International Journal of
Mathematical Modelling & Computations
Vol. 03, No. 02, 2013, 131- 147

The Elzaki Homotopy Perturbation Method for Partial Differential Equations

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Received: 5 March 2013; Accepted: 29 June 2013.

Abstract. In this paper, Elzaki Homotopy Perturbation Method is employed for solving linear and nonlinear differential equations with a variable coefficient. This method is a combination of Elzaki transform and Homotopy Perturbation Method. The aim of using Elzaki transform is to overcome the deficiencies that mainly caused by unsatisfied conditions in some semi-analytical methods such as Homotopy Perturbation Method, Variational Iteration Method and Adomian Decomposition Method. The approximate solutions obtained by means of Elzaki Homotopy Perturbation Method were compared in a wide range of problem's domain with those results obtained by Homotopy Perturbation Method. The comparison shows a precise agreement between the exact solutions and the obtained results by this new method as an applicable one, which needs less computation and is much easier and more convenient than others. So, it can be widely used in engineering and other branches of science.

Keywords: Elzaki Transform; Homotopy Perturbation Method; Iteration Method; Convergence Analysis.

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1. Introduction

The importance of obtaining the exact or approximate solutions of linear and nonlinear partial differential equations in physics and mathematics is still a significant problem that needs new methods. As we know, most new linear and nonlinear equations do not have a precise analytic solution. So, beside numerical methods that have largely been used to handle these problems, there are also analytic techniques

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to solve these equations. Some of the classic analytic methods are: perturbation techniques [8, 22], expansion method and Hirota bilinear method [15, 23].

In recent years, many researchers have paid attention to study the solutions of partial differential equations by using various methods such as Adomian Decomposition Method (ADM) [5, 16], He's semi-inverse method [24], Homotopy Perturbation Method (HPM) [11, 12, 21], Variational Iteration Method (VIM) [1, 7, 19] and Variational Iteration Method using He's Polynomials (VIMHP) [17, 18]. Most of these methods have their inbuilt deficiencies like the calculation of Adomian's polynomials, the Lagrange multiplier, divergent results and huge computational work. The Homotopy Perturbation Method was formulated by taking the full advantage of the standard homotopy and perturbation methods and has been modified later by some scientists to obtain more accurate results, rapid convergence and to reduce amount of computation [13, 20]. But, as mentioned above, it has some shortages that need to be modified. This paper introduces an improvement on HPM and considers the effectiveness of the Elzaki Homotopy Perturbation Method (EHPM) in solving partial differential equations. Our proposed method, gives accurate results in wide range via one or two iteration steps.

2. Homotopy Perturbation Method (HPM)

The essential idea of HPM is to introduce a homotopy parameter, say p, which takes the values from 0 to 1. When $p = 0$, the system of equations is in a sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of "deformation", the solution of each stage is "close" to that at the previous stage of "deformation". Eventually, at $p = 1$, the system takes the original form of equation and the final stage of "deformation" gives the desired solution. The embedding parameter can be considered as an expanding parameter [9, 10]. To illustrate the basic concept of Homotopy Perturbation Method, consider the following nonlinear system of differential equations:

$$A(u) = f(r), \qquad r \in \Omega \tag{1}$$

with boundary conditions

$$B(u, \frac{\partial u}{\partial n}), \qquad r \in \Gamma \tag{2}$$

where A is a differential operator, B is a boundary operator, f(r) is a known analytic function, and Γ is the boundary of the domain Ω . Generally speaking, the operator A can be divided into two parts L and N, where L is a linear and N is a nonlinear operator. Therefore, (1) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

We construct a homotopy $v(r,p) : \Omega \times [o,1] \to R^n$, which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p\ [A(v) - f(r)] = 0, \quad p \in [0,1] \;\; r \in \Omega \tag{4}$$

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{5}$$

where u_0 is an initial approximation of (1). In this method, using the homotopy parameter p, we have the following power series presentation for v,

$$v = v_0 + pv_1 + p^2 v_2 + \ldots. \tag{6}$$

The approximate solution can be obtained by setting $p = 1$, i.e.

