International Journal of
Mathematical Modelling & Computations
Vol. 05, No. 04, Fall 2015, 347- 360

Dynamic Complexity of a Three Species Competitive Food Chain Model with Inter and Intra Specific Competition

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Abstract. The present article deals with the inter specific competition and intra-specific competition among predator populations of a prey-dependent three component food chain model consisting of two competitive predator sharing one prey species as their food. Stability analysis including local and global stability of the equilibria has been carried out in order to examine the dynamic behaviour of the system. As a result, intra-specific competition among predator populations can establish global coexistence. The ecological implications of both the analytical and numerical findings are discussed at length towards the end.

Received: 3 June 2015, Revised: 2 August 2015, Accepted: 5 September 2015.

Keywords: Food chain, Inter and Intra-specific Competition, Global Stability, Hopf-bifurcation, Lyapunov Function.

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1. Introduction

In 1934, Gause competitive exclusion principle states that two predator species can not coexist for long time on a single prey species which was supported by the experiments on Paramecium cultures [8]. But in 1969, Ayala's experimental results on two species of Drosophila upon a single prey showed that competitive coexistence is possible in nature [4]. Until now, various competition models have been proposed

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to explain Ayala's experimental result [1, 3, 12, 18, 22]. Many natural factors such as inter specific competition [23], heterogeneity [6], stoichiometric principles [16], intra specific competition of competitors [13, 19], prey refuge [13, 20] etc are considered in various competitive models so as to get strong coexistence.

Intra-specific competition among a species mainly reduces the predation rates of that species. They compete with each other for their common resources such as food, shelter, environment etc by aggressive displays, posturing, fighting, infanticide and cannibalism [15].

Gakkhar et.al [7] proposed a competitive model consisting of two competitive predators sharing one prey as follows:

$$\frac{dX}{dT} = rX\left(1 - \frac{X}{K}\right) - \frac{M_1XY_1}{A_1 + X} - \frac{M_2XY_2}{A_2 + X}, \quad X(0) > 0, \tag{1a}$$

$$\frac{dY_1}{dT} = \frac{E_1M_1XY_1}{A_1 + X} - D_1Y_1 - \gamma_1Y_1Y_2, \quad Y_1(0) > 0, \tag{1b}$$

$$\frac{dY_2}{dT} = \frac{E_2M_2XY_2}{A_2 + X} - D_2Y_2 - \gamma_2Y_1Y_2, \quad Y_2(0) > 0, \tag{1c}$$

where X is the population density of prey, Y_1 and Y_2 are the population densities of respective predators; r is biotic potential and K is the environmental carrying capacity of prey species; M_1, M_2 are predation co-efficients; E_1, E_2 are conversion factors; A_1, A_2 are half-saturation constants; D_1 and D_2 are the natural death rate of predators and γ_1, γ_2 are co-efficients of inter-specific competition between two predator species. Here, they showed that the persistence is not possible for two competing predators sharing a single prey species for the system (1). Sarwardi et.al [21] studied the above model (1) after a modification by introducing constant proportion of prey refuge and support the result of Gause competitive exclusion principle. Recently, Haque et.al [11] studied a simple food chain model with intra specific competition among predator and top-predator species. This study showed that intra specific competition among predators could be beneficial for predator's survival. The influence of intra specific competition in the dynamical behaviour of simple food chain model motivates us to take a further attempt to establish the global coexistence of the competitive food chain model system (1) by taking into account the intra specific competition among predator populations.

2. Mathematical Model Formulation

In this article, we modify the model (1) to get strong coexistence of the system incorporating intra specific competition among two competitive predator species. We thus obtain the following system

$$\frac{dX}{dT} = rX\left(1 - \frac{X}{K}\right) - \frac{M_1XY_1}{A_1 + X} - \frac{M_2XY_2}{A_2 + X}, \quad X(0) > 0, \tag{2a}$$

$$\frac{dY_1}{dT} = \frac{E_1M_1XY_1}{A_1 + X} - D_1Y_1 - \gamma_1Y_1Y_2 - H_1Y_1^2, \quad Y_1(0) > 0, \tag{2b}$$

$$\frac{dY_2}{dT} = \frac{E_2M_2XY_2}{A_2 + X} - D_2Y_2 - \gamma_2Y_1Y_2 - H_2Y_2^2, \quad Y_2(0) > 0, \tag{2c}$$

where r, K, M_1, M_2, E_1, E_2, A_1, A_2, D_1, D_2, γ_1, γ_2 are defind earlier; H_1 and H_2 are the intra-specific competition among predator species.

Using the transformation $x = \frac{X}{K}$, $y_1 = \frac{Y_1}{KE_1}$, $y_2 = \frac{Y_2}{KE_1E_2}$, $t = rT$, the system takes the following form

$$\frac{dx}{dt} = x(1-x) - \frac{a_1xy_1}{1+b_1x} - \frac{a_2xy_2}{1+b_2x}, \quad x(0) > 0, \tag{3a}$$

$$\frac{dy_1}{dt} = \frac{a_1xy_1}{1+b_1x} - d_1y_1 - c_1y_1y_2 - h_1y_1^2, \quad y_1(0) > 0, \tag{3b}$$

$$\frac{dy_2}{dt} = \frac{a_2xy_2}{1+b_2x} - d_2y_2 - c_2y_1y_2 - h_2y_2^2, \quad y_2(0) > 0, \tag{3c}$$

where

$$a_1 = \frac{KM_1E_1}{rA_1}, \quad d_1 = \frac{D_1}{r}, \quad c_1 = \frac{\sigma_1KE_1E_2}{r}, \quad b_1 = \frac{K}{A_1}, \quad h_1 = \frac{H_1K_1E_1}{r},$$
$$a_2 = \frac{KM_1E_1E_2}{rA_2}, \quad d_2 = \frac{D_2}{r}, \quad c_2 = \frac{\gamma_2KE_1}{r}, \quad b_2 = \frac{K}{A_2}, \quad h_2 = \frac{H_2K_1E_1E_2}{r}.$$

The present article is organized as follows. In Section 2 we state the formulation of the model under consideration and its assumptions. Section 3 contains some preliminary results. Then in Section 4 the model with intra-specific competition is analyzed, identifying its equilibria, giving conditions for their feasibility, stability and bifurcation. Numerical simulation has been carried out in Section 5. Finally, the article concludes with a discussion of the results obtained.

3. Preliminary Results

3.1. Existence and positive invariance

For $t > 0$, letting $X \equiv (x, y_1, y_2)^T$, $F : R^3 \to R^3$, $F = (f_1, f_2, f_3)^T$, system (3) can be rewritten as $\frac{dX}{dt} = F(X)$. Here $f_i \in C^\infty(R)$ for $i = 1, 2, 3$; where $f_1 = x(1-x) - \frac{a_1xy_1}{1+b_1x} - \frac{a_2xy_2}{1+b_2x}$, $f_2 = \frac{a_1xy_1}{1+b_1x} - d_1y_1 - c_1y_1y_2 - h_1y_1^2$, $f_3 = \frac{a_2xy_2}{1+b_2x} - d_2y_2 - c_2y_1y_2 - h_2y_2^2$. Since the vector function F is a smooth function of the variables (x, y_1, y_2) in the positive octant $\Omega = \{(x, y_1, y_2) : x > 0, y_1 > 0, y_2 > 0\}$, the local existence and uniqueness of the solution hold.

