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Hybrid Functions Approach and Piecewise Constant Function by Collocation Method for the Nonlinear Volterra-Fredholm Integral Equations

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Abstract. In this work, we will compare two approximation method based on hybrid Legendre and Block-Pulse functions and a computational method for solving nonlinear Fredholm-Volterra integral equations of the second kind which is based on replacement of the unknown function by truncated series of well known Block-Pulse functions (BPfs) expansion.

Keywords: Hybrid functions, Nonlinear Integral Equation, Block-Pulse Function.

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1. Introduction

Integral equation has been one of the principal tools in various areas of applied mathematics, physics and engineering. Integral equation is encountered in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, antenna synthesis problem, communication theory, mathematical economics, population genetics, radiation, the particle transport problems of astrophysics and reactor theory, fluid mechanics, etc. Many of these integral equations are nonlinear, see ([17]-[3]). Some computational methods for approximating the solution of linear and nonlinear integral equations are known. The classical method of successive approximation for Fredholm-Hammerstein integral equations was introduced in [16]. Brunner in

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[4] applied a collocation type method and Ordokhani in [13] applied rationalized Haar function to nonlinear Volterra-Fredholm-Hammerstein integral equations. A variation of the Nystrom method was presented in [13]. A collocation type method was developed in [19]. The asymptotic error expansion of a collocation type method for volterra-Hammerstein integral equations has been considered in [**?**]. The aim of this work is to present two numerical methods for approximating the solution of nonlinear Fredholm-Volterra integral equation of the form:

$$u(x) = f(x) + \lambda_1 \int_0^x k_1(x,s)\psi_1(s,u(s))\, ds + \lambda_2 \int_0^1 k_2(x,s)\psi_2(s,u(s))\, ds, \tag{1}$$

where the parameters λ_1, λ_2 and functions $f(x), \psi_1(s,u(s)), \psi_2(s,u(s))$ and $k_1(x,s), k_2(x,s)$ are known and in $L^2[0,1)$ and $u(x)$ is an unknown function. In this work we suppose $\psi_1(s,u(s)) = (u(s))^\alpha$ and $\psi_2(s,u(s)) = (u(s))^\beta$ where α, β are positive integers.

2. Hybrid Functions

We use the Hybrid Legendre and Block-Pulse functions as basis for reducing these NV-FIEs to a system of nonlinear algebraic equations. We present Hybrid Legendre and Block-Pulse useful properties such as operational matrix of integration, product matrix, integration of the cross product and coefficient matrix and use them for transform our NV-FIE. As showed in our examples our method in analogy to existed methods works better. This paper is organized as follows: In subsection 2.1 we introduce hybrid functions and its properties. In Subsection 2.2 we apply these set of Hybrid functions for approximating the solution of NV-FIEs. Convergence analysis is given in Subsection 2.3.

2.1 *Definition of hybrid functions of Block-Pulse and Legendre*

Consider the Legendre polynomials $L_m(x)$ on the interval $[1,1]$

$$L_0(x) = 1, L_1(x) = x, (m+1)L_{m+1}(x) = (2m+1)xL_m(x) - mL_{m-1}(x),$$

such that $m = 1, 2, 3, \cdots$
Set $\{L_m(x) : m = 0, 1, \cdots\}$ in Hilbert space $L^2[-1,1]$ is a complete orthogonal set. A set of Block-Pulse functions $b_i(x), i = 1, 2, \cdots, n$ and the orthogonal set of hybrid functions

$$h_{ij}(x), i = 1, 2, \cdots, n, \quad j = 0, 1, \cdots, m-1,$$

that produces by Legendre polynomials and Block-Pulse functions on $[0,1)$ are defined as follows respectively:

$$b_i(x) = \begin{cases} 1, & \frac{(i-1)}{n} \leqslant x < \frac{i}{n} \\ 0, & \text{otherwise} \end{cases}$$

$$h_{ij}(x) = \begin{cases} L_j(2nx - 2i + 1), & \frac{(i-1)}{n} \leqslant x < \frac{i}{n} \\ 0, & \text{otherwise} \end{cases}$$

