum_{r=0}^{1} \theta_{1,M-1,r} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{r=0}^{M-1} \theta_{M-1,0,r} & \sum_{r=0}^{M-1} \theta_{M-1,1,r} & \cdots & \sum_{r=0}^{M-1} \theta_{M-1,M-1,r} \end{pmatrix}$$

5- Bernoulli wavelet operational matrix of the fractional integration

We now derive the Bernoulli wavelet operational matrix of the fractional integration. Let

$$I^q \Psi(t) = P^{(q)} \Psi(t) \tag{4.6}$$

where matrix $P^{(q)}$ is called the Bernoulli wavelet operational matrix of the fractional integration. Using Eq. (4.1) and Eq. (2.12) we get

$$I^q \Psi(t) = I^q \Phi B(t) = \Phi I^q B(t) = \Phi F^{(q)} B(t) \tag{4.7}$$

From Eq. (4.6) and Eq. (4.7) we have

$$P^{(q)} \Psi(t) = \Phi F^{(q)} B(t) \tag{4.8}$$

From $\Psi(t) = \Phi B(t)$, So

$$P^{(q)} \Phi B(t) = \Phi F^{(q)} B(t) \tag{4.9}$$

Then, the Bernoulli wavelet operational matrix of the fractional integration $P^{(q)}$ is given by

$$P^{(q)} = \Phi F^{(q)} \Phi^{-1} \tag{4.10}$$

5-1- Expanding the integral part of the main equation via Bernoulli wavelets

Consider Eq. (1.1), the two-variable function $k \in L^2([0,1)^2)$ can be approximated as:

$$k(x,t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{l_1=0}^{M-1} \sum_{m=1}^{2^{k-1}} \sum_{l_2=0}^{M-1} k_{i,j}\psi_{n,l_1}(x)\psi_{m,l_2}(t)$$

Or in matrix form

$$k(x,t) \simeq \Psi^T(x)K\Psi(t),\ K = [k_{ij}] \tag{5.1}$$

According to [40].

5-2- Expanding the nonlinear part under integral of the main equation

By using Eq. (6.5), we have $f(x) \cong aB_{m'}(x)$. From the disjoint property of BPFs, we have:

$$[f(x)]^2 \cong [aB_{m'}(x)]^2 = [a_0b_0(x) + \cdots + a_{m'-1}b_{m'-1}(x)]^2 = a_0^2b_0(x) + \cdots + a_{m'-1}^2b_{m'-1} \tag{5.2}$$

So

$$[f(x)]^2 \cong [a_0^2 + a_1^2 + \cdots + a_{m'-1}^2]B_{m'}(x) = \hat{a}_2B_{m'}(x) \tag{5.3}$$

And it is easy to show by induction that:

$$[f(x)]^p \cong [a_0^p + a_1^p + \cdots + a_{m'-1}^p]B_{m'}(x) = \hat{a}_pB_{m'}(x) \tag{5.4}$$

By using of above calculations, we have:

$$\int_0^1 k(x,t)[f(t)]^p dt \tag{5.5}$$

$$= \int_0^1 \Psi^T(x)K\Psi(t)\hat{a}_pB_{m'}(t)dt$$

$$= \int_0^1 \Psi^T(x)K\Psi(t)B_{m'}^T(t)\hat{a}_p^T dt \tag{5.6}$$

$$= \int_0^1 \Psi^T(x)K\Phi B(t)B_{m'}^T(t)\hat{a}_p^T dt$$

$$= \int_0^1 \Psi^T(x)K\Phi E B_{m'}(t)B_{m'}^T(t)\hat{a}_p^T dt$$

$$= \Psi^T(x)K\Phi E \int_0^1 B_{m'}(t)B_{m'}^T(t)\hat{a}_p^T dt \tag{5.7}$$

Now, we simplify the integral part of above calculations:

$$\int_0^1 B_{m'}(t)B_{m'}^T(t)\hat{a}_p^T dt$$

$$= \int_0^1 \begin{bmatrix} b_0(t) & \cdots & 0 \\ \vdots & \ddots & \dots \\ 0 & \cdots & b_{m'-1}(t) \end{bmatrix} \begin{bmatrix} a_0^p \\ \vdots \\ a_{m'-1}^p \end{bmatrix} dt$$

$$= \int_0^1 \left[a_0^pb_0(t), a_1^pb_1(t), \cdots, a_{m'-1}^pb_{m'-1}(t)\right]^T dt$$

$$= \frac{1}{m'}\left[a_0^p, a_1^p, \cdots, a_{m'-1}^p\right]^T = \frac{1}{m'}\hat{a}_p \tag{5.8}$$

