International Journal of
Mathematical Modelling & Computations
Vol. 04, No. 04, Fall 2014, 365- 376

Tension Trigonometric Splines Interpolation Method for Solving a Linear Boundary Value Problem

J. Dabounou, O. EL Khayyari* and A. Lamnii

Faculty of Science and Technology, University Hassan first, Settat, Morocco.

Abstract. By using the trigonometric uniform splines of order 3 with a real tension factor, a numerical method is developed for solving a linear second order boundary value problems (2VBP) with Dirichlet, Neumann and Cauchy types boundary conditions. The moment at the knots is approximated by central finite-difference method. The order of convergence of the method and the theory is illustrated by solving test examples. Experimental results demonstrate that our method is more effective for the problems where the exact solution is trigonometric or hyperbolic.

Received: 12 Janurary 2014, Revised: 20 March 2014, Accepted: 28 April 2014.

Keywords: Trigonometric B-splines, Tension factor, Boundary value problems.

Index to information contained in this paper

1. Introduction

In this paper, where the second derivative is approximated by the three-point central difference scheme, and where we need to build an approximate equation for each type of boundary condition, we study a method based on the trigonometric B-splines of order 3 (lower order) with a tension factor ρ for constructing numerical solutions to second-order boundary value problems (2BVPs) of the form:

$$y^{(2)}(\theta) + f(\theta)y'(\theta) + g(\theta)y(\theta) = p(\theta), \tag{1}$$

*Corresponding author. Email: omar-khayari@hotmail.fr.

subject to the three boundary conditions:

$$Dirichlet : y(a) = a_0,\ y(b) = b_0, \tag{2}$$

$$Neumann : y'(a) = a_1,\ y'(b) = b_1, \tag{3}$$

$$Cauchy : y(a) = a_0,\ y'(b) = b_1, \tag{4}$$

where $f(\theta)$, $g(\theta)$ and $p(\theta)$) are given continuous functions defined in the bounded interval $[a, b]$, a_i and b_i, $i = 0, 1$ are real constants.
In many engineering applications including the study of beam deflections, heat flow, and various dynamic systems, we need solving the second order two-point boundary value problems. Much attention have been given to solve the second-order boundary value problems (2VBP) with Dirichlet, Neumann and Cauchy types boundary conditions, which have application in various branches of applied sciences. These problems are generally arise in the mathematical modeling of viscoelastic flows [10]. A spline has been widely applied for the numerical solutions of some ordinary and partial differential equations in the numerical analysis [5, 7, 16]. Many authors have used numerical and approximate methods to solve second and third BVPs. Recently in [19] M.M. Rahman et al. solved linear differential equation numerically by Galerkin method with Hermite polynomial as trial functions. Lima and Carpentier in [18] obtained a Numerical solution of a singular boundary value problem in non-Newtonien fluid mechanics. Feng and Li in [13] solved a second order Neumann boundary value problem with singular nonlinearity for exact three positive solutions. In a series of paper by Dabounou et al. [8, 9] BVPs of order fourth and sixth were solved using third, and fourth order hyperbolic splines. The numerical solution of the boundary value problem by splines interpolation and quasi-interpolation has been considered by many authors; see, [14–17] for example, and the references given in these papers. Despite the importance of the trigonometric and hyperbolic B-splines, little attention has been paid to developing efficient numerical methods based on these B-splines for solving the boundary value problem, except those of Refs. [8, 9, 12] for special cases. This motivates us to use trigonometric B-splines(tension) of order 3 (lower order) to solve these problems. We apply tension trigonometric B-splines of order 3 to develop a new numerical method for obtaining efficient approximations to the solution of boundary value problem. The new method is of order first for arbitrary a real tension factor $\rho > 0$.

The paper has been arranged in the following way. In Section 2, we give a explicit representation of trigonometric B-splines(tension) of order 3. The interpolation trigonometric B-splines is developed in Section 3. Solutions of (2VBP) with Dirichlet, Neumann and Cauchy types boundary conditions are presented in Section 4. To illustrate our algorithm, numerical examples with various values of a real tension factor ρ are presented in Section 5.

2. Explicit Representation of Trigonometric B-Splines (Tension) of Order 3

To develop the numerical solutions based on trigonometric B-splines (tension) of order 3 for the solution of second order linear Dirichlet, Neumann and Cauchy boundary value problem equations, let k one intergre such that $k \geqslant 1$. Let $n_k =$

$3.2^k + 2$ and $h_k = \frac{b-a}{n_k-2}$. We consider a general uniform mesh

$$\begin{cases} \theta_{-2}^k = \theta_{-1}^k = \theta_0^k = a, \\ \theta_i^k = a + ih_k, i = 1...n_k - 3, \\ \theta_{n_k-2}^k = \theta_{n_k-1}^k = \theta_{n_k} = b, \end{cases} \tag{5}$$