$$u = \sum_{i=0}^{\infty} v_i = v_0 + v_1 + v_3 + \ldots. \tag{7}$$

The convergence of series (7) is discussed in [14]. The method considers the nonlinear term $N[v]$ as

$$N(v) = \sum_{i=0}^{+\infty} p^i H_i = H_0 + pH_1 + p^2 H_2 + \cdots,$$

where H_n's are the so-called He's polynomials [6], which can be calculated by using below formula

$$H_n(v_0, v_1, \cdots, v_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left(N \left(\sum_{i=0}^{n} p^i v_i \right) \right)_{p=0}, \quad n = 0, 1, 2, \cdots.$$

3. Basic idea of EHPM

The purpose of this section is to discuss the use of Elzaki transform algorithm in HPM (EHPM) for solving differential equations. Consider general inhomogeneous nonlinear equation with initial and boundary conditions is given below:

$$L[u(x,t)] + N[u(x,t)] = g(x,t), \tag{8}$$
$$\frac{\partial^j u}{\partial x^j}(0,t) = y_j(t), \quad j = 0, 1, \cdots, n-1$$

and initial conditions

$$\frac{\partial^j u}{\partial t^j}(x,0) = z_j(x), \quad j = 0, 1, \cdots, m-1$$

where $L = \frac{d^m}{dt^m}; m \in N$ is linear operator, $N[u(x,t)]$ represents the non-linear terms and $g(x,t)$ is the source term. The methodology consists of applying Elzaki transform with respect to t on both sides of Equation (8)

$$\mathbb{E}[Lu(x,t)] + \mathbb{E}[Nu(x,t)] = \mathbb{E}[g(x,t)]. \tag{9}$$
$$\mathbb{E}\left[\frac{\partial^j u}{\partial x^j}(0,t)\right] = \mathbb{E}[y_j(t)], \quad j = 0, 1, \cdots, n-1.$$

If we denote $\mathbb{E}[u(x,t)] = T(x,v)$ and $\mathbb{E}\left[\frac{\partial^j u}{\partial x^j}(0,t)\right] = \overline{y}_j(v)$, using the differential property of Elzaki transform and initial conditions, we get

$$\frac{T(v)}{v^m} - \sum_{k=0}^{m-1} z_k v^{2-m+k} + \mathbb{E}[Nu(x,t)] = \mathbb{E}[g(x,t)]. \tag{10}$$

So, we can write Equation (10) as

$$\mathcal{L}\ T(x,v) + \mathcal{N}\ T(x,v) = h(x,v), \tag{11}$$
$$\frac{\partial^j T}{\partial x^j}(0,v) = \overline{y}_j(v), \quad j = 0, 1, \cdots, n-1$$

where $\mathcal{L}$ is new linear operator, $\mathcal{N}$ is new nonlinear operator and $h(x,v)$ is a source term of equation after applying Elzaki transform and considering initial conditions. The next step in EHPM is constructing the appropriate homotopy for Equation (11) and supposing the solution as a power series. Therefore,

$$T = \lim_{p \to 1} \sum_{i=0}^{\infty} p^i \bar{v}_i(x,v) = \bar{v}_0(x,v) + \bar{v}_1(x,v) + \bar{v}_2(x,v) + \ldots. \tag{12}$$

Applying the inverse Elzaki transform on both sides of Equation (12), gives the solution of Equation (8) as

$$u(x,t) = \lim_{p \to 1} \sum_{i=0}^{\infty} p^i v_i(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + \ldots. \tag{13}$$

For more information about Elzaki transform refer to [2].