Then, we have

$$\int_0^1 k(x,t)[f(t)]^p dt = \frac{1}{m'}\Psi^T(x)K\Phi E\hat{a}_p \tag{5.9}$$

by substituting Eq. (5.9) in Eq. (1.1)

$$\Psi^T(x)c - \lambda\frac{1}{m'}\Psi^T(x)K\Phi E\hat{a}_p = \Psi^T(x)g \tag{5.10}$$

by using of collocation points as zeros of Chebyshev polynomials, we can be reduced Eq. (5.10) to a system of algebraic equations that be solved by numerical method.

6- Numerical solution

In this section we present the numerical solution for a class of nonlinear fractional Fredholm integro-differential equations.

$$I^q f(x) - \lambda\int_0^1 k(x,t)[f(t)]^p dt \quad p > 1 \tag{6.1}$$

Now, Let:

$$D^q f(x) \cong c^T\Psi(x) \tag{6.2}$$

For simplicity, we can assume that $\delta_i = 0$ in Eq. (1.1). Hence by using Eq. (4.6) we have

$$f(x) \cong c^TP^{(q)}\Psi(x) \tag{6.3}$$

According to Eq. (2.12), from above equation we get:

$$f(x) \cong c^TP^{(q)}\Phi B(x) \tag{6.4}$$

By using of expanding $B(x)$ via $B_{m'}(x)$, we have

$$f(x) \cong c^T P^{(q)} \Phi E B_{m'}(x) \qquad (6.5)$$

Define

$$a = [a_0, a_1, \dots, a_{m'-1}] = c^T P^{(q)} \Phi E \qquad (6.6)$$

Therefore, it can be written

$$f(x) \cong a B_{m'}(x) \qquad (6.٧)$$

7- Convergence of method

In this section, we first prove that when k or m tends to infinity, the approximation of a function by Bernoulli wavelet bases converges to the function itself, then we converge the method.

Lemma 7.1. Assumption $f \in c^m[0,1]$ and

$$f(t) \simeq f_0(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) = C^T \Psi(t) \qquad (7.1)$$

If $s_m = \text{span}\{\beta_0(t), \beta_1(t), \dots, \beta_{m-1}(t)\}$ is a base for space of system, we can extend $f(t)$ as follow:

$$f_m(t) = \sum_{i=0}^{m-1} a_i \beta_i(t) = A^T B(t) \qquad (7.2)$$

that $f_m(t)$ is a function approximation of $f(t)$ at distance $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$ then

$$\|f - f_0\|_2 \le \frac{1}{m!\sqrt{2m+1}} \sup_{t\in[0,1]} \left|f^{(m)}(t)\right|. \qquad (7.3)$$

Proof. We know $\{1, t, t^2, \dots, t^{m-1}\}$ is a basis for the space of polynomials of degree $m-1$. We define the Taylor expansion $f(t)$ as follows:

$$\tilde{f}(t) = f(0) + t f'(0) + \cdots + \frac{t^{m-1}}{(m-1)!} f^{(m-1)}(0) \qquad (7.4)$$

We have

$$\left|f(t) - \tilde{f}(t)\right| \le \frac{t^m}{m!} \sup_{t\in I_{k,n}} \left|f^{(m)}(t)\right| \qquad (7.5)$$

where $I_{k,n} = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right], n = 1,2,\dots,2^{k-1}$.

According to the definition 2-norm:

$$\|f(t) - f_0(t)\|_2^2 = \int_0^1 (f(t) - C^T\psi(t))^2 dt$$

$$= \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (f(t) - A^T B(t))^2 dt$$

$$\le \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \{f(t) - \tilde{f}(t)\}^2 dt$$

$$\le \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left\{\frac{t^m}{m!} \sup_{t\in I_{k,n}} \left|f^{(m)}(t)\right|\right\}^2 dt$$

$$\le \int_0^1 \left\{\frac{t^m}{m!} \sup_{t\in[0,1]} \left|f^{(m)}(t)\right|\right\}^2 dt \qquad (7.6)$$

Therefore

$$\|f(t) - f_0(t)\|_2^2 \le \frac{1}{(m!)^2(2m+1)} \left(\sup_{t\in[0,1]} \left|f^{(m)}(t)\right|\right)^2 \qquad (7.7)$$

