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A New Modified Homotopy Perturbation Method for Solving Linear Second-Order Fredholm Integro-Differential Equations

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Abstract. In this paper, we tried to accelerate the rate of convergence in solving second-order Fredholm type Integro-differential equations using a new method which is based on Improved homotopy perturbation method (IHPM) and applying accelerating parameters. This method is very simple and the result is obtained very fast.

Keywords: Improved homotopy perturbation method, Laplace transform, Fredholm integro-differential equation.

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1. Introduction

The aim of this work is to decrease the size of operation and simplify the calculations. In fact, this paper is an enhanced version of [8] which applied simple accelerating parameters for solving second-order Fredholm type Integro-differential equations. In [8] the exact solution was obtained but with complex calculations, while in this work, accelerating parameters have been modified to decrease this complexity and reduce the convergence time. Both these works are based on Homotopy Perturbation Method (HPM) [4, 6] and Improved version of it [10, 12]. Consider the second-order Fredholm type Integro-differential equation,

$$y''(x) = my'(x) + ny(x) + \int_a^b k(x,t)y(t)\,\mathrm{d}(x) + f(x) \qquad a \leqslant x \leqslant b \tag{1}$$

with the following initial conditions

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$$y(0) = A, \qquad y'(0) = B \tag{2}$$

Let

$$L(u) = u''(x) - mu'(x) - nu(x) - \int_a^b k(x,t)u(t)\,\mathrm{d}(x) - f(x) = 0 \tag{3}$$

with the solution $u(x) = y(x)$. Define $H(u,p)$ by

$$H(u,p) = \ F(u), \qquad H(u,1) = L(u) \tag{4}$$

where $\mathrm{F}(u)$ is a functional operator with solution, say, u_0, which can be obtained easily. We may choose a convex homotopy

$$H(u,p) = (1-p)\,F(u) + pL(u) = 0 \tag{5}$$

and continuously trace an implicitly defined curve from a starting point $H(u_0,0)$ to a solution $H(y,1)$ [9, 11]. The embedding parameter p monotonically increase from 0 to 1 as the trivial problem $\mathrm{F}(u) = 0$ continuously deformed to the original problem $L(u) = 0$. The HPM uses the homotopy parameter $p\epsilon[0,1]$ as an expanding parameter [2], to obtain

$$u(x) = \sum_{i=0}^{\infty} p^i u_i(x) = u_0(x) + pu_1(x) + p^2 u_2(x) + \ \ldots \tag{6}$$

When $p \to 1$,(5) corresponds to (3) and becomes the approximate solution of (3), i.e.

$$y(x) = \lim_{p\to 1} u(x) = \sum_{i=0}^{\infty} u_i(x). \tag{7}$$

It is well known that the series (7) is convergent in most cases, and also the rate of convergence depend on $L(u)$. Taking $\mathrm{F}(u) = u''(x) - mu'(x) - nu(x) - f(x)$ and substituting (6) to (5) and equating the terms with identical power of p, we have

$$p^0:\ u''_0(x) - mu'_0(x) - nu_0(x) - f(x) = 0, \quad u_0(0) = A,\ u'_0(0) = B \tag{8}$$

$$p^n:\ u''_n(x) - mu'_n(x) - nu_n(x) - \int_a^b k(x,t)\,u_{n-1}(t)\,dt = 0,\ u_n(0) = 0, \tag{9}$$

$$n = 1, 2, \ldots$$

In the next section the new method is presented.

2. The new method

In this section we aim Laplace transforms [8] and combines it with IHPM to be able to solve second-order Fredholm type integro-differential equations with the following kernel $K(x,\ t) = \sum_{i=1}^{N} g_i(x)\, h_i(t)$.
At the first step, we consider $k(x,t) = g(x)\,h(t)$, so we define a new convex homotopy perturbation [1] as

$$H(u,p,m) = (1-p)\,F(u) + pL(u) + p\,(1-p)\,\mathrm{m}k^{*}s = 0, \tag{10}$$

Where

$$F(u) = u''(x) - mu'(x) - nu(x) - f(x)$$
$$Ł(u) = u''(x) - mu'(x) - nu(x) - \int_a^b k(x,t)\,u(t)\,dt - f(x) = 0$$

and

$$k^{*}s = \int_a^b k(x,t)\,u_0(t)dt,$$

hence we can write

$$(1-p)\big(u''(x) - mu'(x) - nu(x) - f(x)\big) + p\Big(u''(x) - mu'(x) - nu(x) - \int_a^b g(x)h(t)u(t)\,\mathrm{d}d - f(x)\Big) + mp(1-p)k^{*}s = 0,$$

or

$$u''(x) - mu'(x) - nu(x) - f(x) - pg(x)\int_a^b h(t)u(t)\,\mathrm{d}t + mpk^{*}s - mp^2k^{*}s = 0 \tag{11}$$

