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Interpolation by Hyperbolic B-spline Functions

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Abstract. In this paper we present a new kind of B-splines, called hyperbolic B-splines generated over the space spanned by hyperbolic functions and we use it to interpolate an arbitrary function on a set of points. Numerical tests for illustrating hyperbolic B-spline are presented.

Keywords: Interpolation, Algebraic Hyperbolic B-spline, Hyperbolic B-spline, Polynomial B-spline, Trigonometric B-spline.

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1. Introduction

B-spline basis is important base of the polynomial space spanned by $\{1, t, \cdots, t^k\}$ in which k is an arbitrary positive integer. In recent years, several new spline curve and surface scheme have been proposed for geometric modeling in CAGD. For instance, Nouisser et al. [14] introduced 2π-periodic trigonometric. Maes and Bultheel [13] presented a normalized spherical B-splines. In this paper, we begin by considering the hyperbolic B-splines generated on the space

$$\Gamma_k = \begin{cases} \text{span}\{\{\sinh(2l\phi), \cosh(2l\phi)\}_{l=1}^{[\frac{k-1}{2}]} \cup \{1\}\}, & k \text{ is odd} \\ \text{span}\{\{\sinh(2l-1)\phi, \cosh(2l-1)\phi\}_{l=1}^{[\frac{k}{2}]}\}, & k \text{ is even} \end{cases}$$

(for $k > 1$). We call such splines hyperbolic B-splines of order k. Finally, some numerical examples are given.

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Polynomial B-splines are the basic basis which are contained in many literatures, See [5, 16]. Recently in many works the authors try to use various types of B-splines on spheres [1, 6, 9, 10, 13, 14, 17].
The paper is organized as follows: In Sections 2, 3 and 4, we recall polynomial B-splines, 2π-periodic trigonometric B-splines and algebraic hyperbolic B-spline basis (AH B-splines), respectively. In section 5, we introduce the hyperbolic B-splines. The construction of interpolation by hyperbolic B-splines are given in Section 6. There are some numerical examples in Section 7.

2. Polynomial B-splines

Let $\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots$ be a sequence of knots on $\mathbb{R}$. The B-splines of order zero are piecewise constants defined by

$$B_i^0(x) = \begin{cases} 1, & t_i < x \leqslant t_{i+1} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

and those of order $k > 0$ are defined recursively in terms of those of order $k-1$ by

$$B_i^k(x) = \left(\frac{x - t_i}{t_{i+k} - t_i}\right) B_i^{k-1}(x) + \left(\frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}}\right) B_{i+1}^{k-1}(x) \tag{2}$$

They satisfy the following properties:

1. $\forall x \notin [t_i, t_{i+k+1}]$, $B_i^k(x) = 0$
2. $\forall x \in \mathbb{R}$, $B_i^k(x) \geqslant 0$
3. $B_i^k \in C^{k-1}(\mathbb{R})$

Definition 2.1 *The support of a function is the closure of the set of points in which the function is not zero:*

$$\text{supp} f = \overline{\{x : \ f(x) \neq 0\}}$$

3. 2π-Periodic Trigonometric B-Splines

Suppose that $J = [0, 2\pi]$. Let m be an odd positive integer and let $Y = \{t_i\}$ be the 2π-periodic partition J defined by

$$t_m = 0 < t_{m+1} < \cdots < t_M < \cdots < t_{M+m-1} < t_{M+m} = 2\pi,$$

such that $0 < t_{i+m-1} - t_i \leqslant \pi$, $t_i = t_{M+i} - 2\pi$, and $t_{m+i} = t_{M+m+i} - 2\pi$ for $i = 1, \cdots, m-1$.
The 2π-periodic trigonometric B-splines N_i^m of order m associated with the partition X are defined by [14]:

$$N_i^1(\phi) = \begin{cases} 1, & t_i \leqslant \phi < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

and for $m > 1$,

$$N_i^m(\phi) = \frac{s(\phi - t_i)}{s(t_{m+i-1} - t_i)} N_i^{m-1}(\phi) + \frac{s(t_{i+m} - \phi)}{s(t_{i+m} - t_{i+1})} N_{i+1}^{m-1}(\phi)$$

where $s(x) = \sin\left(\frac{x}{2}\right)$ and $c(x) = \cos\left(\frac{x}{2}\right)$. They satisfy the following properties:

1. $N_i^m \in C^{m-2}(R)$, and for all ϕ we have $N_{i+M}^m(\phi) = N_i^m(\phi - 2\pi)$.
2. $N_i^m(\phi)$ is a piecewise trigonometric function, i.e., $N_i^m(\phi)|[t_j, t_{j+1}] \in \Gamma_m$, where

$$\Gamma_m = \begin{cases} < 1, s(2\phi), c(2\phi), \cdots, s((m-1)\phi), c((m-1)\phi) >, & \text{m is odd} \\ < s(\phi), c(\phi), \cdots, s((m-1)\phi), c((m-1)\phi) >, & \text{m is even} \end{cases}$$

   is the space of trigonometric polynomials of order m.
3. $N_i^m(\phi) \geqslant 0$ and supp $N_i^m = [t_i, t_{i+m}]$.
4. The family $\{N_i^m, i = 1, \cdots, M + m - 1\}$ forms a basis for the space $\tilde{S}_m = \{g \in C^{m-2}([0, 2\pi]); g|[t_i, t_{i+1}] \in \Gamma_m\}$

4. Algebraic Hyperbolic B-splines Basis (AH B-splines)

Let X be a give knot sequence $\{x_i\}_{-\infty}^{+\infty}$ with $x_i \leqslant x_{i+1}$ we first give a set of initial functions by [20]

$$M_{i,2}(x) = \begin{cases} \frac{\sinh(x - x_i)}{\sinh(x_{i+1} - x_i)}, & x_i < x \leqslant x_{i+1} \\ \frac{\sinh(x_{i+2} - x)}{\sinh(x_{i+2} - x_{i+1})}, & x_{i+1} < x \leqslant x_{i+2} \\ 0, & \text{otherwise} \end{cases}$$

We define that $\frac{0}{0} = 0$. Then the algebraic B-spline basic functions of order k in space $\Gamma_{k-1} = \text{span}\{1, x, x^{k-3}, \sinh x, \cosh x\}$ can be defined recursively. as:

$$M_{i,k}(x) = \int_{-\infty}^{x} \left(\delta_{i,k-1} M_{i,k-1}(s) - \delta_{i+1,k-1} M_{i+1,k-1}(s)\right) ds \quad k \geqslant 3 \tag{3}$$

where $\delta_{i,k} = \frac{1}{\int_{-\infty}^{+\infty} M_{i,k}(x)dx}$.
If $M_{i,k}(x) = 0$, $\delta_{i,k} = \infty$ and $\delta_{i,k} M_{i,k}(x) = 0$.
We have from (3) the following:

$$\int_{-\infty}^{x} \delta_{i,k} M_{i,k}(s) ds = \begin{cases} 0, & x \leqslant x_i \\ \geqslant 0, & x_i < x < x_{i+k} \\ 1, & x \geqslant x_{i+k} \end{cases}$$

Properties of the AH B-spline basis:

(1) $M_{i,k}(x) = 0$, $x \notin [x_i, x_{i+k}]$

(2) $\sum_{-\infty}^{+\infty} M_{i,k}(x) = 1$ for all $k \geqslant 3$ and all x.

(3) $M_{i,k}(x) = 0$ if and only if $x_i = x_{i+1} = \cdots = x_{i+k}$

(4) $M'_{i,k}(x) = \delta_{i,k-1} M_{i,k-1}(x) - \delta_{i+1,k-1} M_{i+1,k-1}(x)$

5. Hyperbolic B-Splines

In this section we introduce hyperbolic B-splines. The hyperbolic B-splines T_i^k of order k associated with the partition X are defined by:

$$T_i^1(\phi) = \begin{cases} 1, & t_i \leqslant \phi < t_{i+1} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

and for $k > 1$,

$$T_i^k(\phi) = \frac{s(\phi - t_i)}{s(t_{k+i-1} - t_i)} T_i^{k-1}(\phi) + \frac{s(t_{i+k} - \phi)}{s(t_{i+k} - t_{i+1})} T_{i+1}^{k-1}(\phi) \tag{5}$$

where $s(x) = \sinh(x)$. They satisfy the following properties:

1. For $k \geqslant 2$, $T_i^k \in C^{k-2}(\mathbb{R})$
2. $T_i^k(\phi)$ is a piecewise hyperbolic function.
3. $T_i^k(\phi) \geqslant 0$.
4. supp $T_i^k = [t_i, t_{i+k}]$
5. $T_i^k \in \Gamma_k$

$$\Gamma_k = \begin{cases} \text{span}\{\{\sinh(2l\phi), \cosh(2l\phi)\}_{l=1}^{[\frac{k-1}{2}]} \cup \{1\}\}, & k \quad \text{is odd} \\ \text{span}\{\{\sinh(2l-1)\phi, \cosh(2l-1)\phi\}_{l=1}^{[\frac{k}{2}]}\}, & k \quad \text{is even} \end{cases} \tag{6}$$

is the space of hyperbolic polynomials of order k.

Let $X = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. The graphs of the basis functions T_i^2, T_i^3 and T_i^4 are shown in Figures 1,2, and 3.