We introduce the trigonometric (tension) splines space of order 3 is defined as follows

$$\mathcal{V}_k = \{s \in \mathcal{C}^1(I) : \ s_{[\theta_i^k, \theta_{i+1}^k]} \in \Gamma_3\} \text{ where } \Gamma_3 = span\{1, \cos(\rho\theta), \sin(\rho\theta)\}.$$

where ρ is a positif real tension factor ($\rho > 0$).
The dimension of $\mathcal{V}_k$ is n_k and we denote by $\phi_{i,k},\ i = 0, 1, ..., n_k - 5$, the third-order trigonometric (tension) B-splines:

$$\phi_{i,k}(\theta) = C_k \begin{cases} -1 + \cos(\rho(\theta - \theta_i^k)), & \theta \in [\theta_i^k, \theta_{i+1}^k[; \\ 2(\cos(\rho h_k) - \cos(\rho \frac{h_k}{2}) \cos(\rho(\theta_{i+1}^k + \frac{h_k}{2} - \theta)), & \theta \in [\theta_{i+1}^k, \theta_{i+2}^k[; \\ -1 + \cos(\rho(\theta - \theta_{i+3}^k)), & \theta \in [\theta_{i+2}^k, \theta_{i+3}^k[; \\ 0, & \text{otherwise.} \end{cases}$$

The respective left and right hand side boundary trigonometric B-splines are

$$\phi_{-2,k}(\theta) = C_k \begin{cases} 2(-1 + \cos(\rho(h_k - \theta + a))), & \theta \in [\theta_0^k, \theta_1^k[; \\ 0, & \text{otherwise.} \end{cases}$$

$$\phi_{-1,k}(\theta) = C_k \begin{cases} 1 + 2\cos(\rho h_k) - 2\cos(\rho(a + h_k - \theta)) - \cos(\rho(-a + \theta)), & \theta \in [\theta_0^k, \theta_1^k[; \\ -1 + \cos(\rho(a + 2h_k - \theta)), & \theta \in [\theta_1^k, \theta_2^k[; \\ 0, & \text{otherwise.} \end{cases}$$

$$\phi_{n_k-4,k}(\theta) = C_k \begin{cases} -1 + \cos(\rho(b - 2h_k - \theta)), & \theta \in [\theta_{n_k-4}^k, \theta_{n_k-3}^k[; \\ 1 + 2\cos(\rho h_k) - \cos(\rho(b - \theta)) - 2\cos(\rho(b - h_k - \theta)), & \theta \in [\theta_{n_k-3}^k, \theta_{n_k-2}^k[; \\ 0, & \text{otherwise.} \end{cases}$$

$$\phi_{n_k-3,k}(\theta) = C_k \begin{cases} 2(-1 + \cos(\rho(b - h_k - \theta))), & \theta \in [\theta_{n_k-3}^k, \theta_{n_k-2}^k[; \\ 0, & \text{otherwise.} \end{cases}$$

where $C_k = \frac{1}{2(-1+\cos(\rho h_k))}$.

The trigonometric B-splines(tension) of order 3 possess all the desirable properties of classical polynomial B-splines, see [20]. In this paper, we limit ourselves to list some of them

- $\phi_{i,k}(\theta)$ is supported on the interval $[\theta_i^k, \theta_{i+1}^k]$;
- Positivity : $\phi_{i,k}(\theta) \geqslant 0, \quad \forall \theta \in [\theta_i^k, \theta_{i+1}^k]$;
- Partition of unity: $\displaystyle\sum_{i=-2}^{n_k-3} \phi_{i,k}(\theta) = 1$.

Table 1. The values of $\phi_{i,k}(\theta)$ and $\phi'_{i,k}(\theta)$ at the knots

	θ_i^k	θ_{i+1}^k	θ_{i+2}^k	θ_{i+3}^k	*else*
$\phi_{i,k}(\theta)$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$\phi'_{i,k}(\theta)$	0	$\frac{\rho}{2\tan(\rho\frac{h_k}{2})}$	$\frac{-\rho}{2\tan(\rho\frac{h_k}{2})}$	0	0

3. Trigonometric Interpolation Method

The key point of the discretization of our scheme is how to get an approximate formula of $y^{(2)}(\theta_j^k)$ and $y'(\theta_j^k)$ by using Taylor series expansion.