3.2. Boundedness

The solutions of system (3) which initiate in R^3_+ are uniformly bounded.

Proof: Define a positive definite function

$$\Omega(t) = x(t) + y_1(t) + y_2(t). \tag{4}$$

From definition, $\Omega(t)$ is differentiable in some maximal interval $(0, t_b)$. For an arbitrary $\eta > 0$, the time derivative of (4) along the solution of the system (3) is

$$\frac{d\Omega}{dt} + \eta\Omega = x(\eta + 1 - x) + y_1(\eta - d_1 - h_1y_1) + y_2(\eta - d_2 - h_2y_2) - (c_1 + c_2)y_1y_2$$
$$\leqslant \frac{(\eta+1)^2}{4} + \frac{(\eta-d_1)^2}{4h_1} + \frac{(\eta-d_2)^2}{4h_2}.$$

Hence we can find $\mu > 0$ such that

$$\frac{d\Omega}{dt} + \eta\Omega \leqslant \mu \quad \forall \quad t \in (0, t_b).$$

Applying a theory of differential equation [5], we get

$$0 < \Omega(x, y_1, y_2) < \frac{\mu}{\eta}(1 - e^{-\eta t)} + \Omega(x(0), y(0), z(0))e^{-\eta t} \quad \forall \quad t \in (0, t_b)$$

and for $t_b \to \infty$, $0 < \Omega(x, y_1, y_2) < \frac{\mu}{\eta}$. Hence all the solutions of system (3) that initiate at $(x(0), y_1(0), y_2(0))$ lie in R^3_+ and are confined in the compact region

$$\Gamma = \{(x, y_1, y_2) \in R^3_+; x(t) + y_1(t) + y_2(t) = \frac{\mu}{\eta} + \varepsilon, \quad \forall \quad \varepsilon > 0\}. \tag{5}$$

3.3. Dissipativeness
If $a_1 > d_1(1 + b_1)$ and $a_2 > d_2(1 + b_2)$, then the system (3) is dissipative.
Proof : We obtain from the equation (3a) of the system that

$$\limsup_{t\to+\infty} x(t) \leqslant 1.$$

From the equation (3b), we have

$$\limsup_{t\to+\infty} y_1(t) \leqslant \frac{1}{h_1}\left[\frac{a_1}{1 + b_1} - d_1\right] = \widetilde{y}_1,$$

where $\widetilde{y}_1$ denotes an upper bound of $y_1(t)$ and is positive if $a_1 > d_1(1 + b_1)$.
Again, from the equation (3c), we obtain

$$\limsup_{t\to+\infty} y_2(t) \leqslant \frac{1}{h_2}\left[\frac{a_2}{1 + b_2} - d_2\right] = \widetilde{y}_2,$$

where $\widetilde{y}_2$ denotes an upper bound of $y_2(t)$ and is positive if $a_2 > d_2(1+b_2)$. Hence, the claim.

3.4. Equilibria and their feasibility
System (3) has the following six equilibria $E_i(x_i, y_{1i}, y_{2i})$, $i = 0, 1, .., 5$. E_0 is the origin, $E_1 \equiv (1, 0, 0)$. For E_2, we have $x_2 = 0$, $y_{12} = \frac{c_1d_2-d_1h_2}{h_1h_2-c_1c_2}$, $y_{22} = \frac{c_2d_1-d_2h_1}{h_1h_2-c_1c_2}$. E_2 will be feasible if $c_1d_2 > d_1h_2$, $c_2d_1 > d_2h_1$, $h_1h_2 > c_1c_2$. But multiplication of first two inequalities gives $h_1h_2 < c_1c_2$ which contradicts the third inequality. Hence E_2 is infeasible. For E_3, we have

$$y_{23} = 0, \quad y_{13} = \frac{(1 - x_3)(1 + b_1x_3)}{a_1},$$

and x_3 is a positive root of the cubic equation

$$h_1b_1^2x^3 + h_1b_1(2 - b_1)x^2 + (a_1^2 - d_1b_1a_1 - 2h_1b_1 + h_1)x - (a_1d_1 + h_1) = 0.$$

For E_4, we have

$$y_{14} = 0, \quad y_{24} = \frac{(1 - x_4)(1 + b_2x_4)}{a_2},$$

and x_4 is a positive root of the cubic equation

$$h_2b_2^2x^3 + h_2b_2(2 - b_2)x^2 + (a_2^2 - d_2b_2a_2 - 2h_2b_2 + h_2)x - (a_2d_2 + h_2) = 0.$$

For the coexistence equilibrium E_5, the population levels are $y_{25} = -\frac{B_1}{B_2}$, $y_{15} = -\frac{B_3}{B_4}$, where

$$\begin{aligned}
B_1 &= c_2a_1x_5 - c_2d_1 - c_2d_1b_1x_5 + h_1b_1x_5d_2 + h_1b_1{x_5}^2d_2b_2 + c_2b_2{x_5}^2a_1 - c_2b_2x_5d_1 \\
&\quad - c_2b_2{x_5}^2d_1b_1 + h_1d_2b_2x_5 - h_1a_2x_5 + h_1d_2 - h_1b_1{x_5}^2a_2, \\
B_2 &= -c_2c_1 - c_2c_1b_1x_5 - c_2b_2x_5c_1 - c_2b_2{x_5}^2c_1b_1 + h_1h_2 + h_1h_2b_2x_5 + h_1b_1x_5h_2 \\
&\quad + h_1b_1{x_5}^2h_2b_2, \\
B_3 &= h_2b_2{x_5}^2d_1b_1 - h_2b_2{x_5}^2a_1 - d_2b_2{x_5}^2c_1b_1 + a_2{x_5}^2c_1b_1 - d_2b_2x_5c_1 \\
&\quad + h_2b_2x_5d_1 - h_2a_1x_5 + h_2d_1b_1x_5 + a_2x_5c_1 - d_2c_1b_1x_5 - d_2c_1 + h_2d_1, \\
B_4 &= (1 + b_2x_5)\left(-c_2c_1 - c_2c_1b_1x_5 + h_1h_2 + h_1b_1x_5h_2\right),
\end{aligned}$$

and x_5 is a root of the equation of

$$\sum_{i=0}^{5} Q_i x^i = 0, \tag{6}$$

where $Q_5 = -c_2{b_2}^2c_1{b_1}^2 + h_2{b_2}^2h_1{b_1}^2$,

$Q_4 = c_2{b_2}^2c_1{b_1}^2 - 2\,c_2{b_2}^2c_1b_1 + 2\,h_2h_1b_2{b_1}^2 - 2\,c_2c_1b_2{b_1}^2 - h_2{b_2}^2h_1{b_1}^2 + 2\,h_2{b_2}^2h_1b_1$,

$Q_3 = {a_2}^2h_1{b_1}^2 + 2\,c_2{b_2}^2c_1b_1 + 2\,c_2c_1b_2{b_1}^2 - c_2c_1{b_1}^2 + h_2{b_2}^2h_1 + h_2h_1{b_1}^2 - c_2b_2a_1a_2b_1 - 2\,h_2h_1b_2{b_1}^2 + 4\,h_2b_2h_1b_1 - 4\,c_2b_2c_1b_1 + h_2{b_2}^2{a_1}^2 - d_2b_2h_1{b_1}^2a_2 + d_2{b_2}^2b_1a_1c_1 - 2\,h_2{b_2}^2h_1b_1 - h_2{b_2}^2a_1d_1b_1 - c_2{b_2}^2c_1 - a_2b_1a_1b_2c_1 + c_2b_2d_1{b_1}^2a_2$,