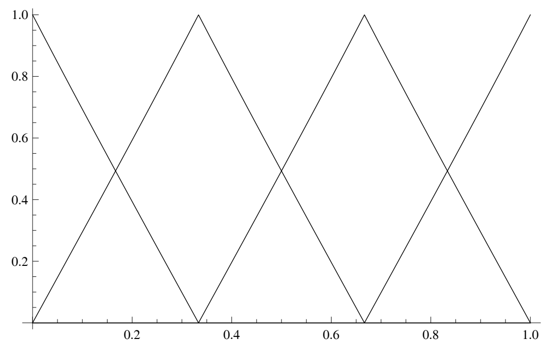

Figure 1. The hyperbolic B-splines T_i^3.

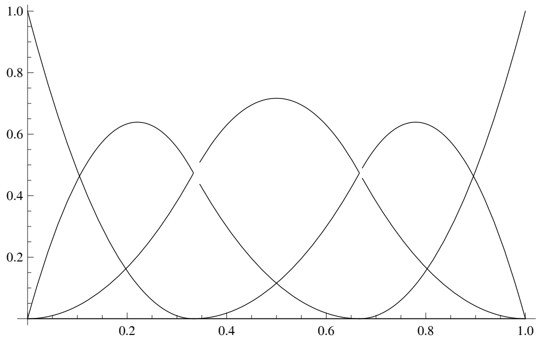

Figure 2. The hyperbolic B-splines T_i^3.

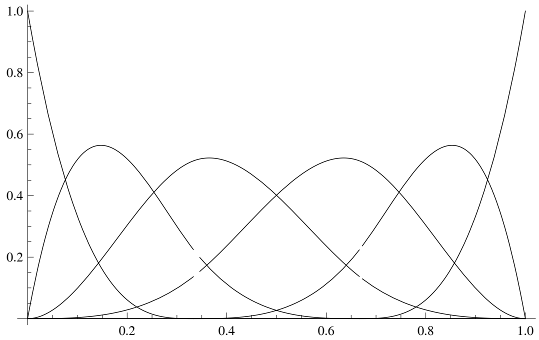

Figure 3. The hyperbolic B-splines T_i^4.

6. Interpolation by Hyperbolic B-Splines

Suppose that the support points $\{(x_i, y_i)\}_{i=1}^m$ are known in which x_i's can be the same as t_i's. We consider the interpolating function as

$$q_k(x) := \sum_{i=1}^{m+k-1} c_i^k T_i^k(x) \tag{7}$$

in which T_i^k's are given in (4) and (5). By considering the interpolation condition we should have

$$q_k(x_j) = f(x_j) \qquad j = 1, \cdots, m \tag{8}$$

If the $k-1$ additional conditions are chosen suitably [5], then the interpolation problem has a good solution.

If for the support points we have $x_i \neq t_j$ for at least one i and one j, then we should solve a least square problem to find the control points.

7. Numerical Examples

In this section we compute the interpolating function on support data or for a given function.

In our examples we consider the set of nodes as X.

Example 7.1 Let $X = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. We interpolate the following function:

$$f(x) = 2\sinh(x) - \frac{1}{5}\cosh(x)$$

by

$$q_2(x) = \sum_{i=2}^{m-1} c_i T_i^2(x)$$

in which c_i's can be easily computed. The error function is shown in Figure 4.

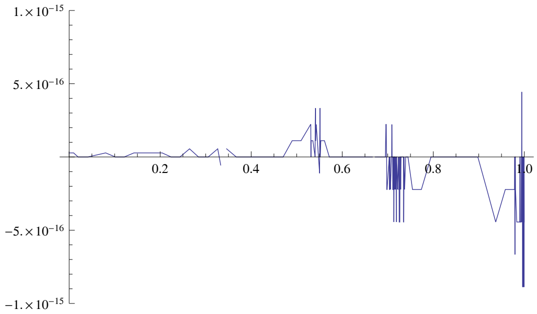

Figure 4. The error function $f(x) - q_3(x)$.

Example 7.2 Let $X = \{0, 1, 2, 3\}$. The error of interpolating function of

$$f(x) = e^{x+2} - e^{-x+1}$$

is shown in Figure 5.

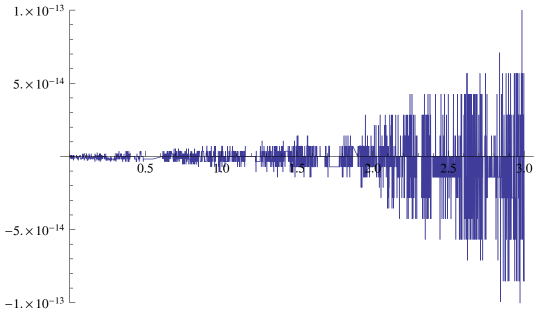

Figure 5. The error function $f(x) - q_3(x)$.

8. Conclusion

In this work we introduced hyperbolic B-splines and used it to interpolate a function. The interpolation method by using T_i^k is exact for any function in Γ_k.

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