According to Schoenberg-Whitney theorem (see [11]), for a given function $y(\theta)$ sufficiently smooth there exists a unique trigonometric spline

$$s(\theta) = \sum_{i=-2}^{n_k-3} \mu_i \phi_{i,k}(\theta) \in \mathcal{V}_k$$

satisfying the interpolation conditions:

$$s(\theta_j^k) = y(\theta_j^k), \quad j = 0, 1, \cdots, n_k - 3; \tag{6}$$

$$s'(a) = y'(a), \quad s'(b) = y'(b). \tag{7}$$

For $j = 0, 1, \cdots, n_k - 3$, let $m_j = s'(\theta_j^k)$ and for $j = 0, 1, \cdots, n_k - 3$, let M_j the approximate moment at the knot θ_j^k, setting :

$$M_j = \frac{s(\theta_j^k + h_k) - 2s(\theta_j^k) + s(\theta_j^k - h_k)}{h_k^2}.$$

By using the Taylor series expansion we have:

$$m_j = s^{'}(\theta_j^k) = y^{'}(\theta_j^k) - \frac{1}{180} h_k^4 y^{(5)}(\theta_j^k) + O(h_k^6); \tag{8}$$

$$M_j = y^{''}(\theta_j^k) + \frac{1}{12} h_k^2 y^{(4)}(\theta_j^k) + \frac{1}{360} h_k^4 y^{(6)}(\theta_j^k) + O(h_k^6); \tag{9}$$

By Table 1 and this equations, we get:

Table 2. The approximation values of $y(\theta_j^k)$, $y^{'}(\theta_j^k)$ and $y^{''}(\theta_j^k)$.

	$y(\theta_j^k)$	$y^{'}(\theta_j^k)$	$y^{''}(\theta_j^k)$
Approximate value	$s(\theta_j^k)$	m_j	M_j
Representation in μ_j	$\frac{\mu_{j-1}+\mu_{j-2}}{2}$	$\frac{\rho(\mu_{j-1}-\mu_{j-2})}{2\tan(\rho\frac{h_k}{2})}$	$\frac{\mu_{j-3}-\mu_{j-2}-\mu_{j-1}+\mu_j}{2h_k^2}$
Error order	$O(h_k^4)$	$O(h_k^4)$	$O(h_k^2)$

4. Formulation of 2 BVP in Matrix form

In this section, we study a method based on trigonometric (tension) B-splines of order 3 for constructing numerical solutions to linear Dirichlet, Neumann and Cauchy boundary value problem 2 BVP of the form (1).

4.1 *Trigonometric (Tension) Solution with Dirichlet Boundary Condition*

We define a trigonometric (tension) spline interpolant $s(\theta) = \sum_{i=-2}^{n_k-3} \mu_i \phi_{i,k}(\theta)$ satisfying boundary conditions (2.2) and $\widetilde{s}(\theta) = \sum_{i=-2}^{n_k-3} \widetilde{\mu}_i \phi_{i,k}(\theta)$ be the approximate spline of $s(\theta)$. To do this we discretize (1) at θ_j^k for $j = 1, 2, \cdots, n_k - 3$, we get the spline approximation as :

$$y^{(2)}(\theta_j^k) + f(\theta_j^k)y'(\theta_j^k) + g(\theta_j^k)y(\theta_j^k) = p(\theta_j^k), \quad j = 1, 2, \cdots, n_k - 3. \tag{10}$$

Substituting $y^{(2)}(\theta_j^k)$, $y'(\theta_j^k)$ and $y(\theta_j^k)$ in equation (10) and using the following notations:

$$f_j = f(\theta_j^k), \quad g_j = g(\theta_j^k) \text{ and } p_j = p(\theta_j^k),$$

we arrive at the following linear system :

$$\frac{\mu_{j-3} - \mu_{j-2} - \mu_{j-1} + \mu_j}{2h_k^2} + f_j \frac{\rho(\mu_{j-1} - \mu_{j-2})}{2\tan(\rho\frac{h_k}{2})} + g_j \frac{\mu_{j-1} + \mu_{j-2}}{2} = p_j + O(h_k^2) \tag{11}$$

Consequently,

$$(\mu_{j-3} - \mu_{j-2} - \mu_{j-1} + \mu_j) + \alpha_{j,k}(\mu_{j-1} - \mu_{j-2}) + \beta_{j,k}(\mu_{j-1} + \mu_{j-2}) = \gamma_{j,k} + \xi_k. \tag{12}$$

Where $\alpha_{j,k} = f_j \frac{\rho h_k^2}{\tan(\rho\frac{h_k}{2})}$, $\beta_{j,k} = g_j h_k^2$, $\gamma_{j,k} = 2h_k^2 p_j$, and $\xi_k = O(2h_k^4)$.
By dropping ξ_k from (12), we yield a linear system with $n_k - 3$ linear equations in n_k unknowns $\mu_j,\ j = -2, -1, \cdots, n_k - 3$. So three more equations are needed.
On the other hand, by using the Dirichlet boundary conditions (2), we get

$$\begin{cases} y(a) = a_0; \\ y(b) = b_0. \end{cases} \text{Thus, } \begin{cases} \mu_{-2} = a_0; \\ \mu_{n_k-3} = b_0. \end{cases} \tag{13}$$