Any function $u(x) \in L^2[0,1)$ can be expanded as $u(x) = \Sigma_{i=1}^{\infty}\Sigma_{j=0}^{\infty} c_{ij}h_{ij}(x)$, where the hybrid coefficients are given by

$$c_{ij} = \frac{(u(x), h_{ij}(x))}{(h_{ij}(x), h_{ij}(x))}\ , \quad i = 1, 2, ..., \infty\ , \quad j = 0, 1, ..., \infty,$$

so that $(.,.)$ denotes the inner product. If $u(x)$ is piecewise constant or may be approximated as piecewise constant, then the sum may be terminated after nm terms, that is $u(x) \cong \Sigma_{i=1}^{n}\Sigma_{j=0}^{m-1} c_{ij}h_{ij}(x) = C^T\mathbf{h}(x)$, where

$$C = [c_{10}, ..., c_{1,m-1}, c_{20}, ..., c_{2,m-1}, ..., c_{n0}, ..., c_{n,m-1}]^T, \tag{2}$$

$$\mathbf{h}(x) = [h_{10}(x), ..., h_{1,m-1}(x), h_{20}(x), ..., h_{2,m-1}(x), ..., h_{n,m-1}(x)]^T. \tag{3}$$

We can also approximate the function $k(x,s) \in L^2([0,1) \times [0,1))$ as follows:

$$k(x,s) \cong \mathbf{h}^T(x)k\mathbf{h}(s),$$

so that

$$K_{ij} = \frac{\Big(\mathbf{h}_{(i)}(x), \big(k(x,s), \mathbf{h}_{(j)}(s)\big)\Big)}{\Big(\mathbf{h}_{(i)}(x), \mathbf{h}_{(i)}(x)\Big)\Big(\mathbf{h}_{(j)}(s), \mathbf{h}_{(j)}(s)\Big)}\ , \quad i,j = 1, 2, ..., nm.$$

The integration of the vector $\mathbf{h}(x)$ defined in (3) is given by

$$\int_0^x \mathbf{h}(x')dx' \cong P\mathbf{h}(x), \tag{4}$$

where P is the $nm \times nm$ operational matrix for integration and is given in [7] in details.
The integration of the cross product of two hybrid function vectors $\mathbf{h}(x)$ can be obtained as

$$D = \int_0^1 h(x)h^T(x)dx = \begin{bmatrix} L & 0 & \cdots & 0 \\ 0 & L & \cdots\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L \end{bmatrix},$$

where matrix L is a $m \times m$ diagonal matrix that can be seen in [7].
It is always necessary to evaluate the product of $\mathbf{h}(x)$ and $\mathbf{h}^T(x)$ that be called the product matrix of hybrid functions. Let

$$\mathbf{H}(x) = \mathbf{h}(x)\mathbf{h}^T(x), \tag{5}$$

where $\mathbf{H}(x)$ is $nm \times nm$ matrix. By multiplying the matrix $\mathbf{H}(x)$ in vector C that defined in (2) we obtain

$$\mathbf{H}(x)C = \widetilde{C}\mathbf{h}(x), \tag{6}$$

where $\widetilde{C}$ is $nm \times nm$ matrix and called the coefficient matrix. Basic multiplication properties of arbitrary two hybrid function $h_{ij}(x)$ and $h_{kl}(x)$ are described in [6].

