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Hybrid of Rationalized Haar Functions Method for Solving Differential Equations of Fractional Order

Y. Ordokhani[a,*] and N. Rahimi[b]

[a,b]*Department of Applied Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran.*

Abstract. In this paper, we implement numerical solution of differential equations of fractional order based on hybrid functions consisting of block-pulse function and rationalized Haar functions. For this purpose, the properties of hybrid of rationalized Haar functions are presented. In addition, the operational matrix of the fractional integration is obtained and is utilized to convert computation of fractional differential equations into some algebraic equations. We evaluate application of present method by solving some numerical examples.

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1. Introduction

Recently, differential equations of fractional order have paid attention so much, since this equations are applied in many science such as fluid mechanics, viscoelasticity, biology, physics and engineering [1, 2]. Usually fractional differential equations are difficult to solve and do not have exact analytical solutions, therefore several numerical methods have been presented to solve fractional differential equations, Legendre wavelet method [3, 4], Haar wavelet [5], Chebyshev wavelets method [6], fractional differential method (FDM), Adomian decomposition method

*Corresponding author. Email: ordokhani@alzahra.ac.ir

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(ADM), the variational iteration method (VIM) [7, 8] and other methods [9 − 12] but a few articles there are that report hybrid functions to solve fractional equations. In this work hybrid of block-pulse and rationalized Haar (HRH) functions method are used to solve fractional differential equations. By using rationalized Haar functions to solve fractional differential equations to obtain high accurancy number of term be very large then for reduce term and time we apply HRH functions.

The article is organized as follows:

In section 2, some necessary fundamentals of fractional calculus are pointed out. In section 3, we present the properties of HRH functions required for our subsequent development. In section 4, we describe solution of fractional differential equations and the proposed method are used to estimate the unknown function $y(x)$ and in section 5, we give some numerical examples to demonstrate the accuracy of the proposed method.

2. Fundamentals of Fractional Calculus

In this section, we give some definitions and preliminaries in the fractional calculus theory.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geqslant 0$ is defined as [2, 13]

$$I^{\alpha}y(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}y(t)\,dt, \alpha > 0, x > 0,$$

$$I^0 y(x) = y(x),$$

whrere $\Gamma(.)$ is Gamma function.

It has the following properties:

$$I^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}, \quad \gamma > -1.$$

Definition 2.2. The Caputo definition of fractional derivative operator is given by [10, 14]

$${}_{*}D^{\alpha}y(x) = I^{n-\alpha}{}_{*}D^{n}y(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}y^{(n)}(t)\,dt,$$

where $n-1 \leqslant \alpha < n, \quad n \in N, \quad x > 0.$

It has following properties

$${}_{*}D^{\alpha}I^{\alpha}y(x) = y(x),$$

$$I^{\alpha}{}_{*}D^{\alpha}y(x) = y(x) - \sum_{k=0}^{n-1} y^{(k)}(0^{+})\frac{x^k}{k!}, \quad x > 0.$$

3. Properties of Hybrid Functions

3.1 *Hybrid Functions of Block-Pulse and Rationalized Haar Functions*

The HRH Functions $h_{nr}(x)$, $n = 1, 2, ..., N$, $r = 1, 2, ..., M-1$, $M = 2^{\beta+1}$, $\beta = 1, 2, ...,$ where n, r are the order of block-pulse functions and rationalized Haar functions respectively. They are defined on the interval $[0, 1)$ as [15]

$$h_{nr}(x) = \begin{cases} h_r(Nx - n + 1), & \frac{n-1}{N} \leqslant x < \frac{n}{N}, \\ 0, & otherwise. \end{cases} \tag{1}$$

In equation (1), $h_r(x)$ are the orthogonal set of rationalized Haar functions and can be defined on the interval $[0, 1)$ as [16]

$$h_r(x) = \begin{cases} 1, & J_1 \leqslant x < J_{\frac{1}{2}}, \\ -1, & J_{\frac{1}{2}} \leqslant x < J_0, \\ 0, & otherwise, \end{cases} \tag{2}$$

where, $J_u = \frac{j-u}{2^i}$, $\quad u = 0, \frac{1}{2}, 1$.