To obtain unique solution we propose the following formula

$$y'_{j-1} + 4y'_j + y'_{j+1} = \frac{3}{h_k}(y_{j+1} - y_{j-1}) + O(h_k^4) \tag{14}$$

which can be easily demonstrated using a Taylor series expansion.
Combining the scheme (14) with Table 2, we can construct an approximate formulae for $y^{(2)}(a)$, as follows

$$y^{(2)}(a) = -4M_1 - M_2 + \frac{3}{h_k}(m_2 - m_0) + O(h_k^4). \tag{15}$$

For smaller h_k, we have $\tan(\frac{h_k}{2}) \sim \frac{h_k}{2}$ and by turning (15) the coefficients are determined as follows

$$(2\mu_{-2} - 3\mu_{-1} - \mu_0 + 3\mu_1 - \mu_2) + 2h_k f_0(\mu_{-1} - \mu_{-2}) = 2h_k^2(p_0 - g_0 a_0) + O(h_k^4) \tag{16}$$

Take (12) and (16), we get $n_k - 2$ linear equations with $\mu_i,\ i = -1, 0, \cdots, n_k - 5, n_k - 4$, as unknowns since μ_{-2}, and μ_{n_k-3} have been yielded from (13).
Let $C = [\mu_{-1}, \mu_0, \cdots, \mu_{n_k-5}, \mu_{n_k-4}]^T$, $\widetilde{C} = [\widetilde{\mu}_{-1}, \widetilde{\mu}_0, \cdots, \widetilde{\mu}_{n_k-5}, \widetilde{\mu}_{n_k-4}]^T$, $D = [d_1, d_2, \cdots, d_{n_k-2}]^T$, $E = [e_1, e_2, \cdots, e_{n_k-2}]^T$. We can write our method in matrix form as :

$$(A_1 + \lambda_k A_2 F + h_k^2 BG)C = D + E; \tag{17}$$

$$(A_1 + \lambda_k A_2 F + h_k^2 BG)\widetilde{C} = D, \tag{18}$$

with

$$\lambda_k = \frac{\rho h_k^2}{\tan(\rho\frac{h_k}{2})}$$

and where A_1 and A_2 are the following $(n_k-2)\times(n_k-2)$ matrix:

$$A_1=\begin{pmatrix} -3 & -1 & 3 & -1 & & & & \\ -1 & -1 & 1 & 0 & & & & \\ 1 & -1 & -1 & 1 & & & & \\ & \cdots & \cdots & \cdots & \cdots & & & \\ & & \cdots & \cdots & \cdots & \cdots & & \\ & & & & 1 & -1 & -1 & 1 \\ & & & & & 1 & -1 & -1 & 1 \\ & & & & & & 1 & -1 & -1 \end{pmatrix}, \qquad A_2=\begin{pmatrix} 1 & 0 & & & & \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & \cdots & \cdots & \cdots & \cdots & \\ & & \cdots & \cdots & \cdots & \cdots \\ & & & & -1 & 1 \\ & & & & & -1 & 1 \\ & & & & & & -1 & 1 \end{pmatrix}$$

and where F,G, B and D are the following matrix

$$F=\begin{pmatrix} f_0 & & & & \\ & f_1 & & & \\ & & f_2 & & \\ & & & \ddots & \\ & & & & f_{n_k-3} \end{pmatrix}, G=\begin{pmatrix} 0 & & & & \\ & g_1 & & & \\ & & g_2 & & \\ & & & \ddots & \\ & & & & g_{n_k-3} \end{pmatrix}, B=\begin{pmatrix} 0 & 0 & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & & 1 & 1 \end{pmatrix},$$

$$D=\begin{pmatrix} \gamma_{0,k}-2\beta_{0,k}a_0+2h_kf_0\mu_{-2}-2\mu_{-2} \\ \gamma_{1,k}-\mu_{-2} \\ \gamma_{2,k} \\ \vdots \\ \gamma_{n_k-4,k} \\ \gamma_{n_k-3,k}-\mu_{n_k-3} \end{pmatrix}$$

and $e_i=O(2h_k^4), i=1,2,\cdots,n_k-2$.

After solving the linear system (28), $\widetilde{\mu}_i,\ i=-1,0,\cdots,,n_k-5,,n_k-4,\ \widetilde{\mu}_{-2}=\mu_{-2}$, and $\widetilde{\mu}_{n_k-3}=\mu_{n_k-3}$ will be used together to get the approximation spline solution $\widetilde{s}(\theta)=\sum_{i=-2}^{n_k-3}\widetilde{\mu}_i\phi_{i,k}(\theta)$.

