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On Coneigenvalues of a Complex Square Matrix

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Abstract. In this paper, we show that a matrix $A \in M_n(\mathbf{C})$ that has n coneigenvectors, where coneigenvalues associated with them are distinct, is condiagonalizable. And also show that if all coneigenvalues of conjugate-normal matrix A be real, then it is symmetric.

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1. Introduction

In section 2 of this paper, we recall some definitions and propositions of similarity transformation. In sections 3 and 4 we present definitions and propositions that are need for consimilarity transformation, and recall coneigenvalue and coneigenvector definitions of matrices, and prove that if $| \mu_1 |, | \mu_2 |, \dots, | \mu_k |$ are distinct coneigenvalues of $A \in M_n(\mathbf{C})$, then $\{x^{(1)}, x^{(2)}, \dots, x^{(k)}\}$ is a linearly independent set, where $| \mu_i |$ is a coneigenvalue associated with coneigenvector $x^{(i)}, \quad i = 1, 2, \dots, k$. this proposition imply that if a matrix $A \in M_n(\mathbf{C})$ has n distinct coneigenvalue associated with coneigenvectors, then A is condiagonalizable. Also show that if all coneigenvalues of conjugate-normal matrix A be real, then it is symmetric.

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2. Similarity and diagonalizable

Recall that a matrix $B \in M_n(\mathbf{C})$ is said to be similar to a matrix $A \in M_n(\mathbf{C})$ if there exists a nonsingular matrix $S \in M_n(\mathbf{C})$ such that $B = S^{-1}AS$. Also recall that, if the matrix $A \in M_n(\mathbf{C})$, is similar to a diagonal matrix, then A is said to be diagonalizable.

Since diagonal matrices are especially simple and have very nice properties, it is of interest to know for which $A \in M_n(\mathbf{C})$, there is a diagonal matrix in the similarity equivalence class of A, that is, which matrices are similar to diagonal matrices.

Theorem 2.1 *Let $A \in M_n(\mathbf{C})$, Then A is diagonalizable if and only if there is a set of n linearly independent vectors, each of which is an eigenvector of A.*

Lemma 2.2 *Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are eigenvalues of $A \in M_n(\mathbf{C})$, no two of which are the same, and suppose that $x^{(i)}$ is an eigenvector associated with $\lambda_i, i = 1, ..., k$. Then $\{x^{(1)}, x^{(2)}, \ldots, x^{(k)}\}$ is a linearly independent set.*

Theorem 2.3 *If $A \in M_n(\mathbf{C})$, has n distinct eigenvalues, then A is diagonalizable.*

For the proofs of these properties, we refer the reader to [2].

In the next section we are going to construct consimilarity from similarity (see [1], [3], [4] and [5]) .

3. Coneigenvalue and coneigenvector

The most important quantities related to similarity transformations of a matrix are its eigenvalues and eigenvectors. Now, that we deal with consimilarity transformations (The transformation $A \longrightarrow S^{-1}A\overline{S}$ is called a consimilarity transformation by the consimilarity nonsingular matrix S), we should instead speak of 'con'-analogues of these quantities. Recall that:

Definition 3.1 *matrices $A, B \in M_n(\mathbf{C})$ are said to be consimilar if $B = S^{-1}A\overline{S}$ for a nonsingular matrix S.*

In this section we recall coneigenvalue defination of a matrix, that for a matrix $A \in M_n(\mathbf{C})$ exist only n coneigenvalue. The coneigenvalues of A are preserved by any consimilarity transformation.

To give an exact definition, we introduce the matrices

$$A_L = \overline{A}A \quad and \quad A_R = A\overline{A} = \overline{A_L}.$$

Although the products AB and BA need not be similar in general, A_L is always similar to A_R (see [[2], p. 246, Problem 9 in Section 4.6]). Therefore in the subsequent discussion of their spectral properties, it will be sufficient to refer to one of them, say, A_L. The spectrum of A_L has two remarkable properties:

1. It is symmetric with respect to the real axis. Moreover, the eigenvalues λ and $\overline{\lambda}$ are of the same multiplicity.

2. The negative eigenvalues of A_L (if any) are necessarily of even algebraic multiplicity.

For the proofs of these properties, we refer the reader to [[2], pp. 252.253].

Definition 3.2 *Let*

$$\lambda(A_L) = \{\lambda_1, ..., \lambda_n\}$$

be the spectrum of A_L. The coneigenvalues of A are the n scalars $\mu_1, ..., \mu_n$ defined as follows: If $\lambda_i \in \lambda(A_L)$ does not lie on the negative real axis, then the corresponding coneigenvalue μ_i is defined as a square root of λ_i with nonnegative real part and the multiplicity of μ_i is set to that of λ_i :

$$\mu_i = \lambda_i^{\frac{1}{2}}, Re\mu_i \geqslant 0.$$

With a real negative $\lambda_i \in \lambda(A_L)$, we associate two conjugate purely imaginary coneigenvalues

$$\mu_i = \pm\lambda_i^{\frac{1}{2}}$$

the multiplicity of each being half the multiplicity of λ_i.