4. Convergence Analysis

In this section, we study the convergence of Elzaki Homotopy Perturbation Method. Consider equation (11)

$$\mathcal{L}\ T(x,v) + \mathcal{N}\ T(x,v) = h(x,v). \tag{14}$$

If we denote $h(x,v) = f(r)$, we can construct a homotopy $\bar{v}(r,p) : \Omega \times [o,1] \to R^n$, which satisfies

$$H(\bar{v},p) = (1-p)[\mathcal{L}(\bar{v}) - \mathcal{L}(T_0)] + p\ [A(\bar{v}) - f(r)] = 0, \quad p \in [0,1] \quad r \in \Omega \tag{15}$$

$$H(\bar{v},p) = \mathcal{L}(\bar{v}) - \mathcal{L}(T_0) + p\mathcal{L}(T_0) + p\ [\mathcal{N}(\bar{v}) - f(r)] = 0. \tag{16}$$

We can write Equation (16) in the following form

$$\mathcal{L}(\bar{v}) = \mathcal{L}(T_0) + p\ [f(r) - \mathcal{N}(\bar{v}) - \mathcal{L}(T_0)]. \tag{17}$$

Applying the inverse operator, $\mathcal{L}^{-1}$, to both sides of Equation (17), we obtain

$$\bar{v} = T_0 + p\ [\mathcal{L}^{-1} f(r) - \mathcal{L}^{-1}\mathcal{N}(\bar{v}) - T_0]. \tag{18}$$

Suppose that

$$\bar{v} = \sum_{i=0}^{\infty} p^i \bar{v}_i \tag{19}$$

substituting (19) into the right-hand side of Equation (18) gives:

$$\bar{v} = T_0 + p\ [\mathcal{L}^{-1} f(r) - \mathcal{L}^{-1}\mathcal{N}[\sum_{i=0}^{\infty} p^i \bar{v}_i)] - T_0]. \tag{20}$$

If $p \to 1$, the exact solution may be obtained by using

$$\begin{aligned} T &= \lim_{p \to 1} \bar{v} \\ &= \mathcal{L}^{-1} f(r) - \mathcal{L}^{-1}\mathcal{N}[\sum_{i=0}^{\infty} \bar{v}_i] \\ &= \mathcal{L}^{-1} f(r) - \sum_{i=0}^{\infty} (\mathcal{L}^{-1}\mathcal{N})(\bar{v}_i), \end{aligned}$$

and finally by applying inverse Elzaki transform, the solution of Equation (8) can be obtained. To study the convergence of the method, let us state the following Theorem.

THEOREM 4.1 (Sufficient Condition of Convergence)*Suppose that X and Y are Banach spaces and $\mathcal{N} : X \to Y$ is a contractive nonlinear mapping, that is*

$$\forall\ w, w^* \in X : ||\mathcal{N}(w) - \mathcal{N}(w^*)|| \leqslant \gamma\ ||w - w^*|| \quad ,0 < \gamma < 1 \tag{21}$$

then, according to Banach's fixed point theorem $\mathcal{N}$ has a unique fixed point T, *that is $\mathcal{N}(T) = T$:*
Assume that the sequence generated by homotopy perturbation method can be written as

$$W_n = \mathcal{N}(W_{n-1}), \qquad W_{n-1} = \sum_{i=0}^{n-1} w_i, \qquad n = 1, 2, 3, \dots \tag{22}$$

and suppose that $W_0 = w_0 \in B_r(w)$ where $B_r(w) = w^* \in X \mid ||w^* - W|| < r$, then we have

$$(i) W_n \in B_r(w),$$
$$(ii) \lim_{n\to\infty} W_n = w.$$

Proof

(i) By inductive approach, for $n \to 1$, we have

$$||W_1 - w|| = ||\mathcal{N}(W_0) - \mathcal{N}(w)|| \leqslant \gamma\ ||w_0 - w||.$$

Assume that $||W_{n-1} - w|| \leqslant \gamma^{n-1}\ ||w_0 - w||$, as induction hypothesis, then

$$||W_n - w|| = ||\mathcal{N}(W_{n-1}) - \mathcal{N}(w)|| \leqslant \gamma\ ||W_{n-1} - w|| \leqslant \gamma^n\ ||w_0 - w||.$$