Substituting (6) in (11) and equating the terms with equal power of p, we obtain

$$p^0:\ u''_0(x) - mu'_0(x) - nu_0(x) = f(x), \qquad u_0(0) = A,\ u'_0(0) = B,$$

whose solution with Laplace Transformation is

$$u_0(x) = \mathcal{L}^{-1}\left\{\frac{F(p) + pA + B - mA}{p^2 - mp - n}\right\} \tag{12}$$

$$p^1: \; u''_1(x) - mu'_1(x) - nu_1(x) - \int_a^b k(x,t)\, u_0(t)\; \mathrm{d}t + mk^*s = 0, \quad u_1(0)=0,\; u'_1(0)=0,$$

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$$u''_1(x) - mu'_1(x) - nu_1(x) = (1-\mathrm{m})\, k^*s, \qquad u_1(0)=0,\; u'_1(0)=0,$$
$$k^*s = \int_a^b k(x,t)\, u_0(t)\; \mathrm{d}t,$$

$$u_1(x) = (1-m)\, \mathcal{L}^{-1}\left\{\frac{K^*(p)}{p^2 - mp - n}\right\} \tag{13}$$

$$p^2: u''_2(x) - mu'_2(x) - nu_2(x) - \int_a^b k(x,t)\, u_1(t)\; \mathrm{d}t - mk^*s = 0, \qquad u_2(0)=0,\; u'_2(0)=0$$
$$u''_2(x) - mu'_2(x) - nu_2(x) = (1-m)\int_a^b k(x,t)\, \mathcal{L}^{-1}\left\{\frac{K^*(p)}{p^2 - mp - n}\right\}(t)\; \mathrm{d}t + mk^*s$$

where

$$\alpha = \int_a^b h(t)\mathcal{L}^{-1}\left\{\frac{K^*(p)}{p^2 - mp - n}\right\}\, \mathrm{d}t,$$

from which we obtain

$$\begin{aligned} u''_2(x) - mu'_2(x) - nu_2(x) &= (1-m)\,\alpha g(x) + mg(x)\int_a^b h(t)\, u_0(t)\; \mathrm{d}t \\ &= [(1-m)\,\alpha + mk_1{}^*s]\, g(x) \end{aligned}$$

Which

$$k_1{}^*s = \int_a^b h(t) u_0(t)\; \mathrm{d}t \tag{14}$$

so we obtain

$$u_2(x) = [(1-m)\,\alpha + mk_1{}^*s]\, \mathcal{L}^{-1}\left\{\frac{G(p)}{p^2 - mp - n}\right\}, \tag{15}$$

$$\begin{cases} p^3: \quad u''_3(x) - mu'_3(x) - nu_3(x) - \int_a^b k(x,t)\, u_2(t)\; \mathrm{d}t = 0, \\ u_2(0)=0,\; u'_2(0)=0, \end{cases}$$

and in general

$$\begin{cases} u''_n(x) - mu'_n(x) - nu_n(x) - g(x)\int_a^b h(t)\, u_{n-1}(t)\, dt = 0, \\ u_n(0) = 0,\ u'_n(0) = 0, \quad n = 3, 4, \ldots \end{cases}$$

Now we find m such that $u_2(x) = 0$, since if $u_2(x) = 0$ then $u_3(x) = u_4(x) = u_5(x) = \ldots = 0$, and the exact solution will be obtained as $u(x) = u_0(x) + u_1(x)$, hence for all values of $x$we should have

$$[(1-m)\,\alpha + mk_1{}^* s] = 0,$$

Or

$$m = \frac{\alpha}{\alpha - k_1{}^* s} = \frac{\int_a^b h(t)\mathcal{L}^{-1}\left\{\frac{K^*(p)}{p^2-mp-n}\right\} \mathrm{d}t}{\int_a^b h(t)\mathcal{L}^{-1}\left\{\frac{K^*(p)}{p^2-mp-n}\right\} \mathrm{d}t - \int_a^b h(t) u_0(t)\, \mathrm{d}t} \tag{16}$$