$Q_2 = -c_2b_2a_1a_2 - h_2h_1{b_1}^2 + 2\,{a_2}^2h_1b_1 + c_2d_1{b_1}^2a_2 + 2\,h_2h_1b_1 + 4\,c_2b_2c_1b_1 - a_2b_1a_1c_1 + 2\,d_2b_1a_1b_2c_1 + 2\,c_2b_2d_1a_2b_1 - a_2a_1b_2c_1 - d_2h_1{b_1}^2a_2 + d_2{b_2}^2a_1c_1 + 2\,h_2b_2h_1 - 2\,c_2c_1b_1 - 2\,c_2b_2c_1 - 2\,h_2a_1b_2d_1b_1 - h_2{b_2}^2h_1 - h_2{b_2}^2a_1d_1 - c_2a_1a_2b_1 - 2\,d_2b_2h_1a_2b_1 + 2\,h_2b_2{a_1}^2 - 4\,h_2b_2h_1b_1 + c_2c_1{b_1}^2 + c_2{b_2}^2c_1$,

$Q_1 = -2\,h_2a_1b_2d_1 + 2\,d_2a_1b_2c_1 + 2\,c_2d_1a_2b_1 - 2\,h_2h_1b_1 - d_2b_2h_1a_2 + c_2b_2d_1a_2 - c_2c_1 - 2\,d_2h_1a_2b_1 - 2\,h_2b_2h_1 + 2\,c_2c_1b_1 + h_2{a_1}^2 + d_2b_1a_1c_1 + 2\,c_2b_2c_1 - h_2d_1b_1a_1 - a_2a_1c_1 + h_2h_1 + {a_2}^2h_1 - c_2a_1a_2$,

$Q_0 = d_2a_1c_1 + c_2d_1a_2 + c_2c_1 - d_2h_1a_2 - h_2d_1a_1 - h_2h_1$.

A fifth degree equation has five roots in the complex domain. We now find sufficient conditions for it to have at least a positive root. Since the degree of the equation is odd, by Descartes' rule of sign, we get a real root and correspondingly a linear factor of the polynomial. The fifth degree polynomial can then be factorized as

$$\begin{aligned}
\sum_{i=0}^{5} Q_i x^i &= Q_5(x + p)(x^4 + A_1x^3 + A_2x^2 + A_3x + A_4) \\
&= Q_5[x^5 + (p + A_1)x^4 + (pA_1 + A_2)x^3 + (pA_2 + A_3)x^2 \\
&\quad + (pA_3 + A_4)x + pA_4]
\end{aligned} \tag{7}$$

where p is to be determined. By equating coefficients of like powers of x on the left and the right, we find $A_1 + p = \frac{Q_4}{Q_5}$, $A_2 + pA_1 = \frac{Q_3}{Q_5}$, $A_3 + pA_2 = \frac{Q_2}{Q_5}$, $A_4 + pA_3 = \frac{Q_1}{Q_5}$, $pA_4 = \frac{Q_0}{Q_5}$, from which we have $p = \frac{Q_0}{Q_5A_4}$. One root of equation (6) is thus found as $x_5 = -p$. By imposing conditions $p < 0$, we obtain $x_5 > 0$. This ensures that the feasible coexistence equilibrium is unique. As for example, for the set of parameter values $a_1 = 14.0$, $a_2 = 9.1$, $b_1 = 13.2$, $b_2 = 7.5$, $d_1 = 0.9$, $d_2 = 0.9$, $c_1 = 0.001$, $c_2 = 0.001$, $h_1 = 0.1$, $h_2 = 0.2$, E_5 becomes $(0.4521503585, 0.08218809244, 0.1846819947)$.

4. System Behaviour Around Boundary Equilibria

The Jacobian matrix of the system at any arbitrary point is given by

$$J(x, y, z) = \begin{bmatrix} G_{11} & -\frac{a_1 x}{1+b_1 x} & -\frac{a_2 x}{1+b_2 x} \\ \frac{a_1 y_1}{(1+b_1 x)^2} & G_{22} & -c_1 y_1 \\ \frac{a_2 y_2}{(1+b_2 x)^2} & -c_2 y_2 & G_{33} \end{bmatrix},$$

where $G_{11} = 1 - 2x - \frac{a_1 y_1}{(1+b_1 x)^2} - \frac{a_2 y_2}{(1+b_2 x)^2}$,
$G_{22} = \frac{a_1 x}{1+b_1 x} - d_1 - c_1 y_2 - 2h_1 y_1$,
$G_{33} = \frac{a_2 x}{1+b_2 x} - d_2 - c_2 y_1 - 2h_2 y_2$.

4.1. System behaviour near the origin

Theorem 1. E_0 is unstable.

Proof: The eigenvalues of the jacobian matrix J_0 at E_0 are 1, $-d_1$, $-d_2$. Hence, the proof.

4.2. System behaviour near the equilibrium $E_1(1, 0, 0)$

Theorem 2. E_1 is locally asymptotically stable if $a_1 < d_1(1+b_1)$ and $a_2 < d_2(1+b_2)$.

Proof: The eigenvalues of the jacobian matrix J_1 at E_1 are -1, $\frac{a_1}{1+b_1} - d_1$, $\frac{a_2}{1+b_2} - d_2$. Hence E_1 is locally asymptotically stable if the conditions $a_1 < d_1(1 + b_1)$ and $a_2 < d_2(1 + b_2)$ are satisfied.

4.3. System behaviour near the equilibrium $E_3(x_3, y_{13}, 0)$

Theorem 3.

(i) E_3 is locally asymptotically stable if $a_1 b_1 y_{13} < (1 + b_1 x_3)^2$ and $a_2 x_3 < (d_2 + c_2 y_{13})(1 + b_2 x_3)$.

(ii) The system experiences Hopf-bifurcation around E_3 for $a_1 = a_1^{[1HB]}$, where $a_1^{[1HB]} = \frac{(x_3+h_1 y_{13})(1+b_1 x_3)^2}{b_1 x_3 y_{13}}$.

Proof: (i) The jacobian matrix J_3 evaluated at E_3 is given by $J_3 = (c_{ij})_{3\times 3}$, where $c_{11} = -x_3 + \frac{a_1 b_1 x_3 y_{13}}{(1+b_1 x_3)^2}$, $c_{12} = -\frac{a_1 x_3}{1+b_1 x_3} < 0$, $c_{13} = -\frac{a_2 x_3}{1+b_2 x_3} < 0$, $c_{21} = \frac{a_1 y_{13}}{(1+b_1 x_3)^2} > 0$, $c_{22} = -h_1 y_{13} < 0$, $c_{23} = -c_1 y_{13} < 0$, $c_{31} = 0$, $c_{32} = 0$, $c_{33} = \frac{a_2 x_3}{1+b_2 x_3} - d_2 - c_2 y_{13}$. Its eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left[c_{11} + c_{22} \pm \sqrt{(c_{11} + c_{22})^2 - 4(c_{11}c_{22} - c_{12}c_{21})}\right], \quad \lambda_3 = c_{33}. \tag{8}$$

If we assume $c_{11} < 0$ and $c_{33} < 0$ then $\lambda_3 < 0$ and $\lambda_{1,2}$ both are either negative or complex numbers with negative real parts. Hence, E_3 is locally asymptotically stable if $c_{11} < 0$ and $c_{33} < 0$, that is, $a_1 b_1 y_{13} < (1 + b_1 x_3)^2$ and $a_2 x_3 < (d_2 + c_2 y_{13})(1 + b_2 x_3)$.