Using (i), we have

$$||W_n - w|| \leqslant \gamma^n\ ||w_0 - w|| \leqslant \gamma^n r < r \Rightarrow W_n \in B_r(w).$$

(ii) Because of $||W_n - w|| \leqslant \gamma^n \, ||w_0 - w||$ and $\lim_{n\to\infty} \gamma^n = 0$, $\lim_{n\to\infty} ||W_n - w|| = 0$ that is,

$$\lim_{n\to\infty} W_n = w.$$

■

5. Applications

To illustrate the proposed method for partial differential equations the following examples are considered. First of all, we solve these equations with HPM and then solve them with the help of EHPM.

Example 5.1 Consider the following nonlinear wave equation [4]

$$\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^2 u}{\partial t^2} = 1 - \frac{x^2+t^2}{2}, \qquad 0 \leqslant x, t \leqslant 1 \tag{23}$$

Subject to boundary conditions

$$u(0,t) = \frac{t^2}{2}, \qquad \frac{\partial}{\partial x}u(x,0) = 0. \tag{24}$$

Exact solution of this equation is

$$u(x,t) = \frac{x^2+t^2}{2}.$$

HPM approach

Substituting Equation (23) into (5) and supposing $u(x,0) = \frac{x^2}{2}$, we have an equation system including *(n + 1)* equations to be simultaneously solved; n is the order of p in Equation (6). Comparing the like-power coefficient of p gives:

$$
\begin{aligned}
p^0 &: \frac{d^2\nu_0}{dx^2} - \frac{d^2u_0}{dx^2} = 0,\\
p^1 &: \frac{d^2\nu_1}{dx^2} - (\nu_0\nu_{0_{tt}}) - \frac{x^2+t^2}{2} = 0,\\
&\vdots
\end{aligned} \tag{25}
$$

Solving differential Equation (25),we have

$$
\begin{aligned}
\nu_0 &= \frac{x^2}{2},\\
\nu_1 &= \frac{x^2(6t^2+x^2)}{24},\\
\nu_2 &= x^6(\frac{1}{240}t^2 + \frac{1}{2688}x^2),\\
&\vdots
\end{aligned} \tag{26}
$$

The solution of Equation (23) can be obtained as $u(x,t) = \sum_{i=0}^{\infty}\nu_i$.

EHPM approach

Applying the Elzaki transform on both sides of Equation (23), and using initial conditions, we get

$$T - v^2\mathbb{E}[u\frac{\partial^2 u}{\partial t^2}] = -v^6 + (1-\frac{t^2}{2})v^4 + \frac{t^2}{2}v^2. \tag{27}$$

Applying inverse Elzaki transform yields

$$u = \frac{x^2}{2} - \frac{t^2x^2}{4} - \frac{x^4}{24} + \frac{t^2}{2} + \mathbb{E}^{-1}[v^2\mathbb{E}(uu_{tt})]. \tag{28}$$

Now, we apply the homotopy perturbation method

$$
\begin{aligned}
u(x,t) &= \sum_{i=0}^{\infty}\nu_i(x,t)\\
&= \frac{x^2}{2} - \frac{t^2x^2}{4} - \frac{x^4}{24} + \frac{t^2}{2} + p\ \mathbb{E}^{-1}[v^2\mathbb{E}[\sum_{i=0}^{\infty}H_i(\nu)]],
\end{aligned} \tag{29}
$$

where $H_n(\nu)$ are He's polynomials. The modified recursive relation is given below