Now, we consider the general case

$$(x,t) = \sum_{i=1}^{N} g_i(x)\, h_i(t)\,.$$

Here we choose the convex homotopy as follow:

$$H(u,p,m) = (1-p)f(u) + pL(u) + p(1-p)\sum_{i=1}^{N} m_i k^* s_i = 0 \tag{17}$$

$$u''_0(x) - mu'_0(x) - nu_0(x) = f(x)\,, \qquad u_0(0) = A,\ u'_0(0) = B,$$

and the solution is

$$u_0(x) = \mathcal{L}^{-1}\left\{\frac{F(p) + pA + B - mA}{p^2 - mp - n}\right\} \tag{18}$$

$$u''_1(x) - mu'_1(x) - nu_1(x) = \sum_{i=1}^{n} \int_a^b k_i(x,t)\, u_0(t)\, dt - m_i k^* s_i)$$

so we have

$$u_1(x) = \sum_{i=1}^{n} \left[(1-m_i)\mathcal{L}^{-1}\left\{\frac{k^*{}_i(P)}{p^2 - mp - n}\right\}\right]$$

$$
\begin{aligned}
u''_2(x) - mu'_2(x) - nu_2(x) &= \sum_{i=1}^{n} \left(\int_a^b k_i(x,t)\, u_1(t)\, dt + m_i k^* s_i \right) \\
&= \sum_{i=1}^{n} \left(\int_a^b k_i(x,t) \left(\sum_{j=1}^{n} \left[(1-m_j)\, \mathcal{L}^{-1} \left\{ \frac{k^*{}_j(P)(t)}{p^2 - mp - n} \right\} \right] \right) \mathrm{d}t \right. \\
&\quad \left. + m_i k^* s_i \right) \qquad (19)
\end{aligned}
$$

$\vdots$

$$
u''_n(x) - mu'_n(x) - nu_n(x) = \sum_{i=1}^{n} \left(\int_a^b k_i(x,t)\, u_{n-1}(t)\ \mathrm{d}t \right),
$$

We try to fine the parameter $\mathrm{m}_i, i = 1, 2, \ldots, N$,such that $u_2(x) = u_3(x) = \ldots = 0$, hence from (18) for every $x \epsilon [a,b]$, we should have

$$
\begin{aligned}
u_2(x) = g_1(x) & \left[(1-m_1)\,\alpha_1 + k^* s_1 m_1 \pm \sum_{i \neq 1}^{n} (1-m_i)\,\alpha_i \right] \\
\pm\, g_2(x) & \left[(1-m_2)\,\beta_2 + k^* s_2 m_2 \pm \sum_{i \neq 2}^{n} (1-m_i)\,\beta_i \right] \qquad (20) \\
\pm \ldots \pm g_n(x) & \left[(1-m_n)\,\gamma_n + k^* s_n m_n \pm \sum_{i=1}^{n-1} (1-m_i)\,\gamma_i \right]
\end{aligned}
$$

For having $u_2(x) = 0$, we should solve the system of equations

$$
\begin{cases}
(k^* s_1 \pm \alpha_1) m_1 - \sum\limits_{i \neq 1}^{n} m_i \alpha_i = \alpha_1 \pm \sum\limits_{i \neq 1}^{n} \alpha_i \\
(k^* s_2 \pm \beta_2) m_1 - \sum\limits_{i \neq 2}^{n} m_i \beta_i = \beta_2 \pm \sum\limits_{i \neq 2}^{n} \beta_i \\
\quad \cdot \\
\quad \cdot \\
\quad \cdot \\
(k^* s_1 \pm \gamma_n) m_1 - \sum\limits_{i=1}^{n-1} m_i \gamma_i = \gamma_n \pm \sum\limits_{i=1}^{n-1} \gamma_i
\end{cases}
$$

where

$$k^* s_i = \sum_{i=1}^{n} \int_a^b h_i(t) \, u_0(t) \, \mathrm{d}t, \tag{21}$$

$$\alpha_i = \int_a^b h_1(t) \left[\sum_{i=1}^{n} \left[\mathcal{L}^{-1} \left\{ \frac{k^*{}_i(P)}{p^2 - mp - n} \right\} \right] \right], \tag{22}$$

$$\beta_i = \int_a^b h_2(t) \left[\sum_{i=1}^{n} \left[\mathcal{L}^{-1} \left\{ \frac{k^*{}_i(P)}{p^2 - mp - n} \right\} \right] \right], \tag{23}$$

$$\gamma_i = \int_a^b h_n(t) \left[\sum_{i=1}^{n} \left[\mathcal{L}^{-1} \left\{ \frac{k^*{}_i(P)}{p^2 - mp - n} \right\} \right] \right]. \tag{24}$$

3. Numerical examples

Example 3.1

$$u''(x) = x - \sin x - \int_0^{\frac{\pi}{2}} xt u(t) \, \mathrm{d}t, \qquad u(0) = 0, \quad u'(0) = 1 \tag{25}$$

with the exact solution $u(x) = sinx$ [3].

$$p^0 : \; u''_0(x) = x - \sin x, \qquad u_0(0) = 0 \quad , u'_0(0) = 1,$$

Using Laplace Transport are obtained

$$u_0(x) = \sin x + \frac{x^3}{6},$$
$$p^1 : \; u''_1(x) = (-1 - m)) \, k^* s, \qquad u_1(0) = 0, u'_1(0) = 0,$$