(ii) From (8), we see that λ_3 is real, λ_1 and λ_2 will be purely imaginary if and only if there is a $a_1 = a_1^{[1HB]}$ such that $a_1^{[1HB]} = \frac{(x_3+h_1 y_{13})(1+b_1 x_3)^2}{b_1 x_3 y_{13}}$. But for $i = 1, 2$,

$$\text{Re}\left(\frac{d\lambda_i}{da_1}\right)\Big|_{[a_1 = a_1^{[1HB]}]} = \frac{b_1 x_3 y_{13}}{(1 + b_1 x_3)} \neq 0.$$

Therefore, the system enters into Hopf-bifurcation at E_3 for $a_1 = a_1^{[1HB]}$.

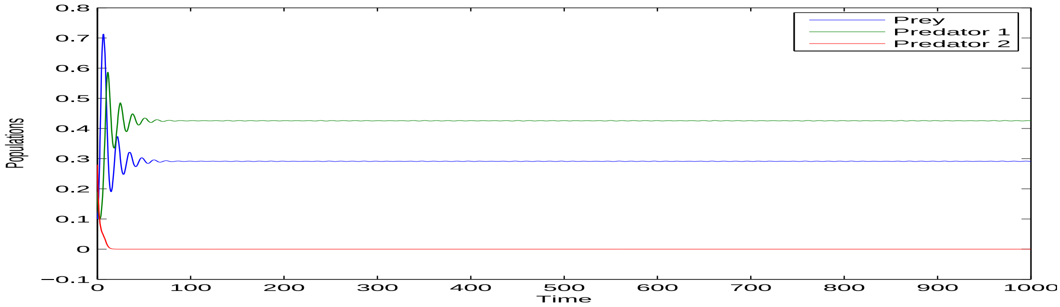


Figure 1. Stability around the equilibrium E_3 for $a_1 = 2.5 < a_1^{[1HB]} = 2.880393$. The other parameter values are $a_2 = 0.9$, $b_1 = 1.72$, $b_2 = 0.6$, $d_1 = 0.4$, $d_2 = 0.5$, $c_1 = 1.0$, $c_2 = 0.4$, $h_1 = 0.2$, $h_2 = 0.1$. Here $a_1b_1y_{13} = 1.829735481 < (1+b_1x_3)^2 = 2.252825522$ and $a_2x_3 = 0.2621205760 < (d_2 + c_2y_{13})(1 + b_2x_3) = 0.7873248148$.

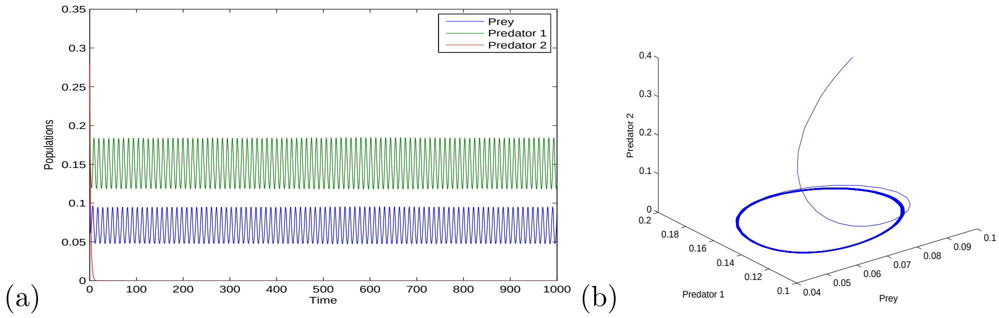


Figure 2. (a) 2D view of Hopf-bifurcation behaviour of the system (3) around the equilibrium E_3 for $a_1 = 7.0 > a_1^{[1HB]} = 2.880393$. (b) 3D phase portrait. The other parameter values are same as in Figure 1.

4.4. System behaviour near the equilibrium $E_4(x_4, 0, y_{24})$

Theorem 4.

(i) E_4 is locally asymptotically stable if $a_2b_2y_{24} < (1 + b_2x_4)^2$ and $a_1x_4 < (d_1 + c_1y_{24})(1 + b_1x_4)$.

(ii) The system experiences Hopf-bifurcation around E_4 for $a_2 = a_2^{[2HB]}$, where $a_2^{[2HB]} = \frac{(x_4+h_2y_{24})(1+b_2x_4)^2}{b_2x_4y_{24}}$.

Proof: The proofs are similar with the proofs of Theorem 3.

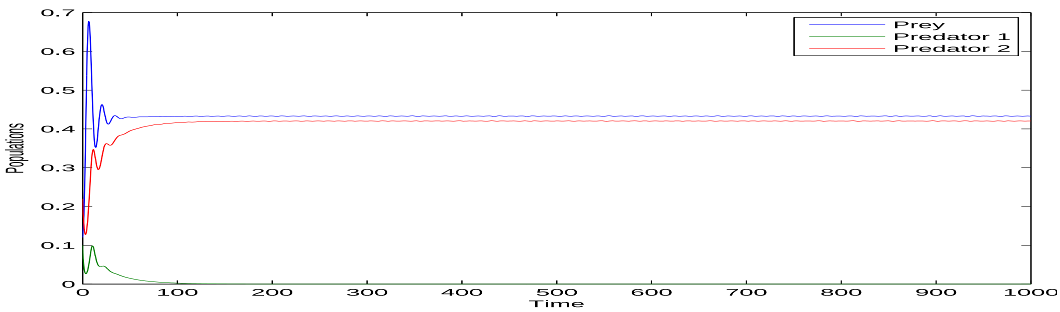


Figure 3. Stability around the equilibrium E_4 for $a_2 = 2.4 < a_2^{[2HB]} = 2.88037$. The other parameter values are $a_1 = 4.0$, $b_1 = 1.5$, $b_2 = 1.8$, $d_1 = 1.0$, $d_2 = 0.5$, $c_1 = 0.2$, $c_2 = 0.02$, $h_1 = 0.3$, $h_2 = 0.2$. Here $a_2b_2y_{24} = 1.815956550 < (1 + b_2x_4)^2 = 3.166729035$ and $a_1x_4 = 1.732290146 < (d_1 + c_1y_{24})(1 + b_1x_4) = 1.788294819$.

(a) (b)

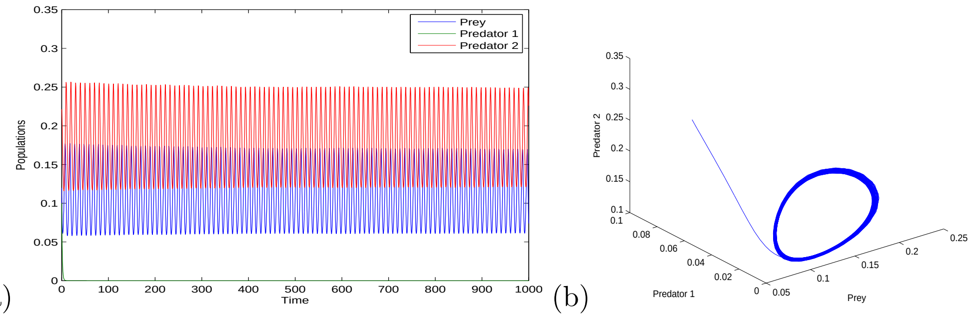


Figure 4. (a) 2D view of Hopf-bifurcation behaviour of the system (3) around the equilibrium E_4 for $a_2 = 6.0 > a_2^{[2HB]} = 2.88037$. (b) 3D phase portrait. The other parameter values are same as in Figure 3.