The value of r is defined by two parameters i and j as

$$r = 2^i + j - 1, \quad i = 0, 1, 2, 3, ..., \quad j = 1, 2, 3, ..., 2^i,$$

$h_0(x)$ is defined for $i = j = 0$ and given by

$$h_0(x) = 1, \quad 0 \leqslant x < 1. \tag{3}$$

since $h_{nr}(x)$ is the combination of rationalized Haar functions and block-pulse functions which are both complete and orthogonal, thus the set of hybrid functions are complete orthogonal set. The orthogonality property of HRH functions is given by [15]

$$\int_0^1 h_{nr}(x) h_{n'r'}(x)\, dx = \begin{cases} \frac{2^{-i}}{N}, & n = n', r = r', \\ 0, & otherwise, \end{cases}$$

where
$r = 2^i + j - 1, \quad r' = 2^{i'} + j' - 1.$

3.2 *Function Approximation*

A function $f(x) \in L^2([0, 1])$ may be expanded into HRH functions as [15]

$$f(x) = \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} c_{nr} h_{nr}(x), \tag{4}$$

where c_{nr} given by

$$c_{nr} = \frac{< f, h_{nr} >}{\|h_{nr}\|^2} = 2^i N \int_0^1 f(x) h_{nr}(x)\, dx,$$

and $< .,. >$ denote the inner product.
If, the infinite series in equation (4) is truncated, then equation (4) can be written as

$$f(x) \simeq \sum_{n=1}^{N} \sum_{r=0}^{M-1} c_{nr} h_{nr}(x) = C^T H(x), \tag{5}$$

The HRH function coefficient vector C and RH function vector $H(x)$ are defined as

$$C = [c_{10}, c_{11}, ..., c_{1M-1} | c_{20}, c_{21}, ..., c_{2M-1} | ... | c_{N0}, c_{N1}, ..., c_{NM-1}]^T, \tag{6}$$

$$H(x) = [H_1^T(x) | H_2^T(x) | ... | H_N^T(x)]^T, \tag{7}$$

where

$$H_i^T(x) = [h_{i0}, h_{i1}, ..., h_{iM-1}], \qquad i = 1, 2, ..., N.$$

Taking the Newton-Cotes nodes as following [17]

$$x_i = \frac{2i-1}{2MN}, \qquad i = 1, 2, ..., MN. \tag{8}$$

$$\begin{aligned} \phi_{MN} &= [H(\tfrac{1}{2MN}), H(\tfrac{3}{2MN}), ..., H(\tfrac{2MN-1}{2MN})], \\ &= diag(\hat{\phi}_{M\times M}, \hat{\phi}_{M\times M}, ..., \hat{\phi}_{M\times M}), \end{aligned} \tag{9}$$

where $\hat{\phi}_{M\times M}$ is M-square Haar matrix ([16]).
For example if $M = 2$ and $N = 3$ we have

$$\phi_{23} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Using equation (5) we get

$$[f(\frac{1}{2MN}), f(\frac{3}{2MN}), ..., f(\frac{2MN-1}{2MN})] \simeq C^T \phi_{MN}. \tag{10}$$

3.3 *Operational Matrix of the Fractional Integration*

The integration of the vector $H(x)$ defined in equation (7) can be defined as [15]

$$\int_0^x H(t)\,dt \simeq PH(x), \tag{11}$$

where P is the $MN \times MN$ operational matrix of integration.
In this section, we want to derive the HRH functions operational matrix of the fractional integration. For this purpose, we consider an m-set of block-pulse functions as

$$b_i(x) = \begin{cases} 1, \frac{i}{m} \leqslant x < \frac{i+1}{m}, i = 0, 1, 2, ..., m-1, \\ 0, \quad otherwise, \end{cases}$$

where $m = MN$.
The function $b_i(x)$, are disjoint and orthogonal. That is

$$b_i(x)b_j(x) = \begin{cases} 0, & i \neq j, \\ b_i(x), & i = j, \end{cases}$$

$$\textstyle\int_0^1 b_i(x)b_j(x)\,dt = \begin{cases} 0, & i \neq j, \\ \frac{1}{m}, & i = j. \end{cases}$$

Rationalized Haar functions can be expanded into an m-set of block-pulse functions. Similarly, HRH functions can be expanded into their block-pulse functions as

$$H(x) = \phi_{MN} B(x), \tag{12}$$

where $B(x) = [b_0(x), b_1(x), ..., b_{m-1}(x)]^T$ and ϕ_{MN} is a $MN \times MN$ matrix defined in equation (9).
In [18], the block-pulse operational matrix of the fractional integration F^α are given as follows:

$$I^\alpha B(x) \simeq F^\alpha B(x), \tag{13}$$

where

$$F^\alpha = \frac{1}{m^\alpha}\frac{1}{\Gamma(\alpha+1)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \xi_1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{14}$$

with $m = MN$ and $\xi_k = (k+1)^\alpha - 2k^{\alpha+1} + (k-1)^{\alpha-1}$.
Now, we obtain the HRH function operational matrix of the fractional integration. Let