The set $\{\mu_1, ..., \mu_n\}$ is called the conspectrum of A and will be denoted by $c\lambda(A)$. For a subspace $L \in M_n(\mathbf{C})$ define

$$\overline{L} = \{\overline{x} | x \in L\},$$

where $\overline{x}$ is the component-wise conjugate of the column vector x.

Definition 3.3 *L is called a coninvariant subspace of A if*

$$A\overline{L} \subseteq L.$$

The fundamental fact on coninvariant subspaces is the following theorem.

Theorem 3.4 *Every matrix $A \in M_n(\mathbf{C}) (n \geqslant 3)$ has a one- or two-dimensional coninvariant subspace.*

proof as given in [1].

Definition 3.5 *Let L is a coninvariant subspace of A and $dimL = 1$, then every nonzero vector $x \in L$ is called a coneigenvector of A.*

If matrix $A \in M_n(\mathbf{C})$ has a coneigenvector x, then there exist a coninvariant subspace L, where $x \in L$, and $A\overline{L} \subseteq L$. Since $dimL = 1$, can suppose that $L = span\{x\}$, this means that $A\overline{x} = \mu x$, for some $\mu \in \mathbf{C}$. in this equation μ is called coefficient associated with coneigenvector x.

Theorem 3.6 *Let $A \in M_n(\mathbf{C})$ has a coneigenvector x, and μ is coefficient associated with x, then $\mid \mu \mid$ is a coneigenvalue of A.*

Proof We know $A\overline{x} = \mu x$. But then

$$\overline{A}A\overline{x} = \overline{A}(\mu x) = \mu\overline{A\overline{x}} = \mu\overline{\mu x} = \mid \mu \mid^2 \overline{x},$$

so $\mid \mu \mid$ is a coneigenvalue of A. (We say $\mid \mu \mid$ is a coneigenvalue associated with the coneigenvector x.) ■

4. Condiagonalizable

Like ordinary similarity, consimilarity is an equivalence relation on $M_n(\mathbf{C})$.

Definition 4.1 *A matrix $A \in M_n(\mathbf{C})$ is said to be condiagonalizable if there exists a nonsingular $S \in M_n(\mathbf{C})$ such that $S^{-1}A\overline{S}$ is diagonal.*

THEOREM 4.2 *Let $A \in M_n(\mathbf{C})$, Then A is condiagonalizable if and only if there is a set of n linearly independent vectors, each of which is a coneigenvector of A.*

Proof If A has n linearly independent coneigenvectors

$$\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\},$$

since $x^{(i)}, i = 1, 2, \ldots, n$ is a coneigenvector of A, then exist a coninvariant subspace L_i, where $x^{(i)} \in L_i$, and $A\overline{L_i} \subseteq L_i$. Since $dim L_i = 1$, can suppose that $L_i = span\{x^{(i)}\}$, this means that $A\overline{x^{(i)}} = \mu_i x^{(i)}$, for some $\mu_i \in \mathbf{C}$. Form a nonsingular matrix S with $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ as columns and calculate

$$S^{-1}A\overline{S} = S^{-1}[A\overline{x^{(1)}}A\overline{x^{(2)}}...A\overline{x^{(n)}}] = S^{-1}[\mu_1 x^{(1)}\mu_2 x^{(2)}...\mu_n x^{(n)}]$$

$$= S^{-1}[x^{(1)}x^{(2)}...x^{(n)}]M = S^{-1}SM = M$$

where

$$M = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix}$$

and $\mu_1, \mu_2, ..., \mu_n$ are coefficients associated with coneigenvectors

$$x^{(1)}, x^{(2)}, \ldots, x^{(n)}.$$

Conversely, suppose that there is a nonsingular matrix S such that $S^{-1}A\overline{S} = M$ is diagonal. Then $A\overline{S} = SM$. This means that A times the ith column of $\overline{S}$ (i.e., the ith column of $A\overline{S}$) is the ith diagonal entry of M times the ith column of S (i.e., the ith column of SM), or $A\overline{S_i} = \mu_i S_i$, where S_i is the ith column of S and μ_i is the ith diagonal entry of M, let $L_i = span\{S_i\}$, then $A\overline{L_i} \subseteq L_i$ and $dim L_i = 1$. This result that ith column of S is an coneigenvector of A. Since S is nonsingular, there are n linearly independent coneigenvector. ■

LEMMA 4.3 *Suppose that $\{x^{(1)}, x^{(2)}, \ldots, x^{(k)}\}$ is a coneigenvectors set of matrix $A \in M_n(\mathbf{C})$, if $|\mu_1|, |\mu_2|, ..., |\mu_k|$ are coneigenvalues of A associated with $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$, no two of which are the same, then $\{x^{(1)}, x^{(2)}, \ldots, x^{(k)}\}$ is a linearly independent set.*

Proof The proof is essentially by contradiction. Suppose that

$$\{x^{(1)}, x^{(2)}, \ldots, x^{(k)}\}$$

is actually a linearly dependent set. Then there is a nontrivial linear combination which produces the 0 vector, and in fact there is such a linear combination with the fewext nonzero coefficients. Suppose that such a minimal linear dependence relation is

$$\alpha_1 x^{(1)} + \alpha_2 x^{(2)} + \ldots + \alpha_r x^{(r)} = 0, \quad r < k \tag{1}$$