2.2 *Outline of the method for NV-FIEs via Hybrid functions*

Consider the nonlinear Volterra-Fredholm integral (1). We put

$$u(x) \cong U^T \mathbf{h}(x), \tag{7}$$

where U is an unknown nm-vector and $\mathbf{h}$(x) is given by (3). Likewise, $k_1(x,s)$, $k_2(x,s)$ and $f(x)$ are expanded into the hybrid functions as follows

$$k_1(x,s) \cong \mathbf{h}^T(x) K_1 \mathbf{h}(s), k_2(x,s) \cong \mathbf{h}^T(x) K_2 \mathbf{h}(s), \tag{8}$$

$$f(x) \cong F^T \mathbf{h}(x), \tag{9}$$

where K_1, K_2 are known nm-matrices and F is a known nm-vector.
After substituting the approximate (7), (8), (9) in (1) we get

$$U^T \mathbf{h}(x) \cong F^T \mathbf{h}(x) + \lambda_1 \mathbf{h}^T(x) K_1 \int_0^x \mathbf{h}(s) \psi_1(s, U^T \mathbf{h}(s)) ds$$

$$+ \lambda_2 \mathbf{h}^T(x) K_2 \int_0^x \mathbf{h}(s) \psi_2(s, U^T \mathbf{h}(s)) ds. \tag{10}$$

Functions $\psi_1(s, U^T \mathbf{h}(s)) = (U^T \mathbf{h}(s))^\alpha$ and $\psi_2(s, U^T \mathbf{h}(s)) = (U^T \mathbf{h}(s))^\beta$ are known which can be expanded into the hybrid functions as

$$(u(s))^\alpha \cong U_\alpha^T \mathbf{h}(s), (u(s))^\beta \cong U_\beta^T \mathbf{h}(s). \tag{11}$$

In the next subsection, we consider computing U_α and U_β in terms of U, which U_α, U_β are mn-vectors whose elements are nonlinear combination of the elements of the vector U. Substitute (11) in (10) produces

$$U^T \mathbf{h}(x) \cong F^T \mathbf{h}(x) + \lambda_1 \mathbf{h}^T(x) K_1 \int_0^x \mathbf{h}(s) \mathbf{h}^T(s) U_\alpha ds$$

$$+ \lambda_2 \mathbf{h}^T(x) K_2 \int_0^1 \mathbf{h}(s) \mathbf{h}^T(s) U_\beta ds. \tag{12}$$

Note that by use of (4) and (6) we have $\int_0^x \mathbf{h}(s) \mathbf{h}^T(s) U_\alpha ds = \int_0^x \widetilde{U_\alpha} \mathbf{h}(s) ds = \widetilde{U_\alpha} P \mathbf{h}(x)$, by this relation and D we get

$$U^T \mathbf{h}(x) \cong F^T \mathbf{h}(x) + \lambda_1 \mathbf{h}^T(x) K_1 \widetilde{U_\alpha} P \mathbf{h}(x) + \lambda_2 \mathbf{h}^T(x) (K_2 D U_\beta). \tag{13}$$

In order to find U we collocate (13) in nm nodal points of Newton-Cotes as

$$x_p = \frac{2p-1}{2nm}, \qquad p = 1, 2, ..., nm. \tag{14}$$

then we have following system of nonlinear equations

$$U^T\mathbf{h}(x_p) \cong F^T\mathbf{h}(x_p) + \lambda_1 \mathbf{h}^T(x_p)K_1\widetilde{U_\alpha}P\mathbf{h}(x_p)$$

$$+\lambda_2\mathbf{h}^T(x_p)(K_2DU_\beta)\ , \qquad p = 1, 2, ..., nm. \tag{15}$$

This nonlinear system of equations can be solved by Newton's method. We used the Mathematica software to solve this nonlinear system. After solving above nonlinear system we can achieve U, then we will have our unknown $u(x)$ as $U^T\mathbf{h}(x)$, that is the approximate solution of NV-FIE (1).