5. System Behaviour Near the Coexistence Equilibrium

The Jacobian matrix J_5 evaluated at E_5 has the components

$$J_{11} = -x_5 + \frac{a_1 b_1 x_5 y_{15}}{(1+b_1x_5)^2}, \quad J_{12} = -\frac{a_1 x_5}{(1+b_1x_5)} < 0, \quad J_{13} = -\frac{a_2 x_5}{1+b_2x_5} < 0, \tag{9}$$
$$J_{21} = -\frac{a_1 x_5}{(1+b_1x_5)^2} > 0, \quad J_{22} = -h_1 y_{15} < 0, \quad J_{23} = -c_1 y_{15} < 0,$$
$$J_{31} = \frac{a_2 y_{25}}{(1+b_2x_5)^2} > 0, \quad J_{32} = -c_2 y_{25} < 0, \quad J_{33} = -h_2 y_{25} < 0.$$

Theorem 5.

(i) The equilibrium point E_5 is locally asymptotically stable if

$$J_{11} < 0, [J_{21}J_{33} - J_{31}J_{23}] < 0, [J_{23}J_{32} - J_{22}J_{33}] < 0, [J_{31}J_{22} - J_{21}J_{32}] < 0. \tag{10}$$

(ii) The local stability ensures its global stability around E_5 if the conditions $a_1y_{15}(1+b_2x_5) + a_2y_{25}(1+b_1x_5) < (1+b_1x_5)(1+b_2x_5)$ and $\{c_1(1+b_1x_5) + c_2(1+b_2x_5)\}^2 < 4h_1h_2(1+b_1x_5)(1+b_2x_5)$ hold.

(iii) The system enters into a Hopf-bifurcation at E_5 for $\lambda = \lambda_i$, for a suitable value $a_1 = a_1^{[3HB]}$ if (10) holds.

Proof:
(i) The characteristic equation of J_5 is $\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$ where
$A_1 = -J_{11} - J_{22} - J_{33}$,
$A_2 = J_{22}J_{33} + J_{11}J_{22} + J_{11}J_{33} - J_{12}J_{21} - J_{23}J_{32} - J_{13}J_{31}$,
$A_3 = J_{12}[J_{21}J_{33} - J_{31}J_{23}] + J_{11}[J_{23}J_{32} - J_{22}J_{33}] + J_{13}[J_{31}J_{22} - J_{21}J_{32}]$.
From the Routh-Hurwitz criterion, the equilibrium point is locally asymptotically stable if $A_1 > 0$, $A_3 > 0$ and $A_1A_2 > A_3$. Now

$$\begin{aligned} A_1A_2 - A_3 = & -(J_{11})^2J_{22} - (J_{11})^2J_{33} - J_{11}(J_{22})^2 - J_{11}(J_{33})^2 - 2J_{11}J_{22}J_{33} \\ & + J_{11}J_{12}J_{21} + J_{11}J_{13}J_{31} + J_{22}J_{12}J_{21} + J_{22}[J_{23}J_{32} - J_{22}J_{33}] \\ & + J_{13}J_{31}J_{33} + J_{33}[J_{23}J_{32} - J_{22}J_{33}] + J_{12}J_{31}J_{23} + J_{13}J_{21}J_{32}. \end{aligned}$$

If we take $J_{11} < 0$, $[J_{21}J_{33} - J_{31}J_{23}] < 0$, $[J_{23}J_{32} - J_{22}J_{33}] < 0$, $[J_{31}J_{22} - J_{21}J_{32}] < 0$ in view of the signs of the Jacobian entries (9) all the Routh-Hurwitz conditions hold. But this requirement amounts to the assumptions (10). Hence, the claim.

(ii) Let $R^3_* = \{(x, y_1, y_2) \in R^3_+, x > 0, y_1 > 0, y_2 > 0\}$ and consider the scalar function $L : R^3_* \to R$ defined by

$$L = k_1\left[x - x_5 - x_5 ln\frac{x}{x_5}\right] + k_2\left[y_1 - y_{15} - y_{15} ln\frac{y_1}{y_{15}}\right] + k_3\left[y_2 - y_{25} - y_{25} ln\frac{y_2}{y_{25}}\right] \quad (11)$$

where k_1, k_2 and k_3 are positive constants to be determined later. The derivative of the above equation (11) along the solution of the system (3) is given by

$$\begin{aligned}\frac{dL}{dt} &= k_1\left[1 - \frac{x_5}{x}\right]\frac{dx}{dt} + k_2\left[1 - \frac{y_{15}}{y_1}\right]\frac{dy_1}{dt} + k_3\left[1 - \frac{y_{25}}{y_2}\right]\frac{dy_2}{dt},\\ &= k_1(x - x_5)\left[1 - x - \frac{a_1 y_1}{1 + b_1 x} - \frac{a_2 y_2}{1 + b_2 x}\right] + k_2(y_1 - y_{15})\left[\frac{a_1 x}{1 + b_1 x} - d_1\right.\\ &\quad \left. - c_1 y_2 - h_1 y_1\right] + k_3(y_2 - y_{25})\left[\frac{a_2 x}{1 + b_2 x} - d_2 - c_2 y_1 - h_2 y_2\right].\end{aligned}$$

At the equilibrium point E_5 of the system (3), we have

$$\begin{aligned}1 = x_5 + \frac{a_1 y_{15}}{1 + b_1 x_5} - \frac{a_2 y_{25}}{1 + b_2 x_5}, \quad d_1 &= \frac{a_1 x_5}{1 + b_1 x_5} - c_1 y_{25} - h_1 y_{15},\\ d_2 &= \frac{a_2 x_5}{1 + b_2 x_5} - c_2 y_{15} - h_2 y_{25}. \qquad (12)\end{aligned}$$