$$I^\alpha H(x) \simeq P^\alpha H(x), \tag{15}$$

where P^{α} is called the HRH functions operational matrix of the fractional integration.
Using equations (12) and (13), we have

$$I^{\alpha}H(x) = I^{\alpha}\phi_{MN}B(x) = \phi_{MN}I^{\alpha}B(x) \simeq \phi_{MN}F^{\alpha}B(x),$$

from (12) and (15), we get

$$P^{\alpha}H(x) = P^{\alpha}\phi_{MN}B(x) = \phi_{MN}F^{\alpha}B(x).$$

Then, P^{α} is given by

$$P^{\alpha} = \phi_{MN}F^{\alpha}\phi_{MN}^{-1}, \tag{16}$$

where, ϕ_{MN}^{-1} is inverse of matrix ϕ_{MN}. then, we have found the operational matrix of fractional integration for HRH functions.
For example if $M = 2$ and $N = 3$ we have

$P^{\alpha} = \frac{1}{6^{\alpha}}\frac{1}{\Gamma(\alpha+1)}\times$

$$\begin{bmatrix} \frac{\xi_1}{2}+1 & -\frac{\xi_1}{2} & \frac{\xi_1}{2}+\xi_2+\frac{\xi_3}{2} & \frac{\xi_1}{2}-\frac{\xi_3}{2} & \frac{\xi_3}{2}+\xi_4+\frac{\xi_5}{2} & \frac{\xi_3}{2}-\frac{\xi_5}{2} \\ \frac{\xi_1}{2} & 1-\frac{\xi_1}{2} & \frac{\xi_3}{2}-\frac{\xi_1}{2} & \xi_2-\frac{\xi_1}{2}-\frac{\xi_3}{2} & \frac{\xi_5}{2}-\frac{\xi_3}{2} & \xi_4-\frac{\xi_3}{2}-\frac{\xi_5}{2} \\ 0 & 0 & \frac{\xi_1}{2}+1 & -\frac{\xi_1}{2} & \frac{\xi_1}{2}+\xi_2+\frac{\xi_3}{2} & \frac{\xi_1}{2}-\frac{\xi_3}{2} \\ 0 & 0 & \frac{\xi_1}{2} & 1-\frac{\xi_1}{2} & \frac{\xi_3}{2}-\frac{\xi_1}{2} & \xi_2-\frac{\xi_1}{2}-\frac{\xi_3}{2} \\ 0 & 0 & 0 & 0 & \frac{\xi_1}{2}+1 & -\frac{\xi_1}{2} \\ 0 & 0 & 0 & 0 & \frac{\xi_1}{2} & 1-\frac{\xi_1}{2} \end{bmatrix},$$

and for $\alpha = 0.5$, the operational matrix is as following

$$P^{0.5} = \begin{bmatrix} 0.4343 & -0.127 & 0.3598 & 0.06024 & 0.2342 & 0.0153 \\ 0.1272 & 0.1799 & -0.0602 & -0.0285 & -0.0153 & -0.0029 \\ 0 & 0 & 0.4343 & -0.1272 & 0.3598 & 0.0602 \\ 0 & 0 & 0.1272 & 0.1799 & -0.0602 & -0.0285 \\ 0 & 0 & 0 & 0 & 0.4343 & -0.1272 \\ 0 & 0 & 0 & 0 & 0.1272 & 0.1799 \end{bmatrix}.$$

4. Solution of Fractional Differential Equation

In this section we consider the solution of fractional differential equations of linear and nonlinear type.

4.1 *Linear Differential Equations of Fractional Order*

Consider the following differential equations of fractional order

$${}_*D^{\alpha}y(x) = a_{1*}D^{\beta_1}y(x) + ... + a_{k*}D^{\beta_k}y(x) + a_{k+1}y(x) + a_{k+2}f(x), \tag{17}$$

$$0 \leqslant x \leqslant 1, \quad n-1 < \alpha \leqslant n, \quad n \in N, \quad 0 < \beta_1 < \beta_2 < ... < \beta_k < \alpha,$$

with initial conditions

$$y^{(i)}(0) = \delta_i, \quad i = 0, 1, 2, 3, ..., n-1, \tag{18}$$

where $a_j, j = 1, 2, ..., k+2$ are real constant coefficients.
To solve this problem, we approximate ${}_*D^{\alpha}y(x)$ as following

$${}_*D^{\alpha}y(x) \simeq C^T H(x), \tag{19}$$

where C is unknown vector and $H(x)$ is HRH functions vector defined in (7).
Also, we let

$$f(x) \simeq F^T H(x). \tag{20}$$

where F is known vector that defined in equation (6). By using properties of Caputo derivative we can expanding other terms as following