4.2 *Trigonometric (Tension) Solution with Neumann Boundary Condition*

Proceeding as above and by using Neumann boundary condition (3) we get:

$$\begin{cases} y'(a)=a_1; \\ y'(b)=b_1. \end{cases} \text{Thus, } \begin{cases} \mu_{-1}-\mu_{-2}=\frac{2a_1\tan(\rho\frac{h_k}{2})}{\rho}; \\ \mu_{n_k-3}-\mu_{n_k-4}=\frac{2b_1\tan(\rho\frac{h_k}{2})}{\rho}. \end{cases} \tag{19}$$

We have also by using (14)

$$y^{(2)}(a)=\frac{1}{h_k}(m_1+2a_1)-\frac{3}{2h_k^2}(s(a+h_k)-s(a))+O(h_k^4) \tag{20}$$

For smaller h_k, we have $\tan(\frac{h_k}{2})\sim\frac{h_k}{2}$ and by turning (20) the coefficients are determined as follows

$$(3\mu_{-2}-4\mu_{-1}+\mu_0+8h_ka_1)+2h_k^2g_0(\mu_{-1}+\mu_{-2})=4h_k^2(p_0-f_0a_1)+O(h_k^4) \tag{21}$$

Take (12),(16) and (24), we yield:

$$(A_1+\lambda_kA_2F+h_k^2BG)C=D+E; \tag{22}$$

$$(A_1+\lambda_kA_2F+h_k^2BG)\widetilde{C}=D, \tag{23}$$

where A_1 and A_2 are the following $n_k\times n_k$ matrix:

$$A_1 = \begin{pmatrix} -1 & 1 & & & & & \\ 3 & -4 & 1 & & & & \\ 1 & -1 & -1 & 1 & & & \\ & \cdots & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & 1 & -1 & -1 & 1 \\ & & & & & -1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & & & & & \\ 0 & 0 & & & & & \\ 0 & -1 & 1 & & & & \\ & \cdots & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & & -1 & 1 & 0 \\ & & & & & 0 & 0 \end{pmatrix}$$

and where F, G ,B ,$\widetilde{C}$ and D are the following matrix

$$F = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & f_1 & & & \\ & & & \ddots & & \\ & & & & f_{n_k-3} & \\ & & & & & 0 \end{pmatrix}, \; G = \begin{pmatrix} 0 & & & & & \\ & g_0 & & & & \\ & & g_1 & & & \\ & & & \ddots & & \\ & & & & g_{n_k-3} & \\ & & & & & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & & & & & \\ & 2 & 2 & & & & \\ & 1 & 1 & & & & \\ & & 1 & 1 & & & \\ & & & \ddots & \ddots & & \\ & & & & 1 & 1 & \\ & & & & & 1 & 1 \\ & & & & & 0 & 0 \end{pmatrix},$$

$\widetilde{C} = [\widetilde{\mu}_{-2}, \widetilde{\mu}_{-1}, \cdots, \widetilde{\mu}_{n_k-5}, \widetilde{\mu}_{n_k-4}, \widetilde{\mu}_{n_k-3}]^T$,

$$D = \begin{pmatrix} \frac{2a_1 \tan(\rho \frac{h_k}{2})}{\rho} \\ 2\gamma_{0,k} - 4h_k^2 f_0 a_1 - 8a_1 h_k \\ \gamma_{1,k} \\ \vdots \\ \gamma_{n_k-3,k} \\ \frac{2b_1 \tan(\rho \frac{h_k}{2})}{\rho} \end{pmatrix}$$

4.3 *Trigonometric (Tension) Solution with Cauchy Boundary Condition*

Using similar techniques as in above section and by transforming the Cauchy boundary condition (4). One can obtain the following results.

$$\begin{cases} y(a) = a_0; \\ y'(b) = b_1. \end{cases} \text{Thus,} \begin{cases} \mu_{-2} = a_0; \\ \mu_{n_k-3} - \mu_{n_k-4} = \frac{2b_1 \tan(\rho \frac{h_k}{2})}{\rho}. \end{cases} \tag{24}$$

We have also by using (14)

$$y^{(2)}(a) = \frac{1}{h_k}(m_1 + 2m_0) - \frac{3}{2h_k^2}(s(a+h_k) - a_0) + O(h_k^4) \tag{25}$$

For smaller h_k, we have $\tan(\frac{h_k}{2}) \sim \frac{h_k}{2}$ and by turning (25) the coefficients can be determined by using the relation

$$(-2\mu_{-2} + \mu_{-1} + \mu_0) + 2\lambda_k f_0(\mu_{-1} - \mu_{-2}) = 4h_k^2(p_0 - g_0 a_0) + O(h_k^4) \tag{26}$$

Take (12),(24) and (26), we yield :

$$(A_1 + \lambda_k A_2 F + h_k^2 BG)C = D + E; \tag{27}$$

$$(A_1 + \lambda_k A_2 F + h_k^2 BG)\widetilde{C} = D, \tag{28}$$

where A_1 and A_2 are the following $(n_k - 1) \times (n_k - 1)$ matrix:

$$A_1 = \begin{pmatrix} 1 & 1 & & & & & \\ -1 & -1 & 1 & & & & \\ 1 & -1 & -1 & 1 & & & \\ & \cdots & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & 1 & -1 & -1 & 1 \\ & & & & & -1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & 0 & & & & & \\ -1 & 1 & & & & & \\ 0 & -1 & 1 & & & & \\ & \cdots & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & & -1 & 1 & 0 \\ & & & & & 0 & 0 \end{pmatrix}$$

and where F, G ,B ,$\widetilde{C}$ and D are the following matrix

$$F = \begin{pmatrix} f_0 & & & & \\ & f_1 & & & \\ & & \ddots & & \\ & & & f_{n_k-3} & \\ & & & & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & & & & \\ & g_1 & & & \\ & & \ddots & & \\ & & & g_{n_k-3} & \\ & & & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & \ddots & & & \\ & & & 1 & 1 & \\ & & & 1 & 1 & \\ & & & & 0 & 0 \end{pmatrix},$$