2.2.1 Evaluating U_α and U_β

For numerical implementation of the method explained in section 2.2, we need to evaluate U_α and U_β, so that the elements of each one are nonlinear combination of the elements of the vector U. From (6) and (7), We have

$$(u(x))^2 \cong (U^T\mathbf{h}(x))(U^T\mathbf{h}(x)) = U^T\mathbf{h}(x)\mathbf{h}^T(x)U = U^T\widetilde{U}\mathbf{h}(x) = U_2\mathbf{h}(x), \tag{16}$$

where the vector $U_2 = U^T\widetilde{U}$ is a mn-row vector, then for $(u(s))^3$ we get

$$(u(x))^3 \cong (U^T\mathbf{h}(x))(U_2\mathbf{h}(x)) = U^T\mathbf{h}(x)\mathbf{h}^T(x)U_2^T = U^T\widetilde{U_2^T}\mathbf{h}(x) = U_3\mathbf{h}(x), \tag{17}$$

Therefore with this method we can approximate $(u(s))^\alpha$ and $(u(s))^\beta$ for arbitrary α and β. Suppose that this method holds for $\alpha - 1$ where $(u(x))^{\alpha-1} = U_{\alpha-1}\mathbf{h}(x)$, we shall obtain it for αas follows

$$(u(x))^\alpha = u(x)(u(x))^{\alpha-1} \cong (U^T\mathbf{h}(x))(U_{\alpha-1}\mathbf{h}(x))$$

$$= U^T\mathbf{h}(x)\mathbf{h}^T(x)U_{\alpha-1}^T = U^T\widetilde{U_{\alpha-1}^T}\mathbf{h}(x) = U_\alpha\mathbf{h}(x), \tag{18}$$

we have similar relation for β. So, the components of U_α and U_β can be computed in terms of components of unknown vector U.

2.3 *Convergence analysis*

We assume the following conditions on k_1, k_2 and ψ_1, ψ_2 for (1).
1. $M_1 \equiv sup_{0\leqslant x,s\leqslant 1}|k_1(x,s)| < \infty, M_2 \equiv sup_{0\leqslant x,s\leqslant 1}|k_2(x,s)| < \infty$;
2. $\psi_1(s,x), \psi_2(s,x)$ are continuous in $s \in [0,1]$ and Lipschitz continuous in $x \in \mathbb{R}$, i.e., there exists constants $C_1, C_2 > 0$ for which

$$|\psi_1(s,x_1) - \psi_1(s,x_2)| \leqslant C_1|x_1 - x_2| \qquad forall \qquad x_1, x_2 \in \mathbb{R},$$

$$|\psi_2(s,x_1) - \psi_2(s,x_2)| \leqslant C_2|x_1 - x_2| \qquad forall \qquad x_1, x_2 \in \mathbb{R},$$

THEOREM 2.1 *The solution of Nonlinear Volterra-Fredholm Integral Equation by using hybrid functions converges if* $0 < \gamma < 1$.

Proof For NV-FIE by assumption $\int_0^x |k_1(x,t)|dt \leqslant \int_0^1 |k_1(x,t)|dt$ for $0 < x < 1$; We see that there exists a constant $\gamma = |\lambda_1|M_1C_1 + |\lambda_2|M_2C_2 > 0$ such that

$$\|u_{nm}(x) - u(x)\| = max_{x\in[0,1]}|u_{nm}(x) - u(x)|$$

$$\leqslant max_{x\in[0,1]}|\lambda_1| \int_0^x |k_1(x,s)||\psi_1(s,u_{nm}(s)) - \psi_1(s,u(s))|ds$$

$$+max_{x\in[0,1]}|\lambda_2| \int_0^1 |k_2(x,s)||\psi_2(s,u_{nm}(s)) - \psi_2(s,u(s))|ds$$

$$\leqslant (|\lambda_1|M_1C_1 + |\lambda_2|M_2C_2)max_{x\in[0,1]}|u_{nm}(x) - u(x)| \leqslant \gamma max_{x\in[0,1]}|u_{nm}(x) - u(x)|.$$

We get $(1-\gamma)\|u_{nm}(x) - u(x)\| \leqslant 0$ and choose $0 < \gamma < 1$, when $n \to \infty$, it implies $\|u_{nm}(x) - u(x)\| \to 0$. ■