In extending a function using the Bernoulli wavelet base, we have two degrees of freedom, which increases the accuracy of the method. One parameter is k and the other m is below the interval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$. When m is fixed and k tends to infinity:

$$\left\|\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]\right\| = \frac{1}{2^{k-1}} \to 0 \qquad (7.8)$$

so

$$\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (f(t) - f_m(t))^2 dt \to 0 \qquad (7.9)$$

Given the inequality proof (7.7)

$$\lim_{k\to\infty} \|f(t) - f_0(t)\|_2 = 0. \qquad (7.10)$$

and if we fix k and move m to infinity according to the equation (7.2)

$$\lim_{m\to\infty} \|f(t)-f_m(t)\|_2 = 0.$$

Therefore, the convergence proof of the approximation of the function is completed by the Bernoulli wavelet. We now consider the convergence of the method.

Theorem 7.1. Assume that $y(x)$ and $y_m(x)$ are the exact and approximate answers of Eq. (1.1) and that $g_m(x)$ is an extension of $g(x)$ in terms of the Bernoulli wavelet and $H(x,y) = \int_0^1 (k(x,t)(y(t))^p)dt$ in Lip-Sheetz condition

$$\| H(x,y) - H(x,z) \| \leq \lambda \| y - z \| \ , \lambda > 0 \quad (7.12)$$

is true and $\frac{\lambda}{\Gamma(q+1)} < 1$ then

$$\|y(x) - y_m(x)\| \leq \frac{E(g)}{\Gamma(q+1)\left(1-\frac{\lambda}{\Gamma(q+1)}\right)} \qquad (7.11)$$

where in $E(g) = \|g(x) - g_m(x)\|$.

Proof. According to the definition of $H(x,y)$, Eq. (1.1) is shown below

$$D^q y(x) = g(x) + H\big(x, y(x)\big) \qquad q > 1 \ (7.12)$$

with the initial conditions ($r \in \mathbb{N}$)

$$y^{(i)}(0) = 0,\ i = 0,1,\dots,r-1$$

where $r-1 < q \leq r$. By taking the integral I^q on the other side of the equation

$$y(x) = \frac{1}{\Gamma(q)}\int_0^x (x-s)^{q-1} g(s)ds$$

$$+\frac{1}{\Gamma(q)}\int_0^x (x-s)^{q-1} H\big(s, y(s)\big)ds \qquad (7.13)$$

On the other hand, $g_m(x)$ is an extension of $g(x)$ and $y_m(x)$ is the approximate answer to the equation (7.4) So

$$\|y - y_m\|$$

$$\leq \left\|\frac{1}{\Gamma(q)}\int_0^x (x-s)^{q-1}(g(s) - g_m(s))ds\right\|$$

$$+\left\|\frac{1}{\Gamma(q)}\int_0^x (x-s)^{q-1}\big(H(s,y(s)) - H(s,y_m(s))\big)ds\right\|$$

According to the norm properties

$$\|y - y_m\| \leq \frac{1}{\Gamma(q)}\int_0^x (x-s)^{q-1}\|g(s) - g_m(s)\|ds$$

$$+\frac{1}{\Gamma(q)}\int_0^x (x-s)^{q-1}\|H(s,y(s)) - H(s,y_m(s))\|ds$$

so

$$\|y - y_m\| \leq \frac{E(g)}{\Gamma(q+1)} + \frac{\lambda}{\Gamma(q+1)}\|y - y_m\|$$

in result

$$\|y(x) - y_m(x)\| \leq \frac{E(g)}{\Gamma(q+1)\left(1 - \frac{\lambda}{\Gamma(q+1)}\right)}$$

and the proof is complete.

8- Numerical examples

Example 8.1. Consider the following nonlinear Fredholm integro-differential equations of fractional order ([28, 36]) :

$$D^\alpha f(x) - \int_0^1 xt[f(t)]^2 dt = 1 - \frac{x}{4},\ 0 \leq x < 1,\ 0 < \alpha \leq 1$$

with supplementary condition$f(0) = 0$.

$f(x) = x$ is the exact solution of the equation in the case of $\alpha = 1$. The error in the case $\alpha = 1$, for different values of k and M, is shown in Table 2 and 3. Note that:

$$\|e_j(x)\|_2 = \left(\int_0^1 e_j^2(x)dx\right)^{1/2} \cong \left(\frac{1}{N}\sum_{i=0}^{N} e_j^2(x_i)\right)^{1/2},$$

where $e_j(x_i) = f(x_i) - f_j(x_i), i = 0,1,\dots,N$. $f(x)$ is the exact solution and $f_j(x)$ is the approximate solution which is obtained by numerical methods. According to [35, 30], in comparison between Chebyshev method and Bernoulli method $j = M(2^{k-1})$ and in comparison, between CAS method and Bernoulli method $j = 2^k(2M+1)$.