$\widetilde{C} = [\widetilde{\mu}_{-1}, \widetilde{\mu}_0, \cdots, \widetilde{\mu}_{n_k-5}, \widetilde{\mu}_{n_k-4}, \widetilde{\mu}_{n_k-3}]^T$,

$$D = \begin{pmatrix} 2\gamma_{0,k} - 4h_k^2 g_0 a_0 + 2\lambda_k f_0 a_0 + 2a_0 \\ \gamma_{1,k} \\ \vdots \\ \gamma_{n_k-3,k} \\ \frac{2b_1 \tan(\rho \frac{h_k}{2})}{\rho} \end{pmatrix}$$

5. Numerical Examples

To justify the accuracy and efficiency of our presented method and compare our computed results we consider the following examples. The solution of the given examples is obtained for different values of k and ρ. The error solution $E = |y - s|_\infty$ where y is the exact solution and s is the approximated solution spline of boundary value problem equation which is given by the suggested method.

Example 5.1 We consider the following Dirichlet boundary-value problem

$$\begin{cases} d^2y/dx^2 - y(x) = 0, & x \in [0,1]; \\ y(0) = 0, \ y(1) = \sinh(x). \end{cases} \tag{29}$$

The exact solution is $y(x) = \sinh(x)$.
The numerical results for different values of k and ρ are presented in the Table 3.

Table 3. Maximum absolute error for Problem (29).

k	Error for $\rho = 10^{-2}$	Error for $\rho = 1$
3	4.958e-003	4.958e-003
4	2.494e-003	2.494e-003
5	1.250e-003	1.250e-003
6	6.262e-004	6.262e-004
7	3.131e-004	3.133e-004
8	1.547e-004	1.567e-004
9	7.620e-005	7.837e-005

Example 5.2 We consider the following Dirichlet boundary-value problem

$$\begin{cases} y^{(2)}(x) - y(x) = -2exp(x), \ x \in [0,1]; \\ y(0) = 1, \ y(1) = 0. \end{cases} \tag{30}$$

The exact solution is $y(x) = (1 - x)\exp(x)$.
We found the following results for different values of k and ρ (Table 4).

Table 4. Maximum absolute error and Order of convergence for Problem (30).

k	Error for $\rho = 10^{-3}$	Error for $\rho = 1$	Order of convergence for $\rho = 1$
3	2.475e-002	2.475e-002	1.1087
4	1.246e-002	1.246e-002	1.0536
5	6.256e-002	6.256e-003	1.0276
6	3.133e-003	3.133e-003	1.0136
7	1.595e-003	1.568e-003	1.0070
8	7.303e-004	7.843e-004	1.0036
9	6.082e-004	3.922e-004	

Example 5.3 We consider the following Dirichlet boundary-value problem

$$\begin{cases} d^2y/dx^2 - dy/dx = -\exp(x-1) - 1, \; x \in [0,1]; \\ y(0) = y(1) = 0. \end{cases} \tag{31}$$

The exact solution is $y(x) = x(1 - \exp(x-1))$.
Result has been shown for different values of k and with various values of tension factor ρ in Table 5.

Table 5. Maximum absolute error for Problem (31).

k	Error for $\rho = 10^{-4}$	Error for $\rho = 1$
3	1.018e-002	1.018e-002
4	5.185e-003	5.190e-003
5	2.599e-003	2.620e-003
6	1.477e-003	1.316e-003
7	7.828e-004	6.597e-004
8	3.302e-004	3.302e-004
9	1.605e-004	1.652e-004

Example 5.4 Consider the following Neumann boundary-value problem discussed in [19].

$$\begin{cases} y^{(2)}(x) + y(x) = \cos(x), \; x \in [0,5]; \\ y'(0) = 0, \;\; y'(5) = 0. \end{cases} \tag{32}$$

for which the exact solution is

$$y(x) = \frac{(-6\cos^2(5) + 2 + 10\cot(5)\sin(10) + 2\cos(10))\cos(x)}{4} + \frac{2\cos^3(x) + 2x\sin(x) + \sin(x)\sin(2x)}{4}.$$

The observed maximum errors in absolute values computed at various points of the interval $[0,5]$, for problem (32), and the convergence order are summarized in Table 6.