$$\begin{aligned} {}_*D^{\beta_1}y(x) &= I^{\alpha-\beta_1}{}_*D^{\alpha}y(x) \simeq C^T P^{\alpha-\beta_1} H(x), \\ {}_*D^{\beta_2}y(x) &= I^{\alpha-\beta_2}{}_*D^{\alpha}y(x) \simeq C^T P^{\alpha-\beta_2} H(x), \\ &\vdots \\ {}_*D^{\beta_k}y(x) &= I^{\alpha-\beta_k}{}_*D^{\alpha}y(x) \simeq C^T P^{\alpha-\beta_k} H(x), \end{aligned} \tag{21}$$

and if $\beta_j = q \in N_0 = \{0, 1, 2, ...\}$, $j = 1, 2, ..., k$, then for expanding ${}_*D^q y(x)$, $q = 0, 1, ..., n-1$, by using equation (19) and properties of Caputo derivative we have

$$\begin{aligned} {}_*D^{n-1}y(x) &= I^{\alpha-n+1}{}_*D^{\alpha}y(x) \simeq C^T P^{\alpha-n+1} H(x) + \delta_{n-1}, \\ {}_*D^{n-2}y(x) &= I^{\alpha-n+2}{}_*D^{\alpha}y(x) \simeq (C^T P^{\alpha-n+2} + \delta_{n-1} e^T \phi_{MN}^{-1} P^1) H(x) + \delta_{n-2}, \\ &\vdots \\ y(x) &\simeq (C^T P^{\alpha} + e^T \phi_{MN}^{-1} \sum_{i=0}^{n-1} \delta_i (P^1)) H(x), \end{aligned} \tag{22}$$

where
$e = (1, 1, ..., 1)^T$, (P^1) is operational matrix of HRH functions defined in equation (11) and $P^0 = I$ is $MN \times MN$-dimensional identity matrix.
By substituting equations $(19)-(22)$ in equation (17), we obtain a system of linear equations can be solved for unknown vector C easily.

4.2 *Nonlinear Differential Equations of Fractional Order*

Consider the nonlinear differential equations of fractional order

$${}_*D^{\alpha}y(x) = f(x, y(x), {}_*D^{\beta_1}y(x), {}_*D^{\beta_2}y(x), ..., {}_*D^{\beta_k}y(x)), \tag{23}$$

$$0 \leqslant x \leqslant 1, \quad n-1 < \alpha \leqslant n, \quad n \in N, \quad 0 < \beta_1 < \beta_2 < ... < \beta_k < \alpha,$$

with initial conditions

$$y^{(i)}(0) = \delta_i, \quad i = 0, 1, 2, 3, ..., n-1. \tag{24}$$

Similarly to linear fractional differential equations we first approximate ${}_*D^{\alpha}y(x)$, ${}_*D^{\beta_j}y(x)$ for $j = 1, 2, ..., k$ and $y(x)$ as equations (19), (21) and (22). Then we substituting these equations in equation (23) and to find solution $y(x)$, we collocate this equation in the Newton-Cotes nodes defined in equation (8). So, we get MN nonlinear equations which can be solved for unknown vector C.

5. Numerical Examples

In this section we applied the HRH functions method to solve some numerical examples.

Example 5.1 Consider the following Relaxation equation of fractional order ([3, 8])

$${}_*Dy(x) - a{}_*D^{\alpha}y(x) - by(x) = 0, \quad x > 0, \quad 0 < \alpha \leqslant 1,$$

with initial condition

$$y(0) = 1.$$

In particular, we assume $a = b = -1$ and in this case when $\alpha = 1$, the exact solution is $y(x) = e^{-\frac{1}{2}x}$.
We have solved this example for $N = 3$ and $M = 8$ and have compared it with VIM method of [8]. The comparison is shown in Table 1.

Table 1. Comparison of the solutions of VIM and HRH for different α of Example 5.1.

x	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$		Exact
	y_{VIM}	y_{HRH}	y_{VIM}	y_{HRH}	y_{VIM}	y_{HRH}	for $\alpha = 1$
0.0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.751577	0.914585	0.662104	0.923203	0.566982	0.935891	0.951229
0.2	0.605730	0.846942	0.543694	0.863216	0.493983	0.883039	0.904837
0.3	0.499314	0.791889	0.463863	0.813022	0.443495	0.836268	0.860708
0.4	0.417812	0.746231	0.404072	0.769780	0.403821	0.793990	0.818731
0.5	0.353711	0.707633	0.356911	0.731715	0.370859	0.755230	0.778801
0.6	0.302381	0.675023	0.318509	0.698085	0.342608	0.719648	0.740818
0.7	0.260714	0.646904	0.286543	0.667854	0.317909	0.686629	0.704688
0.8	0.226518	0.622594	0.259495	0.640587	0.296016	0.655952	0.670320
0.9	0.198191	0.601421	0.236315	0.615841	0.276413	0.627360	0.637628
1.0	0.174535	0.582772	0.216243	0.593195	0.258722	0.600581	0.606531