Using (12), the time derivative of L becomes

$$\begin{aligned}\frac{dL}{dt} &= k_1(x - x_5)\left[-(x - x_5) - \frac{a_1 y_1}{1 + b_1 x} + \frac{a_1 y_{15}}{1 + b_1 x_5} - \frac{a_1 y_2}{1 + b_2 x} + \frac{a_1 y_{25}}{1 + b_2 x_5}\right]\\ &\quad + k_2(y_1 - y_{15})\left[\frac{a_1 x}{1 + b_1 x} - \frac{a_1 x_5}{1 + b_1 x_5} - c_1(y_2 - y_{25}) - h_1(y_1 - y_{15})\right]\\ &\quad + k_3(y_2 - y_{25})\left[\frac{a_2 x}{1 + b_2 x} - \frac{a_2 x_5}{1 + b_2 x_5} - c_2(y_1 - y_{15}) - h_2(y_2 - y_{25})\right],\\ &= k_1(x - x_5)\left[-(x - x_5) - \frac{a_1(y_1 - y_{15})}{1 + b_1 x} + \frac{a_1 y_{15}(x - x_5)}{(1 + b_1 x)(1 + b_1 x_5)} - \frac{a_2(y_2 - y_{25})}{1 + b_2 x}\right.\\ &\quad \left. + \frac{a_2 y_{25}(x - x_5)}{(1 + b_2 x)(1 + b_2 x_5)}\right] + k_2(y_1 - y_{15})\left[\frac{a_1(x - x_5)}{(1 + b_1 x)(1 + b_1 x_5)} - c_1(y_2 - y_{25})\right.\\ &\quad \left. - h_1(y_1 - y_{15})\right] + k_3(y_2 - y_{25})\left[\frac{a_2(x - x_5)}{(1 + b_2 x)(1 + b_2 x_5)} - c_2(y_1 - y_{15}) - h_2(y_2 - y_{25})\right],\\ &= -\left[k_1\left(1 - \frac{a_1 y_{15}}{(1 + b_1 x)(1 + b_1 x_5)} - \frac{a_2 y_{25}}{(1 + b_2 x)(1 + b_2 x_5)}\right)\right](x - x_5)^2\\ &\quad - \left[a_1 k_1 - \frac{a_1 k_2}{1 + b_1 x_5}\right]\frac{(x - x_5)(y_1 - y_{15})}{1 + b_1 x} - \left[a_2 k_1 - \frac{a_2 k_3}{1 + b_2 x_5}\right]\frac{(x - x_5)(y_2 - y_{25})}{1 + b_2 x}\\ &\quad - k_2 h_1(y_1 - y_{15})^2 - (c_1 k_2 + c_2 k_3)(y_1 - y_{15})(y_2 - y_{25}) - k_3 h_2(z - z_3)^2.\end{aligned}$$

Assuming, $k_1 = 1$, $k_2 = (1 + b_1x_5)$ and $k_3 = (1 + b_2x_5)$, we have

$$\begin{aligned}
\frac{dL}{dt} &\leqslant -\left[\left(1 - \frac{a_1y_{15}}{(1+b_1x)(1+b_1x_5)} - \frac{a_2y_{25}}{(1+b_2x)(1+b_2x_5)}\right)\right](x-x_5)^2 \\
&\quad -k_2h_1(y_1-y_{15})^2 - (c_1k_2+c_2k_3)(y_1-y_{15})(y_2-y_{25}) - k_3h_2(z-z_3)^2, \\
&\leqslant -\left[\left(1 - \frac{a_1y_{15}}{(1+b_1x_5)} - \frac{a_2y_{25}}{(1+b_2x_5)}\right)\right](x-x_5)^2 \\
&\quad -k_2h_1(y_1-y_{15})^2 - (c_1k_2+c_2k_3)(y_1-y_{15})(y_2-y_{25}) - k_3h_2(z-z_3)^2, \\
&\leqslant 0, \qquad (13) \\
&\quad \text{if} \quad a_1y_{15}(1+b_2x_5) + a_2y_{25}(1+b_1x_5) < (1+b_1x_5)(1+b_2x_5) \\
&\quad \text{and} \quad (c_1k_2+c_2k_3)^2 < 4k_2k_3h_1h_2 \\
&\quad \text{i,e} \quad \{c_1(1+b_1x_5)+c_2(1+b_2x_5)\}^2 < 4h_1h_2(1+b_1x_5)(1+b_2x_5),
\end{aligned}$$

and $\frac{dL}{dt} = 0$ when $(x, y_1, y_2) = (x_5, y_{15}, y_{25})$. The proof follows from(13) and Lyapunov-Lasale invariance principle [9].

(iii) The Routh-Hurwitz conditions are satisfied, as seen above, if we assume $J_{11} < 0$. To have a Hopf bifurcation, we need however $A_1A_2 = A_3$ for some value of a_1, say $a_1 = a_1^{[3HB]}$. Since $A_2 > 0$ at $a_1 = a_1^{[3HB]}$, for some $a_1 > \epsilon > 0$ there is an interval $(a_1^{[3HB]} - \epsilon, a_1^{[3HB]} + \epsilon)$ in which $A_2 > 0$. Thus in this interval the characteristic equation cannot have real positive roots.
Now, for $a_1 = a_1^{[3HB]}$, the characteristic equation factorizes $(\lambda^2+A_2)(\lambda+A_1) = 0$ to give the three roots $\lambda_1 = i\sqrt{A_2}$, $\lambda_2 = -i\sqrt{A_2}$, $\lambda_3 = -A_1$. These roots are functions of $a_1 \in (a_1^{[3HB]} - \epsilon, a_1^{[3HB]} + \epsilon)$ and can therefore be written as $\lambda_1 = \alpha(a_1) + i\beta(a_1)$, $\lambda_2 = \alpha(a_1) - i\beta(a_1)$, $\lambda_3 = -A_1(a_1)$.

Now we verify the transversality condition

$$Re\left(\frac{d\lambda_i}{da_1}\right)|_{a_1=a_1^{[3HB]}} \neq 0, \quad i = 1, 2.$$

Substituting $\lambda_j = \alpha(a_1) + i\beta(a_1)$, $j = 1, 2$, into the characteristic equation and differentiating w.r.t a_1, we have

$$\omega(a_1)\alpha'(a_1) - \phi(a_1)\beta'(a_1) + \eta(a_1) = 0, \quad \phi(a_1)\alpha'(a_1) + \omega(a_1)\beta'(a_1) + \mu(a_1) = 0,$$

where

$$\omega(a_1) = 3\alpha^2(a_1) + 2A_1(a_1)\alpha(a_1) + A_2(a_1) - 3\beta^2(a_1), \qquad (14)$$

$$\phi(a_1) = 6\alpha(a_1)\beta(a_1) + A_1(a_1)\beta(a_1), \qquad (15)$$

$$\eta(a_1) = \alpha^2(a_1)A_1'(a_1) + A_2'(a_1)\alpha(a_1) + A_3'(a_1) - A_1'(a_1)\beta^2(a_1), \qquad (16)$$

$$\mu(a_1) = 2\alpha(a_1)\beta(a_1)A_1'(a_1) + A_2'(a_1)\beta(a_1). \qquad (17)$$

Since $\phi(a_1)\mu(a_1) + \omega(a_1)\eta(a_1) \neq 0$, we have

$$Re\left(\frac{d\lambda_j}{da_1}\right)|_{a_1=a_1^{[3HB]}} = -\frac{\phi\mu + \omega\eta}{\phi^2 + \omega^2} \neq 0, \quad j = 1, 2, \quad \lambda_3(a_1) = -A_1(a_1) \neq 0.$$

Hence, the claim.

Equilibria	Stability condition	Equilibrium nature
E_0	No condition	Unstable
E_1	$a_1 < d_1(1+b_1), a_2 < d_2(1+b_2)$	LAS
E_2	No condition	LAS
E_3	$a_1 b_1 y_{13} < (1+b_1 x_3)^2$, $a_2 x_3 < (d_2 + c_2 y_{13})(1+b_2 x_3)$	LAS
E_3	$a_1 = \frac{(x_3+h_1 y_{13})(1+b_1 x_3)^2}{b_1 x_3 y_{13}}$	Hopf-bifurcation
E_4	$a_1 x_4 < (d_1 + c_1 y_{24})(1+b_1 x_4)$	LAS
E_5	$a_1 b_1 y_{15} < (1+b_1 x_5)^2$	LAS
E_5	Conditions stated in 5.(ii)	GAS
E_5	$a_1 = \frac{(1+b_1 x_5)^2}{b_1 y_{15}}$	Hopf-bifurcation

Table 1. Schematic representation of our analytical results: LAS ≡ locally asymptotically stable, GAS ≡ Globally asymptotically stable.