Example 5.5 We consider the following Neumann boundary-value problem discussed in [19].

$$\begin{cases} y^{(2)}(x) + y(x) = x^2\exp(-x), \; x \in [0,10]; \\ y'(0) = 0, \;\; y'(10) = 0. \end{cases} \tag{33}$$

The exact solution is $y(x) = -\frac{\cos(10)+99\exp(-10)}{2\sin(10)}\cos(x) - \frac{\sin(x)}{2} + \frac{\exp(-x)(1-x)^2}{2}$.
Result has been shown for different values of k and with various values of ρ in Table 8.

Table 6. Maximum absolute error and Order of convergence for Problem (32).

k	Error for $\rho = 10^{-4}$	Error for $\rho = 1$	Order of convergence for $\rho = 1$
3	2.531e-001	2.531e-001	0.9725
4	1.340e-001	1.340e-001	0.9768
5	6.944e-002	6.944e-002	0.9862
6	3.541e-002	3.541e-002	0.9924
7	1.805e-002	1.789e-002	0.9960
8	9.072e-003	8.993e-003	0.9978
9	8.553e-003	4.509e-003	

Table 7. Error in Galerkin method developed in [19] for Problem (32).

x	Maximum absolute error for $n = 12$
0.1	6.7277e-001
0.2	2.1316e+000
0.3	1.3334e+001
0.4	4.8501e+000
0.5	2.5722e+000

Table 8. Maximum absolute error for Problem (33).

k	Error for $\rho = 10^{-3}$	Error for $\rho = 1$
3	2.062e-001	2.062e-001
4	1.478e-001	1.478e-001
5	9.614e-002	9.614e-002
6	5.700e-002	5.700e-002
7	3.149e-002	3.149e-002
8	1.663e-002	1.663e-002
9	8.557e-003	8.558e-003

Table 9. Error in Galerkin method developed in [19] for Problem (33).

x	Maximum absolute error for $n = 12$
0.1	1.5213e+000
0.2	8.9251e-002
0.3	2.8328e-001
0.4	3.8058e-001
0.5	3.0792e-001

Example 5.6 We consider the following Cauchy boundary-value problem

$$\begin{cases} y^{(2)}(x) + xy(x) = -(x^3 + 3x)\exp(x), \; x \in [0,1]; \\ y(0) = 0, \;\; y'(1) = -exp(1). \end{cases} \tag{34}$$

The exact solution is $y(x) = (-x^2 + x)\exp(x)$.
In Table 10, we give the corresponding errors for different values of k and ρ and the computed convergence orders.

Example 5.7 We consider the following Cauchy boundary-value problem

$$\begin{cases} y^{(2)}(x) - 2y(x) = 2(1 + \tan^2(x)), \; x \in [0,1]; \\ y(0) = 0, \;\; y'(1) = \tan(1). \end{cases} \tag{35}$$

Table 10. Maximum absolute error and Order of convergence for Problem (34).

k	Error for $\rho = 10^{-2}$	Error for $\rho = 1$	Order of convergence for $\rho = 1$
3	2.714e-002	2.715e-002	1.0311
4	1.434e-002	1.434e-002	0.9966
5	7.473e-003	7.473e-003	0.9940
6	3.828e-003	3.828e-003	0.9960
7	1.939e-003	1.939e-003	0.9973
8	9.757e-004	9.763e-004	0.9988
9	4.909e-004	4.898e-004	

The exact solution is $y(x) = (x-1)\tan(x)$.
The results for different values of k and with various values of ρ are summarized in Table 11.

Table 11. Maximum absolute error for Problem (35).

k	Error for $\rho = 10^{-3}$	Error for $\rho = 1$
3	6.328e-002	6.328e-002
4	3.481e-002	3.481e-002
5	2.157e-002	2.157e-002
6	1.515e-002	1.515e-002
7	1.198e-002	1.999e-002
8	1.043e-002	1.041e-002
9	9.566e-003	9.635e-003

The norms of the approximated solutions were compared with those of finite difference (FDM) and cubic B-spline interpolation methods (CBIM). The results from these methods were generated by solving Examples (29) and (31) by using the methods explained in [4] and [6]. The results are shown in Table 12 and Table 13.

Table 12. Max-Norm and L^2-Norm for Problem (29).

Method	Max-Norm	L^2-Norm
FDM(Burden-Faires [4])	5.1880e-005	1.1764e-004
CBIM(Caglar et al. [6])	5.2011e-005	1.1794e-004
Our method(for $\rho = 1$ and $k = 10$)	3.9190e-005	1.0670e-004

Table 13. Max-Norm and L^2-Norm for Problem (31).