Table 2: Comparison between approximate norm-2 of absolute error by using of the two different methods in Example 8.1

	Chebyshev method [36]	Bernoulli method
1	$\|e_8\|_2$	$\|e_8\|_2$
2	$(k=3, M=2)$	$(k=3, M=2)$
3	$2.9700e-007$	$2.7133e-008$
4	$\|e_{16}\|_2$	$\|e_{16}\|_2$
5	$(k=4, M=2)$	$(k=4, M=2)$
6	$1.8610e-008$	$1.9181e-009$
7	$\|e_{32}\|_2$	$\|e_{32}\|_2$
8	$(k=5, M=2)$	$(k=5, M=2)$
9	$1.1645e-009$	$1.6745e-0011$

Table 3: Comparison between approximate norm-2 of absolute error by using of the two different methods in Example 7.1

	CAS method [30]	Bernoulli method
1	$\|e_{12}\|_2$	$\|e_{12}\|_2$
2	$(k=2, M=1)$	$(k=2, M=1)$
3	$2.7133e-003$	$2.8421e-006$
4	$\|e_{24}\|_2$	$\|e_{24}\|_2$
5	$(k=3, M=1)$	$(k=3, M=1)$
6	$6.8179e-004$	$2.2312e-007$
7	$\|e_{48}\|_2$	$\|e_{48}\|_2$
8	$(k=4, M=1)$	$(k=4, M=1)$
9	$1.6745e-005$	$1.2150e-008$

Example 8.2. ([28, 36]) Consider the following nonlinear Fredholm integro-differential equation of order $\alpha = \frac{5}{3}$.

$$D^{\frac{5}{3}}f(x) - \int_0^1 (x+t)^2[f(t)]^3 dt = \frac{6}{\Gamma\left(\frac{1}{3}\right)}\sqrt[3]{x} - \frac{x^2}{7} - \frac{x}{4} - \frac{1}{9}, \quad 0 \le x < 1$$

with these supplementary conditions $f(0) = f'(0) = 0$. the exact solution of the equation is $f(x) = x^2$. The error for different values of k and M, is shown in Table 4 and 5.

Table 4: Comparison between approximate norm-2 of absolute error by using of the two different methods in Example 8.2

	Chebyshev method [36]	Bernoulli method
1	$\|e_8\|_2$	$\|e_8\|_2$
2	$(k=3, M=2)$	$(k=3, M=2)$
3	$3.1863e-005$	$3.5560e-006$
4	$\|e_{16}\|_2$	$\|e_{16}\|_2$
5	$(k=4, M=2)$	$(k=4, M=2)$
6	$6.1566e-006$	$6.2111e-007$
7	$\|e_{32}\|_2$	$\|e_{48}\|_2$
8	$(k=5, M=2)$	$(k=5, M=2)$
9	$2.4897e-007$	$2.5310e-008$

Table 5: Comparison between approximate norm-2 of absolute error by using of the two different methods in Example 8.2

	CAS method [30]	Bernoulli method
1	$\|e_{12}\|_2$	$\|e_{12}\|_2$
2	$(k=2, M=1)$	$(k=2, M=1)$
3	$3.5560e-003$	$3.2752e-004$
4	$\|e_{24}\|_2$	$\|e_{24}\|_2$
5	$(k=3, M=1)$	$(k=3, M=1)$
6	$9.0145e-004$	$7.1300e-005$
7	$\|e_{48}\|_2$	$\|e_{48}\|_2$
8	$(k=4, M=1)$	$(k=4, M=1)$
9	$2.2537e-005$	$2.0078e-006$

9- Conclusion

The most important point of this paper is the convergence of the proposed method and this hybrid approach has a unique approximation that is shown using mathematical principles and matrix theory.

In this paper the Bernoulli wavelet is built and its fractional integration operational matrix is produced in this article. Then we utilize them to solve a class of nonlinear Fredholm integro-differential equation of fractional order. The Bernoulli wavelet is made up of Bernoulli polynomials. It is better suited to the solution of fractional issues. The major advantage of the wavelet approach for solving equations is that the coefficients matrix of algebraic equations is sparse after discretization [32]. Even if the increment size is big, the solution is convergent.

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