$$\begin{aligned}
\nu_0 &= \frac{x^2}{2} - \frac{t^2x^2}{4} - \frac{x^4}{24} + \frac{t^2}{2},\\
\nu_1 &= \mathbb{E}^{-1}[v^2\mathbb{E}[H_0(\nu)]] = \mathbb{E}^{-1}[v^2\mathbb{E}[\nu_0\nu_{0_{tt}}]]\\
&= \frac{t^2x^2}{4} + \frac{x^6(3t^2-7)}{720} + \frac{x^8}{2688} - \frac{x^4(t^2-1)}{24},\\
\nu_2 &= \mathbb{E}^{-1}[v^2\mathbb{E}[H_1(\nu)]] = \mathbb{E}^{-1}[v^2\mathbb{E}[\nu_0\nu_{1_{tt}} + \nu_1\nu_{0_{tt}}]]\\
&= \frac{x^4t^2}{24} - \frac{x^6(8t^2-7)}{720} + \frac{x^8(36t^2-64)}{40320} - \frac{x^{10}(168t^2-519)}{3628800} - \frac{43x^{12}}{1064480},\\
&\vdots
\end{aligned} \tag{30}$$

The other components of the solution can be easily found by using before recursive relation.
Consequently,

$$\begin{aligned}
u(x,t) &= \nu_0 + \nu_1 + \nu_2 + \ldots,\\
&= \frac{t^2}{2} + \frac{x^6(3t^2-7)}{720} + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^8}{2688} - \frac{x^4(t^2-1)}{24} + \ldots.
\end{aligned}$$

Studying Table 1 and Figure 1 show that our proposed technique has an excellent agreement with the exact solution in comparison with HPM.

Table 1. The relative errors for HPM, and $EHPM$ at $x = 0.5$ for Equation (23).

t	RE_{HPM}	RE_{EHPM}
0.00000000	2.0845e-002	1.3857e-006
1.0000e-001	1.3606e-002	1.6170e-006
2.0000e-001	1.0270e-001	2.2154e-006
3.0000e-001	2.1626e-001	2.9781e-006
4.0000e-001	3.2870e-001	3.7333e-006
5.0000e-001	4.2701e-001	4.3935e-006
6.0000e-001	5.0777e-001	4.9359e-006
7.0000e-001	5.7226e-001	4.3935e-006
8.0000e-001	6.2326e-001	5.3691e-006
9.0000e-001	6.6362e-001	5.7116e-006
1.0000e+000	6.9573e-001	6.1983e-006

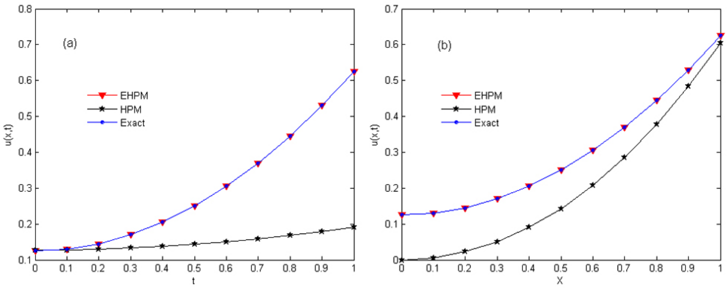

Figure 1. Comparison between the results of EHPM, HPM and the exact solution of Equation (23) at a) $x = 0.5$ and b) $t = 0.5$.

Example 5.2 Consider the following nonlinear differential equation[4]

$$\frac{\partial^2 u}{\partial x^2} - u^2 \frac{\partial^2 u}{\partial t^2} = 2(1 - (x^2 + t^2)^2), \qquad 0 \leqslant x, t \leqslant 1 \tag{31}$$

subject to boundary conditions

$$u(0,t) = t^2, \qquad \frac{\partial}{\partial x} u(x, o) = 0 \tag{32}$$

exact solution of this equation is

$$u(x,t) = x^2 + t^2.$$

HPM approach

Substituting Equation (31) into (5) and supposing $u(x,0) = x^2$, we have an equation system including *(n + 1)* equations to be simultaneously solved; n is the order of p in Equation (6). Comparing the like-power coefficients of p gives:

$$\begin{aligned} p^0 &: \frac{d^2\nu_0}{dx^2} - \frac{d^2u_0}{dx^2} = 0, \\ p^1 &: \frac{d^2\nu_1}{dx^2} - (\nu_0^2 \nu_{0_{tt}}) + 2(x^2 + t^2)^2 = 0, \\ &\vdots \end{aligned} \tag{33}$$

Solving differential Equation (33),we have

$$\begin{aligned} \nu_0 &= x^2, \\ \nu_1 &= \frac{-x^2(15t^4+5t^2x^2+x^4)}{15}, \\ \nu_2 &= -x^8(\frac{1}{135}x^2+\frac{3}{14}t^2), \\ &\vdots \end{aligned} \tag{34}$$

The solution of Equation (31) can be obtained as $u(x,t)=\sum_{i=0}^{\infty}\nu_i$.