Method	Max-Norm	L^2-Norm
FDM(Burden-Faires [4])	1.6265e-004	3.6971e-004
CBIM(Caglar et al. [6])	2.8996e-004	6.6089e-004
Our method(for $\rho = 1$ and $k = 9$)	1.6524e-004	5.1775e-004
Our method(for $\rho = 1$ and $k = 10$)	8.2645e-005	2.5885e-004

6. Conclusion

Despite a lower order of used splines, our method produced better with Neumann boundary conditions (see examples 4 and 5), in comparison with the largest value of n degree of

Hermite polynomials in Galerkin method developed in [19], also the Galerkin method is not applied to any example with Dirichlet boundary condition , for more details see [19]. The purpose of this paper is to present a new simple numerical method, suitable and accurate to solve (2VBP) boundary value problem with Dirichlet, Neumann and Cauchy conditions by using trigonometric splines of order 3 with a tension factor ρ.
Experimental results confirm the first order of convergence. We notice also that the influence of the value of tension factor ρ is remarkable when the value of k exceeds the value 8. The results in this paper can be easily extended to other classes of systems of boundary value problems of higher order and a multi-dimensional boundary problems.

References

[1] M. Amirfakhrian, *Approximation of 3D-Parametric Functions by Bicubic B-spline Functions*, International Journal of Mathematical Modelling & Computations, **2 (3)** (2012) 211-220.
[2] M. Amirfakhrian and H. Nouriani, *Interpolation by Hyperbolic B-spline Functions*, International Journal of Mathematical Modelling & Computations, **1 (3)** (2011) 175-181.
[3] M. Amirfakhrian and S. Katayoon , *Solving Integro-Differential Equation by Using B- Spline Interpolation*, International Journal of Mathematical Modelling & Computations, **3 (3)** (2013) 237-244
[4] R.L. Burden and J.D. Faires, *Numerical Analysis,* 8th ed. Belmont: Brooks/Cole, Cengage Learning, 2005.
[5] N. Çağlar and H. Çağlar, *B-spline solution of singular boundary value problems.* Appl. Math. Comput. **182** (2006) 1509-1513.
[6] N. Çağlar, H. Çağlar and K. Elfaituri, *B-spline interpolation Compared with finite difference, finite element and finite volume methods which applied to two point boundary value problems,* Applied Mathematics and computation. **175 (1)** (2006) 72-79.
[7] H. Çağlar, M. Özer and N. Çağlar, *The numerical solution of the one-dimensional heat equation by using third degree B-spline functions*, Chaos, Solitons & Fractals, **38** (2008) 1197-1201.
[8] J. Dabounou, O. El khayyari and A. Lamnii, *A Hyperbolic Uniform Spline Method for Linear Fourth Order Boundary Value Problem*, Int. Journal of Math. Analysis, **7 (46)** (2013) 2293-2302.
[9] J. Dabounou, O. El khayyari and A. Lamnii, *Solving linear sixth-order boundary value problems by using hyperbolic uniform spline method*, International Journal of Mathematical Modelling & Computations, **3 (3)** (2013) 169-180.
[10] A. R. Davies, A. Karageorghis and T. N. Phillips, *Spectral Galerkin methods for the primary two-point boundary value problem in modelling viscoelastic flows*, Int. J. Numer. Methods Engng, **26** (1988) 647-662.
[11] C. De boor, *A Practical Guide to Spline*, Springer-Verlag, 1978.
[12] O. El khayyari and A. Lamnii, *Numerical Solutions of Second Order Boundary Value Problem by Using Hyperbolic Uniform B-Splines of Order 4*, International Journal of Mathematical Modelling & Computations, (In press).
[13] Y. Feng and G. Li, *Exact three positive solutions to a second-order Neumann boundary value problem with singular nonlinearity*, Arabian J. Sci. Eng. **35** (2010) 189-195.
[14] A. Lamnii, H. Mraoui, D. Sbibih, A. Tijini and A. Zidna, *Sextic spline collocation methods for nonlinear fifth-order boundary value problems*, International Journal of Computer Mathematics, (2011) 1-17.
[15] A. Lamnii, H. Mraoui, D. Sbibih and A. Tijini, *Sextic spline solution of nonlinear fifth-order boundary value problems*, Mathematics and Computers in Simulation, **77** (2008) 237-246.
[16] A. Lamnii and H. Mraoui, *Spline Collocation Method for Solving Boundary Value Problems*, International Journal of Mathematical Modelling & Computations, **3 (1)** (2013) 11-23.
[17] A. Lamnii, H. Mraoui, D. sbibih, A. Tijini and A. zidna, *Spline Solution Of Some Linear Boundary Value Problems*, Applied Mathematics E-Notes, **8** (2008) 171-178.
[18] P. M. Lima and M Carpentier, *Numerical solution of a singular boundary-value problem in non-Newtonien fluid mechanics*, Computer phys. Communica. **126** (2000) 114-120.
[19] M. M. Rahman, M. A. Hossen, M. Nurul Islam and Md. Shajib Ali, *Numerical Solutions of Second Order Boundary Value Problem by Galerkin Method with Hermite Polynomials*, Annals of Pure and Applied Mathematics, **2** (2012) 138-148.
[20] G. Wang, Q. Chen and M. Zhou, *NUAT B-spline curves*, Computer Aided Geometric Design, **21** (2004) 193-205.