We have $r > 1$ because all $x^{(i)} \neq 0$. We may assume for convenience (renumber if necessary) that it involves the first r vectors. We also have

$$A(\overline{\alpha_1 x^{(1)} + \alpha_2 x^{(2)} + \ldots + \alpha_r x^{(r)}}) = \overline{\alpha_1} A\overline{x^{(1)}} + \overline{\alpha_2} A\overline{x^{(2)}} + \ldots + \overline{\alpha_r} A\overline{x^{(r)}}$$

$$= \overline{\alpha_1}\mu_1 x^{(1)} + \overline{\alpha_2}\mu_2 x^{(2)} + \ldots + \overline{\alpha_r}\mu_r x^{(r)} = 0 \quad (2)$$

another dependence relation.Now multiply the relation (1) by $\overline{\alpha_r}\mu_r$ and the relation (2) by α_r and subtract the first relation from the second relation to produce

$$(\alpha_1\overline{\alpha_r}\mu_r - \alpha_r\overline{\alpha_1}\mu_1)x^{(1)} + \cdots + (\alpha_{r-1}\overline{\alpha_r}\mu_r - \alpha_r\overline{\alpha_{r-1}}\mu_{r-1})x^{(r-1)} = 0$$

a third dependence relation, which has fewer nonzero coefficients than the relation (1). This last relation is nontrivial since for i, $\quad 1 \leqslant i \leqslant r-1$

$$\alpha_i\overline{\alpha_r}\mu_r - \alpha_r\overline{\alpha_i}\mu_i = 0 \Rightarrow |\,\alpha_i\,||\,\overline{\alpha_r}\,||\,\mu_r\,| = |\,\alpha_r\,||\,\overline{\alpha_i}\,||\,\mu_i\,| \Rightarrow |\,\mu_r\,| = |\,\mu_i\,|\,.$$

This contradicts the minireality assumption for the dependence relation (1) and completes the proof. ∎

THEOREM 4.4 *If $A \in M_n(\mathbf{C})$ has n coneigenvectors, where coneigenvalues associated with them are distinct, then A is condiagonalizable.*

Proof Suppose that $\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$ is a coneigenvectors set of matrix A. This is a linearly independent set by lemma (4.3), and therefore A is condiagonalizable by theorem (4.2). ∎

5. Coneigenvalue and conjugate-normal matrix

For a conjugate-normal matrix A, matrix

$$\hat{A} = \begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix} \quad (3)$$

is normal. Conjugate-normality means that

$$AA^* = \overline{A^* A},$$

and normality means that

$$AA^* = A^* A.$$

A particular example of conjugate-normal matrices are symmetric matrices.

THEOREM 5.1 *Let $\{\mu_1, ..., \mu_n\}$ be the coneigenvalues set of an $n \times n$ matrix A. Then*

$$\lambda(\hat{A}) = \{\mu_1, ..., \mu_n, -\mu_1, ..., -\mu_n\},$$

where $\lambda(\hat{A})$ be the spectrum of $\hat{A}$.

Proof The assertion desired follows from two observations. First, we have $\hat{A}^2 = A_R \oplus A_L$, which implies that any eigenvalue of $\hat{A}$ is a square root of an eigenvalue of A_L. Second, the characteristic polynomial $\varphi(\lambda)$ of $\hat{A}$ is given by.

$$\varphi(\lambda) = det(\lambda I_{2n} - \hat{A}) = det(\lambda^2 I_n - A_L) = det(\lambda^2 I_n - A_R).$$

Thus, if λ is an eigenvalue of $\hat{A}$, then $-\lambda$ also is an eigenvalue of $\hat{A}$, and both of them have the same multiplicity. ■

Flowing theorem was proved in [[2], theorem(4.1.4)].

THEOREM 5.2 *Let $A \in M_n(\mathbf{C})$ be given. Then A is Hermitian if and only if A is normal and all the eigenvalues of A are real.*

THEOREM 5.3 *Suppose that matrix $A \in M_n(\mathbf{C})$ is conjugate-normal and all the coneigenvalues of A are real, then A is symmetric.*

Proof since A is a conjugate-normal matrix, then $\hat{A}$ is normal. Now if all coneigenvalues of the matrix A are real, then all eigenvalues of $\hat{A}$ are real (by theorem (5.1)), so $\hat{A}$ is a hermitian matrix by theorem (5.2), i.e. $\hat{A}^* = \hat{A}$, this equality implies that

$$\begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix} \tag{4}$$

or

$$\begin{pmatrix} 0 & A^T \\ A^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix} \tag{5}$$

so $A^T = A$, this equality means that A is symmetric.

■

6. Conclusion

Properties of coneigenvalues and coneigenvectors of a matrix, which are considered in this paper, compared with the previous definition of the coneigenvalues (presented at the [2]), are more similar to the general eigenvalues and eigenvectors of a matrix.

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