6. Numerical Simulation

The numerical simulation based on the theoretical findings of the system (3) is illustrated for the purpose of clear understanding of the complex dynamical behaviour of the system. Numerical study of this model is performed by MATLAB 2010a and MAPLE 16. The findings are summarized and represented schematically in Table-1. All these results are verified by means of numerical illustrations of which some chosen ones are exhibited in the figures. The equilibrium E_3 has been shown to be asymptotically stable for a set of parameter values $a_1 = 2.5$, $a_2 = 0.9$, $b_1 = 1.72$, $b_2 = 0.6$, $d_1 = 0.4$, $d_2 = 0.5$, $c_1 = 1.0$, $c_2 = 0.4$, $h_1 = 0.2$, $h_2 = 0.1$ satisfying the conditions of the subsection 4.3.(i) which has been exhibited in Figure 1. When the parameter a_1 exceeds its critical value $a_1^{[1HB]} = 2.880393$ for this set of parameter values which is mentioned in the subsection 4.3.(ii), the present system does experience Hopf bifurcation around E_3 which has been exhibited in Figure 2. Similarly, the stable behaviour of the system (3) at equilibrium E_4 for a set of parameter values $a_2 = 2.4$, $a_1 = 4.0$, $b_1 = 1.5$, $b_2 = 1.8$, $d_1 = 1.0$, $d_2 = 0.5$, $c_1 = 0.2$, $c_2 = 0.02$, $h_1 = 0.3$, $h_2 = 0.2$ satisfying the conditions mentioned in the subsection 4.4.(i) has been displayed in Figure 3. When the parameter a_2 exceeds its critical value $a_2^{[2HB]} = 2.88037$ for this set of parameter values which is mentioned in the subsection 4.4.(ii), the present system does experience Hopf bifurcation around E_4 which has been displayed in Figure 4.

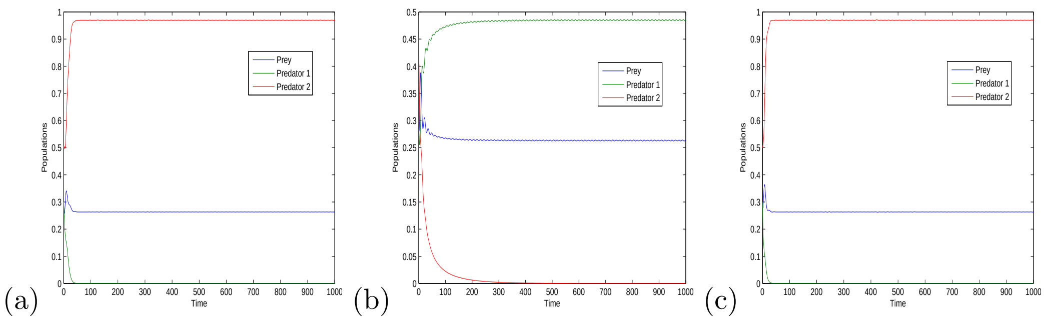
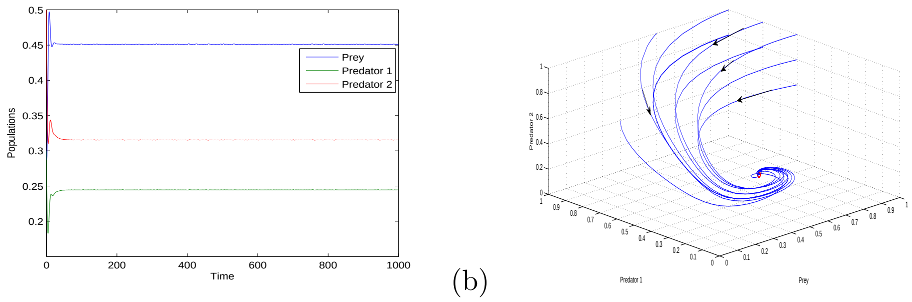


Figure 5. (a) When $h_1 = 0$, $h_2 = 0$, the predator 2 wins the competition and predator 1 tends towards extinction; (b) When $h_1 = 0$, $h_2 = 0.4$, the predator 1 wins the competition and predator 2 tends towards extinction; (c) When $h_1 = 0.5$, $h_2 = 0$, the predator 2 wins the competition and predator 1 tends towards extinction. The other parameter values are $a_1 = 2.0$, $a_2 = 0.8$, $b_1 = 1.2$, $b_2 = 0.2$, $d_1 = 0.4$, $d_2 = 0.2$, $c_1 = 0.2$, $c_2 = 0.02$.

The coexistence interior equilibrium point E_5 has also been found through numerical simulations whose global asymptotically stable behaviour has been depicted in Figure 6. The set of parameter values that has been taken for the results of Figure 6 is indicated in the caption of the figure in which all the three populations are settled at non-zero levels, i.e., towards E_5. Moreover, the chosen set of parameter values satisfies the condition of global stability as mentioned in section 5.1.(ii) and hence the global coexistence of the system (3) around E_5 is ensured which has been depicted in Figure 6.



(a) (b)

Figure 6. (a) When $h_1 = 0.5$, $h_2 = 0.4$, both the competitive predators of the system (3) coexist globally around the equilibrium $E_5(0.4511575297, 0.2446196414, 0.3154052167)$; (b) 3D phase portrait. The other parameter values are same as in Figure 5.

7. Conclusions and Comments

Ecological research mainly involves to find out the whole mechanism through population dynamics operate [14, 17]. It is well known that intra-specific competition terms can greatly affect the outcome of food chain models [2, 10, 11]. For the case of competitive food chain consisting of two competitive predators competing for a common prey resource, our results indicate that intra-specific competition of one competitor not only ensures the long term survival of itself, but also guarantees the existence of the other competitor, which would otherwise be out competed.

Uniform persistence is not possible for the system (3) in absence of intra-specific competitions in predators populations [7]. But introduction of intra-specific competition terms make the system coexistent not only in the sense of uniform persistence, but also in the sense of existence of a globally stable positive equilibrium (Figure 6). For the set of parameter values $a_1 = 2.0$, $a_2 = 0.8$, $b_1 = 1.2$, $b_2 = 0.2$, $d_1 = 0.4$, $d_2 = 0.2$, $c_1 = 0.2$, $c_2 = 0.02$, $h_1 = 0.5$, $h_2 = 0.4$, all the conditions in theorem 5 for global stability i.e $J_{11} = -0.3396751461 < 0$, $[J_{21}J_{33} - J_{31}J_{23}] = -0.01512224496 < 0$, $[J_{23}J_{32} - J_{22}J_{33}] = -0.01559332662 < 0$, $[J_{31}J_{22} - J_{21}J_{32}] = -0.02466570586 < 0$, $a_1 y_{15}(1 + b_2 x_5) + a_2 y_{25}(1 + b_1 x_5) = 0.9223137945 < (1 + b_1 x_5)(1 + b_2 x_5) = 1.680470890$ and $\{c_1(1 + b_1 x_5) + c_2(1 + b_2 x_5)\}^2 = 0.1089544154 < 4h_1 h_2(1 + b_1 x_5)(1 + b_2 x_5) = 1.344376712$ are satisfied. Hence the coexistence equilibrium $E_5(0.4511575297, 0.2446196414, 0.3154052167)$ is globally stable. A possible explanation for this situation is that intra-specific competition in one competitive predator prevents it from reducing the density of the prey below the minimum value needed for the other predator to be able to maintain itself.