3. Piecewise Constant Function by Collocation Method

3.1 *Review of Some Related Papers*

Some computational methods for approximating the solution of linear and nonlinear integral equations are known. The classical method of successive approximation for Fredholm-Hammerstein integral equations was introduced in [16]. Brunner in [4] applied a collocation type method and Ordokhani in [13] applied rationalized Haar function to nonlinear Volterra-Fredholm-Hammerstein integral equations. A variation of the Nystrom method was presented in [10]. A collocation type method was developed in [9]. The asymptotic error expansion of a collocation type method for volterra-Hammerstein integral equations has been considered in [6]. Yousefi in [19] applied Legendre wavelets to a special type of nonlinear Volterra-Fredholm integral equations of the form

$$u(x) = f(x) + \lambda_1 \int_0^x k_1(x,s)F(u(s))ds + \lambda_2 \int_0^1 k_2(x,s)G(u(s))ds, \qquad 0 \leqslant s,x \leqslant 1, \tag{19}$$

where $f(x)$, and $k_1(x,s), k_2(x,s)$ are assumed to be in $L^2(R)$ on the interval $0 \leqslant s,x \leqslant 1$. Yalcinbas in [16] used Taylor polynomials for solving Equation (1) with $F(u) = u^p$ and $G(u) = u^q$. Orthogonal functions and polynomials receive attention in dealing with various problems that one of those in integral equation. The main characteristic of using orthogonal basis is that it reduces these problems to solving a system of nonlinear algebraic equations. The aim of this work is to present a numerical method for approximating the solution of nonlinear Fredholm-Volterra integral equation of the form:

$$u(x) = f(x) + \lambda_1 \int_0^x k_1(x,s)(u(s))^m ds + \lambda_2 \int_0^1 k_2(x,s)(u(s))^n ds, \qquad 0 \leqslant s,x \leqslant 1, \tag{20}$$

where m and n are nonnegative integers and λ_1 and λ_2 are constants. For this purpose we define a n-set of BPfs as

$$b_i(x) = \begin{cases} 1, & \frac{(i-1)}{n} \leqslant x < \frac{i}{n} \\ 0, & \text{otherwise} \end{cases}$$

The functions $b_i(x)$ are disjoint and orthogonal. That is,

$$b_j(x)b_i(x) = \begin{cases} 0, & i \neq j \\ b_i(t), & i = j \end{cases}$$

$$< b_j(x)b_i >= \begin{cases} 0, & i \neq j \\ \frac{1}{n}, & i = j \end{cases}$$

A function $u(x)$ defined over the interval [0, 1) may be expanded as:

$$u(x) = \sum_{i=1}^{\infty} u_i b_i(x), \tag{21}$$

In practice, only n-term of (21) are considered, where n is a power of 2, that is

$$u(x) \cong u_n(x) = \sum_{i=1}^{n} u_i b_i(x), \tag{22}$$

with matrix from:

$$u(x) \cong u_n(x) = u^x b(x), \tag{23}$$

where $\mathbf{u} = [u_1, u_2, ..., u_n]^x$ and

$$\mathbf{b}(x) = [b_1(x), b_2(x), ..., b_n(x)]^x.$$

In a similar manner, $[u(x)]^m$ can be approximated in term of BPfs

$$[u(x)]^m \cong \widetilde{\mathbf{u}}^x \mathbf{b}(x)$$

that we need to calculate vector $\widetilde{u}$ whose elements are nonlinear combination of the elements of the vector $\mathbf{u}$ For this purpose, we can write $u(x) = \mathbf{u}^x \mathbf{b}(x)$ and $[u(x)]^m \cong \widetilde{\mathbf{u}}^x \mathbf{b}(x)$.
So,

$$\widetilde{\mathbf{u}}^x \mathbf{b}(x) = [\mathbf{u}^x \mathbf{b}(x)]^m \tag{24}$$

now using $b_j(x)b_i(x)$ leads to

$$\mathbf{b}(x)\mathbf{b}^x(x) = \begin{bmatrix} b_1(x) & 0 & ... & 0 \\ 0 & b_2(x) & ... & 0 \\ & & \ddots & \\ 0 & ... & 0 & b_n(x) \end{bmatrix}$$