Numerical results in Table 1 show that proposed method is very useful in computing, because in fractional calculus theory, we know when exact solution is unknown, as α $(n-1 \leqslant \alpha < n)$ approaches to positive integer number n, the numerical solution converges to the exact solution of the problem with derivation n ([10]). So, we can conclude HRH function method is better than VIM method since HRH functions has this property but VIM method do not has.

Example 5.2 Next, we consider the following linear equation of fractional order ([8, 9])

$${}_*D^2y(x) - a{}_*D^{\alpha}y(x) - by(x) = g(x), \quad x > 0, \quad 0 < \alpha \leqslant 2,$$

with initial condition

$$y(0) = 0 \quad y'(0) = 0.$$

We assume $a = b = -1$, $\alpha = \frac{1}{2}$ and $g(x) = 8$ for $x \in [0, 1]$. The exact solution is the closed form series solution given in [8]. In Table 2, we present numerical results obtained by HRH functions method and methods of [8, 9] such as FDM method, ADM method and VIM method . We have solved this example for $N = 3$ and $M = 8$ results show that our method is very better than above mentioned methods.

Table 2. Estimated for $N = 6$ and $M = 16$ with comparison methods of [8, 9].

x	y_{FDM}[9]	y_{ADM}[8]	y_{VIM}[8]	y_{HRH}	y_{exact}
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.039473	0.039874	0.039874	0.039848	0.039750
0.2	0.157703	0.158512	0.158512	0.157092	0.157036
0.3	0.352402	0.353625	0.353625	0.347414	0.347370
0.4	0.620435	0.622083	0.622083	0.604751	0.604695
0.5	0.957963	0.960047	0.960047	0.921737	0.921768
0.6	1.360551	1.363093	1.363093	1.290470	1.290457
0.7	1.823267	1.826257	1.826257	1.701990	1.702008
0.8	2.340749	2.344224	2.344224	2.147250	2.147287
0.9	2.907324	2.911278	2.911278	2.616960	2.617001
1.0	3.517013	3.521462	3.521462	3.101840	3.101906

Example 5.3 Let us consider the nonlinear fractional differential equations ([12])

$${}_*D^2 y(x) + g(x)_* D^{\frac{3}{2}} y(x) + y^2(x) = 2 + 2x + x^4,$$

where $g(x) = \Gamma(\frac{3}{2}) x^{\frac{1}{2}}$ and subject to the initial conditions

$$y(0) = y'(0) = 0.$$

The exact solution is $y(x) = x^2$.
We have solved this example for $M = 32$ and different N and have compared it with method of [12] comparison show in Table 3.

Table 3. Absolute error for Example 5.3

x	Method of [12]	Present method $N = 3, M = 32$	Present method $N = 4, M = 32$
0.0	2.74260×10^{-5}	0	0
0.1	4.20794×10^{-5}	2.69071×10^{-5}	1.02893×10^{-5}
0.2	3.76716×10^{-5}	1.93434×10^{-5}	1.58345×10^{-5}
0.3	8.44125×10^{-5}	2.04899×10^{-5}	1.65120×10^{-5}
0.4	3.27010×10^{-5}	3.02811×10^{-5}	1.22849×10^{-5}
0.5	3.61133×10^{-5}	5.26161×10^{-6}	3.12358×10^{-5}
0.6	1.94954×10^{-5}	3.21831×10^{-5}	1.34117×10^{-6}
0.7	2.95780×10^{-5}	2.41830×10^{-5}	1.87026×10^{-5}
0.8	4.92488×10^{-5}	2.46014×10^{-5}	1.89602×10^{-5}
0.9	2.83224×10^{-5}	3.33725×10^{-5}	1.41488×10^{-5}
1.0	7.73238×10^{-5}	7.02804×10^{-6}	4.22857×10^{-6}
CPU	-	$2.5617s$	$4.0370s$

6. Conclusion

In the present work a fractional operational matrix of HRH functions is obtained and is used to estimated numerical solution of linear and nonlinear differential equations of fractional order. In this method time and computations are small, because the matrices ϕ_{MN} and P^α introduce in equations (9) and (16) have many zeros, then proposed method is fast and easy to use. Numerical examples are given to show the efficiency of the present method.

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