EHPM approach

Applying the Elzaki transform on both sides of Equation (31) and using initial conditions, we get

$$T - v^2\mathbb{E}[u^2\frac{\partial^2 u}{\partial t^2}] = -48v^8 - 8t^2v^6 + (2-2t^4)v^4 + t^2v^2. \tag{35}$$

Applying inverse Elzaki transform gives

$$u = \frac{-1}{15}x^6 - \frac{1}{3}t^2x^4 + (1-t^4)x^2 + t^2 + \mathbb{E}^{-1}[v^2\mathbb{E}(u^2u_{tt})]. \tag{36}$$

Now, we apply the homotopy perturbation method

$$\begin{aligned} u(x,t) &= \sum_{i=0}^{\infty}\nu_i(x,t) \\ &= \frac{-1}{15}x^6 - \frac{1}{3}t^2x^4 + (1-t^4)x^2 + t^2 + p\,\mathbb{E}^{-1}[v^2\mathbb{E}[\sum_{i=0}^{\infty}H_i(\nu)]], \end{aligned} \tag{37}$$

where $H_n(\nu)$ are He's polynomials. The modified recursive relation is given below

$$\begin{aligned}
\nu_0 &= \frac{-1}{15}x^6 - \frac{1}{3}t^2x^4 + (1-t^4)x^2 + t^2,\\
\nu_1 &= \mathbb{E}^{-1}[v^2\mathbb{E}[H_0(\nu)]] = \mathbb{E}^{-1}[v^2\mathbb{E}[\nu_0^2\nu_{0_{tt}}]]\\
&= t^4x^2 - \frac{8t^6x^3}{3} + \frac{77t^4x^9}{270} + \frac{17t^2x^{12}}{1485} - \frac{5t^2x^{15}}{875}\\
&- \frac{14t^4x^{13}}{585} - \frac{t^2x^{16}}{1350} + \ldots,\\
\nu_2 &= \mathbb{E}^{-1}[v^2\mathbb{E}[H_1(\nu)]] = \mathbb{E}^{-1}[v^2\mathbb{E}[(\nu_1^2 + 2\nu_0\nu_2)\nu_{0_{tt}} + 2\nu_0\nu_1\nu_{1_{tt}} + \nu_0^2\nu_{2_{tt}}]]\\
&= \frac{14x^3t^4}{3} - 2t^2x^2 - \frac{271t^2x^28}{17413975} + \ldots,\\
&\vdots
\end{aligned} \tag{38}$$

The other components of the solution can be easily found by using before recursive relation and consequently

$$u(x,t) = \nu_0 + \nu_1 + \nu_2 + \ldots.$$

Table 2 and Figure 2 show the ability and accuracy of EHPM to handle the PDEs.

Table 2. The relative errors for HPM, and $EHPM$ at $x = 0.5$ for Equation (31).

t	RE_{HPM}	RE_{EHPM}
0.00000000	4.1956e-003	1.7437e-008
1.0000e-001	4.3425e-002	7.0725e-008
2.0000e-001	1.4592e-002	5.7985e-007
3.0000e-001	2.7948e-002	4.0971e-006
4.0000e-001	4.1687e-002	2.2543e-005
5.0000e-001	5.4418e-002	9.6956e-005
6.0000e-001	6.5779e-002	3.3719e-004
7.0000e-001	7.5904e-002	9.8362e-004
8.0000e-001	8.5092e-002	2.4833e-003
9.0000e-001	9.3644e-001	5.5627e-003
1.0000e+000	1.0182e+000	1.1271e-002