also from $b_i(x)$ we get
$0 \leqslant x < \frac{1}{n}$ implies that $b_1(x) = 0$ and $b_i(x) = 0$ for i=2, ..., n.
$\frac{1}{n} \leqslant x < \frac{2}{n}$ implies that $b_2(x) = 1$ and $b_i(x) = 0$ for $i = 1, ..., n$ and $i \neq 2$.
$\frac{n-1}{n} \leqslant x < 1$ implies that $b_n(x) = 1$ and $b_i(x) = 0$ for i=1,..., n-1. Therefore, simply we obtain

$$\int_0^1 \mathbf{b}(x)\mathbf{b}^x(x)dx = \frac{1}{n}\mathbf{I}, \tag{25}$$

where, $\mathbf{I}$ is the identity matrix of order k. By incorporating these results we have

$$\tilde{u}^u = \tilde{u}^x I = n\int_0^1 \tilde{u}^x b(x)b^x(x)dx = n\int_0^1 [u^x b(x)]^m b^x(x)dx.$$

Hence,

$$\tilde{u}^x = n\int_0^1 [u^x b(x)]^m b^x(x)dx = n\sum_{i=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}} [u^x b(x)]^m b^x(x)dx,$$

$$= n\sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} [\mathbf{u}^x\mathbf{b}(x)]^{m-1}\mathbf{u}^x[\mathbf{b}(x)\mathbf{b}^x(x)]dx. \tag{26}$$

So using (26) leads to

$$\widetilde{\mathbf{u}}^x = n\int_0^{\frac{1}{n}} \left([u_1, u_2, ..., u_n] \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \right)^{m-1} [u_1, u_2, ..., u_n] \begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & 0 & \\ & & & \ddots \\ 0 & & & 0 \end{pmatrix} dx$$

$$+n\int_{\frac{1}{n}}^{\frac{2}{n}} \left([u_1, u_2, ..., u_n] \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)^{m-1} [u_1, u_2, ..., u_n] \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & 0 & \\ & & & \ddots \\ 0 & & & 0 \end{pmatrix} dx + ...$$

$$+n\int_{\frac{n-1}{n}}^{1}\left([u_1,u_2,...,u_n]\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}\right)^{m-1}[u_1,u_2,...,u_n]\begin{pmatrix}0&&&0\\&0&&\\&&0&\\&&&\ddots\\0&&&1\end{pmatrix}dx$$

$$= n\int_0^{\frac{1}{n}} u_1^{m-1}[u_1,0,...,0]dx + n\int_{\frac{1}{n}}^{\frac{2}{n}} u_2^{m-1}[0,u_2,...,0]dx + ...$$

$$= n\int_{\frac{n-1}{n}}^{1} u_m^{m-1}[0,...,0,u_m]dx = [u_1^m, u_2^m, ..., u_m^m].$$

Now for evaluating the integral $\int_0^x \mathbf{b}(x)\mathbf{b}^x(x)dx$ at the collocation points

$$x_j = \frac{j-\frac{1}{2}}{n}, \quad j = 1,2,...,n, \tag{27}$$

we may proceed as follows

$$\int_0^{x_j} \mathbf{b}(x)\mathbf{b}^x(x)dx = \int_0^{\frac{j-\frac{1}{2}}{n}} \mathbf{b}(x)\mathbf{b}^x(x)dx \int_0^{\frac{1}{n}} \mathbf{b}(x)\mathbf{b}^x(x)dx$$

$$+\int_{\frac{1}{n}}^{\frac{2}{n}} \mathbf{b}(x)\mathbf{b}^x(x)dx + ... + \int_{\frac{j-2}{n}}^{\frac{j-1}{n}} \mathbf{b}(x)\mathbf{b}^x(x)dx + \int_{\frac{j-1}{n}}^{\frac{j-\frac{1}{2}}{n}} \mathbf{b}(x)\mathbf{b}^x(x)dx$$