Hence using (13) – (16) in the required order, we get

$$k^* s = (\frac{\pi^5}{960} + 1)x, \qquad m = \frac{-\frac{\pi^5}{960}}{1 + \frac{\pi^5}{960}} \tag{26}$$

so we have,

$$u_1(x) = (-1 - m) \, \mathcal{L}^{-1} \left\{ \frac{k^* s(p)}{p^2 - mp - n} \right\},$$

Hence we obtain,

$$u_1(x) = \frac{-x^3}{6},$$

and the solution will obtained as

$$u(x) = u_0(x) + u_1(x) = \sin x,$$

Example 3.2

$$u''(x) = x - 2 + 60\int_0^1 (x - \mathrm{t})\, u(t)\, \mathrm{d}t, \qquad u(0) = 0, u'(0) = 1, \tag{27}$$

with the exact solution $u(x) = x(x-1)^2$[8].
In this case we have

$$\begin{array}{cccc} f(x) = x - 2, & m = 0, & & n = 0, \\ a = 0, & b = 1, & & g_1(x) = 60x, \\ g_2(x) = -60, & h_1(t) = 1 & \text{and} & h_2(t) = t \end{array}$$

we have

$$p^0 : u''_0(x) = x - 2, \qquad u_0(0) = 0, \qquad u'_0(0) = 1$$

By applying the Laplace transform, we have

$$\begin{aligned} u_0(x) &= \frac{x^3}{6} - x^2 + x, \\ p^1 &: u''_1(x) = (1 - m_1)\, x k^* s_1 - (1 - m_2)\, k^* s_2 \quad u_1(0) = 0, u'_1(0) = 0 \end{aligned} \tag{28}$$

From $(21) - (24)$,we have

$$k^* s_1 = \frac{25}{2}, \quad k^* s_2 = 7, \quad \alpha_1 = \frac{125}{4}, \quad \alpha_2 = 70, \quad \beta_1 = \frac{105}{8}, \quad \beta_2 = 25,$$

From

$$\begin{cases} (\frac{25}{2} - \frac{125}{4}) m_1 + 70 m_2 = 70 - \frac{125}{4} \\ \left(7 + \frac{105}{8}\right) m_2 - 25 m_1 = \frac{105}{8} - 25 \end{cases} \tag{29}$$

So we obtain

$$m_1 = \frac{3}{5}, \; m_2 = \frac{5}{7} \tag{30}$$

now, white replacing (30) in (29) we can write

$$u''_1(x) = 5x - 2, \qquad u_1(0) = 0, \qquad u'_1(0) = 0,$$

So we have,

$$u_1(x) = \frac{5x^3}{6} - x^2,$$

and the solution will obtained as

$$u(x) = u_0(x) + u_1(x) = x\left(x^2 - 1\right),$$

which is the exact solution.

Example 3.3 Consider

$$u''(x) = -2u'(x) - 5u(x) + 3e^{-x}\sin x + \int_{-\pi}^{\pi} e^t u(t)\,\mathrm{d}t,\ u(0) = 0,\ u'(0) = 2, \tag{31}$$

with the exact solution $u(x) = \frac{1}{2}e^{-x}\sin(2x) + e^{-x}\sin\ x$[9].
As in the previous examples we obtain

$$p^0:\ u''_0(x) + 2u'_0(x) + 5u_0(x) = 3e^{-x}\sin x, \qquad u_0(0) = 0,\ u'_0(0) = 2,$$

Now, we apply the Laplace transform, so we have

$$u_0(x) = e^{-x}\mathcal{L}^{-1}\left\{\frac{1}{p^2+4}\right\} + e^{-x}\mathcal{L}^{-1}\left\{\frac{1}{p^2+1}\right\},$$

Hence, we obtain

$$u_0(x) = \frac{1}{2}e^{-x}\sin(2x) + e^{-x}sinx, \qquad u_0(0) = 0,\ u'_0(0) = 2 \tag{32}$$

$$p^1:\ u''_1(x) + 2u'_1(x) + 5u_1(x) = (1-\mathrm{m})\,k^*s,\quad u_1(0) = 0,\ u'_1(0) = 0 \tag{33}$$

where

$$k^*s = \int_{-\pi}^{\pi} e^t(\frac{1}{2}e^{-t}\sin(2t) + e^{-t}\sin t)\,\mathrm{d}t = 0,$$

With placing (34) in (33), we obtain

$$u_1(x) = 0$$

So

$$u(x) = u_0(x) + u_1(x) = 0, \tag{34}$$

It is clear that the exact result is obtained just by one iteration.

4. Conclusion

As it was seen in the previous sections, we successfully obtained the exact solution by modifying the accelerating parameters. Besides, the number of iteration is reduced compared to previous method. The new method is totally effective, stable and error free.

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