Another interesting observed situation for the set of parameter values $a_1 = 14.0$, $a_2 = 9.1$, $b_1 = 13.2$, $b_2 = 7.5$, $d_1 = 0.9$, $d_2 = 0.9$, $c_1 = 0.001$, $c_2 = 0.001$, $h_1 = 0.1$, $h_2 = 0.2$ is that the system (3) is locally asymptotically stable around E_5 but when a_1 is increased, E_5 loses its stability and a Hopf-bifurcation occurs when a_1 passes a critical value (Figure 7).

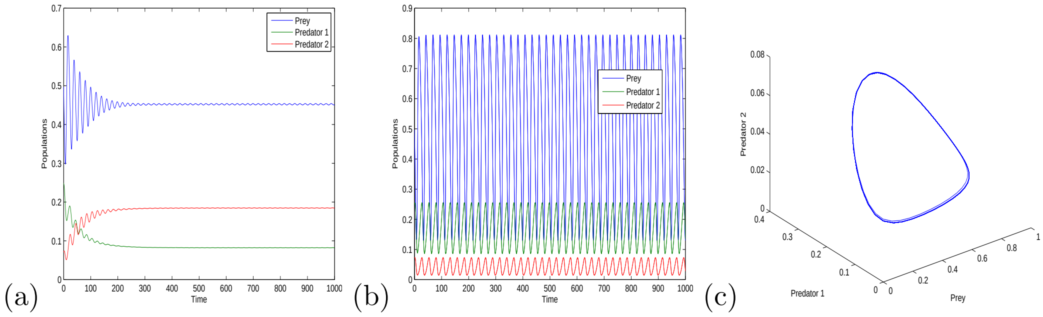


Figure 7. (a) Stable behaviour of the system (3) around the equilibrium $E_5(0.4521503585, 0.08218809244, 0.1846819947)$ for $a_1 = 14.0 < a_1^{[3HB]} = 14.47465$, (b) 2D view of Hopf-bifurcation behaviour of the system (3) around the equilibrium E_5 for for $a_1 = 14.5 > a_1^{[3HB]} = 14.47465$, (b) 3D phase portrait. The other parameter values are $a_2 = 9.1$, $b_1 = 13.2$, $b_2 = 7.5$, $d_1 = 0.9$, $d_2 = 0.9$, $c_1 = 0.001$, $c_2 = 0.001$, $h_1 = 0.1$, $h_2 = 0.2$.

So, the major significant findings of our analysis are as follows:
At a significant level of competition,
(i) Intra specific competition prevents predator extinction from the system and damp predator prey oscillation.
(ii) Strong coexistence of all species is possible due to intra specific competition under appropriate conditions on the environmental parameters.
(iii) These results can be used to make biological control mechanism.
(iv) These results will be helpful in theoretical research of ecology.

Acknowledgment

The authors Dr. Nijamuddin Ali and Prof. Santabrata Chakravarty gratefully acknowledge the financial support in part by Minor Research Project of University Grants Commission, New Delhi, India vide Ref. No. F. No. PSW-021/14-15 (ERO), ID No. WBI-042 dated 03.02.2015 and Special Assistance Programme (SAP-II) sponsored by the University Grants Commission (UGC), New Delhi, India. They are thankful to the unanimous reviewers for making fruitful comments and suggestions in order to improve the quality of the work done.

References

[1] P. A. Abrams and R. D. Holt, *The impact of consumer–resource cycles on the coexistence of competing consumers*, Theor. Pop. Biol., **62** (2002) 281-295.
[2] N. Ali and S. Chakravarty, *Global behaviour of a food chain model consisting of two different predator species*, Int. J. Math. Comp., **26** (2015) 1-19.
[3] R. A. Armstrong and R. McGehee, *Competitive exclusion*, Am. Nat., (1980) 151-170.
[4] F. J. Ayala, *Experimental invalidation of the principle of competitive exclusion*, Nat. Pub. Gr., (1969).
[5] G. Birkhoff and G. C. Rota, *Ordinary differential equations*, Aufl.(Ginn & Co., New York 1962), (1989).
[6] G. J. Butler and P. Waltman, *Bifurcation from a limit cycle in a two predator-one prey ecosystem modeled on a chemostat*, J. Math. Biol., **12** (1981) 295-310.
[7] S. Gakkhar, B. Singh and R. K. Naji, *Dynamical behavior of two predators competing over a single prey*, Biosystems, **90** (2006) 808-817.
[8] G. F. Gause, *The struggle for existence*,Williams and Wilkins, Baltimore,(1934) Online: http://www. ggause. com/Contgau. htm, (1971).
[9] J. K. Hale, *Ordinary Differential Equations*, Wiley-Interscience, New York, (1969).
[10] M. Haque, *Ratio-dependent predator-prey models of interacting populations*, Bull. Math. Biol., **71** (2009) 430-452.
[11] M. Haque, N. Ali and S. Chakravarty, *Study of a tri-trophic prey-dependent food chain model of interacting populations*, Math. Biosci., **246** (2013) 55-71.

[12] S. B. Hsu, S. P. Hubbell and P. Waltman, *A contribution to the theory of competing predators*, Ecol. Monogr., **48** (1978) 337-349.
[13] T. K. Kar, A. Ghorai and S. Jana, *Dynamic consequences of prey refugia in a two-predator-one prey system*, J. Biol. Syst., **21** (2013).
[14] M. Kot, *Elements of mathematical ecology*, Cambridge University Press, (2001).
[15] Y. Kuang, W. F. Fagan and I. Loladze, *Biodiversity, habitat area, resource growth rate and interference competition*, Bull. Math. Biol., **65** (2003) 497-517.
[16] I. Loladze, Y. Kuang, J. J. Elser and W. F. Fagan, *Competition and stoichiometry: coexistence of two predators on one prey*, Theo. Pop. Biol., **65** (2004) 1-15.
[17] R. M. May, *Stability and complexity in model ecosystems*, Princeton University Press, (2001).
[18] S. Muratori and S. Rinaldi, *Remarks on competitive coexistence*, SIAM J. Appl. Math., **49** (1989) 1462-1472.
[19] S. Ruan, A. Ardito, P. Ricciardi and D. L. DeAngelis, *Coexistence in competition models with density-dependent mortality*, C. R. Biologies, **330** (2007) 845-854.
[20] S. Sarwardi, P. K. Mandal and S. Ray, *Analysis of a competitive prey–predator system with a prey refuge*, Biosystems, **110** (2012) 133-148.
[21] S. Sarwardi, P. K. Mandal and S. Ray, *Dynamical behaviour of a two-predator model with prey refuge*, J. Biol. Phys., **39** (2013) 701-722.
[22] H. L. Smith, *The interaction of steady state and Hopf bifurcations in a two-predator-one-prey competition model*, SIAM J. Appl. Math., **42** (1982) 27-43.
[23] R. R. Vance, *The stable coexistence of two competitors for one resource*, Am. Nat., (1985) 72-86.