$$= \begin{pmatrix}\int_0^{\frac{1}{n}} 1dx & & 0\\ & 0 & \\ & & \ddots\\ 0 & & 0\end{pmatrix} + \begin{pmatrix}0 & & 0\\ & \int_{\frac{1}{n}}^{\frac{2}{n}} 1dx & \\ & 0 & \ddots\\ 0 & & 0\end{pmatrix} + ...$$

$$+\begin{pmatrix}0 & & 0\\ \ddots & & \\ & 0 & \\ & \int_{\frac{j-2}{n}}^{\frac{j-1}{n}} 1dx & \\ & 0 & \\ & & \ddots\\ 0 & & 0\end{pmatrix} + \begin{pmatrix}0 & & 0\\ \ddots & & \\ & 0 & \\ & 0 & \\ & \int_{\frac{j-1}{n}}^{\frac{j-\frac{1}{2}}{n}} 1dx & \\ & 0 & \\ & & \ddots\\ 0 & & 0\end{pmatrix} = \frac{1}{n}\mathbf{D}^j \tag{28}$$

where,

$$\mathbf{D}^j = Diag\left[1, 1, ..., \frac{1}{2}, 0, ..., 0\right]_{n\times n},$$

in fact, the diagonal matrix $\mathbf{D}^j, \quad j = 1, 2, ..., n$ is defined as follows :

$$D^j_{mn} = \begin{cases} 1\ , & m = n = 1, 2, \cdots, j-1 \\ \frac{1}{2}\ , & m = n = j \\ 0\ , & m = n = j+1, \cdots, n. \end{cases}$$

Also, $K(x, t) \in L^2[0, 1)^2$ may be approximated as:

$$K(x, t) \cong \sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij} b_i(x) b_j(t),$$

or in matrix form

$$K(x, t) \cong \mathbf{b}^t(x)\mathbf{k}\mathbf{b}(t), \tag{29}$$

where $\mathbf{k} = [K_{ij}]_{1\leqslant i,j\leqslant n}$ and $K_{ij} = n^2 \int_0^1 \int_0^1 K(x, t)B_i(x)B_j(t)dxdt$.

3.2 *Solution of the Nonlinear Fredholm-Volterra Integral Equations*

In order to use BPfs for solving nonlinear Fredholm-Voterra integral equations given in (20), we first approximate the $u(s)$, $f(s)$, $(u(x))^m$, $(u(x))^n$, $k_1(x, s)$ and $k_2(x, s)$ with respect to BPfs

$$u(s) \cong \mathbf{b}^s(s)\mathbf{u}, \tag{30}$$

$$f(s) \cong \mathbf{b}^s(s)\mathbf{f}, \tag{31}$$

$$(u(x))^m \cong \widetilde{\mathbf{u}}_1^s(x)\mathbf{b}(x), \tag{32}$$

$$(u(x))^n \cong \widetilde{\mathbf{u}}_2^s(x)\mathbf{b}(x), \tag{33}$$

$$k_1(x, s) \cong \mathbf{b}^s(s)\mathbf{k}_1\mathbf{b}(x), \tag{34}$$

$$k_2(x, s) \cong \mathbf{b}^s(s)\mathbf{k}_2\mathbf{b}(x), \tag{35}$$

where n-vectors $\mathbf{u}, \mathbf{f}, \widetilde{\mathbf{u}}_1, \widetilde{\mathbf{u}}_2$ and $n \times n$ matrices $\mathbf{k}_1$ and $\mathbf{k}_2$ are BPfs coefficients of $u(s)$, $f(s)$, $(u(x))^m$, $(u(x))^n$, $k_1(x, s)$ and $k_2(x, s)$ respectively. For solving (20), we substitute (30-35) into (20), therefore

$$\mathbf{b}^s(s)\mathbf{u} = \mathbf{b}^s(s)\mathbf{f} + \lambda_1\mathbf{b}^s(s)\mathbf{k}_1\int_0^s \mathbf{b}(x)\mathbf{b}^s(x)dx\widetilde{\mathbf{u}}_1 + \lambda_2\mathbf{b}^s(s)\mathbf{k}_2\int_0^1 \mathbf{b}(x)\mathbf{b}^s(x)dx\widetilde{\mathbf{u}}_2. \tag{36}$$

Table 1. Approximate and exact solution for $u(x)$

x	Hybrid n=8	BPFs n=8	Hybrid n=16	BPFs n=8	Exact
0.1	-1.9904	-1.9847	-1.9901	-1.9876	-1.99
0.2	-1.9605	-1.9505	-1.9601	-1.9532	-1.96
0.3	-1.9105	-1.8857	-1.9101	-1.8905	-1.91
0.4	-1.8406	-1.7905	-1.8401	-1.8122	-1.84
0.5	-1.7507	-1.7650	-1.7501	-1.7666	-1.75
0.6	-1.6408	-1.6650	-1.6402	-1.6589	-1.64
0.7	-1.5108	-1.5091	-1.5102	-1.5080	-1.51
0.8	-1.3607	-1.3205	-1.3601	-1.3342	-1.36
0.9	-1.1905	-1.1103	-1.1901	-1.1297	-1.19

We now collocate (36) at n points s_j , $j = 1, 2, ..., n$ defined by (27) as

$$\mathbf{b}^s(s_j)\mathbf{u} = \mathbf{b}^s(s_j)\mathbf{f} + \lambda_1 \mathbf{b}^s(s_j)\mathbf{k}_1 \int_0^{s_j} \mathbf{b}(x)\mathbf{b}^s(x)dx\widetilde{\mathbf{u}}_1 + \lambda_2 \mathbf{b}^s(s_j)\mathbf{k}_2 \int_0^1 \mathbf{b}(x)\mathbf{b}^s(x)dx\widetilde{\mathbf{u}}_2 \tag{37}$$

by using (25) and (28) and the fact that $\mathbf{b}(s_j) = \mathbf{e}_j$ where $\mathbf{e}_j$ is the j-th column of the identity matrix of order n, (37) may then be restated as

$$u_j = f_i + \frac{\lambda_1}{n}\mathbf{e}_j^t\mathbf{k}_1\mathbf{D}^j\widetilde{\mathbf{u}}_1 + \frac{\lambda_2}{n}\mathbf{e}_j^t\mathbf{k}_2\widetilde{\mathbf{u}}_2 \ , \qquad j = 1, 2, ..., n. \tag{38}$$

(38) gives n nonlinear equations which can be solved for the elements $\widetilde{\mathbf{u}}_1$ using Newtons iterative method.

3.3 *Error in BPfs Approximation*

THEOREM 3.1 *If a differentiable function $u(s)$ with bounded first derivative on $(0, 1)$ is represented in a series of BPfs over subinterval $[\frac{i-1}{n}, \frac{i}{n})$, we have $\|e(s) = O(\frac{1}{n})\|$, where $e(s) = u_n(s) - u(s)$.*

Proof See [15]. ∎

4. Numerical results

Consider the following nonlinear volterra-Fredholm integral equation

$$u(x) = -\frac{1}{30}x^6 + \frac{1}{3}x^4 - x^2 + \frac{5}{3}x - \frac{5}{4} + \int_0^x (x-s)(u(s))^2 ds + \int_0^1 (x+s)u(s)ds,$$

with the the exact solution $u(x) = x^2 - 2$ [13]. Now we can solve the equation with these methods and the results are displayed in the Table 1.

5. Conclusion

In this paper we have presented two methods for the numerical solution of Nonlinear Volterra-Fredholm Integral Equations based on hybrid legendre and Block-Pulse functions. These two methods are Piecewise Constant Function by Collocation Method and Hybrid Functions Approach. The results obtained by these two methods to solve an equation, we reach the conclusion that Hybrid Functions Approach method is more accurate.

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