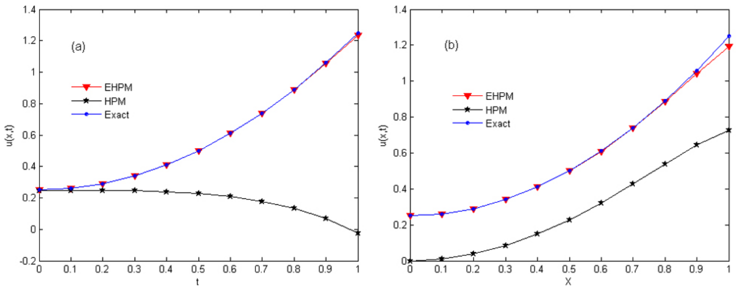

Figure 2. Comparison between the results of EHPM, HPM and the exact solution of Equation (31). at a) $x = 0.5$ and b) $t = 0.5$.

Example 5.3 Consider the linear Klein-Gordon equation[3]

$$u_{tt} - u_{xx} = u, \tag{39}$$

subject to boundary conditions

$$u(0, t) = \cosh(t), \qquad u_x(0, t) = 1, \tag{40}$$

and initial conditions

$$u(x, 0) = 1 + \sin(x), \qquad u_t(x, 0) = 0. \tag{41}$$

The exact solution of this equation is

$$u(x, t) = \sin(x) + \cosh(t). \tag{42}$$

HPM approach

Substituting Equation (39) into (5) and supposing $u(x, 0) = 1 + \sin(x)$, we have an equation system including *(n + 1)* equations to be simultaneously solved; n is the order of p in Equation (6). Comparing the like-power coefficients of p gives:

$$
\begin{aligned}
p^0 &: \frac{d^2\nu_0}{dt^2} - \frac{d^2u_0}{dt^2} = 0,\\
p^1 &: \frac{d^2\nu_1}{dt^2} - \frac{d^2\nu_0}{dx^2} + \nu_0 = 0,\\
p^i &: \frac{d^2\nu_i}{dt^2} - \frac{d^2\nu_{(i-1)}}{dx^2} + \nu_{(i-1)} = 0, \qquad i = 2, 3, \ldots
\end{aligned}
\tag{43}
$$

Solving differential Eqs. (43),we have

$$
\begin{aligned}
\nu_0 &= 1 + \sin(x),\\
\nu_1 &= \frac{-1}{2}t^2(1 + 2\sin(x)),\\
\nu_2 &= \frac{1}{12}t^4(\frac{1}{2} + 2\sin(x)),\\
&\vdots
\end{aligned}
\tag{44}
$$

The solution of Equation (39) can be obtained as $u(x,t) = \sum_{i=0}^{\infty} \nu_i$.

EHPM approach

Applying the Elzaki transform on both sides of Equation (39) and using initial conditions, we get

$$
\frac{\partial^2 T}{\partial x^2} + T(1 - \frac{1}{v^2}) = -(1 + \sin(x)). \tag{45}
$$

Now, applying the homotopy perturbation method on Equation (45) and using $T_0 = xv^2 + \frac{1}{1-v^2}$ as the initial approximation and finally comparing the like-power coefficients p we obtain

$$
\begin{aligned}
\bar{\nu_0} &= xv^2 + \frac{1}{1 - v^2},\\
\bar{\nu_1} &= -\frac{1}{6v^2}(x^3(v^4 - v^2) + 3x^3(v^2 - 1) + 6xv^2 - 6v^2\sin(x)),\\
\bar{\nu_2} &= \frac{v^2 - 1}{120v^4}(x^5(v^4 - v^2) + 5x^3 4(v^2 - 1) + 20x3v^2 - 120xv^2 + 120v^2\sin(x)),\\
&\vdots
\end{aligned}
\tag{46}
$$

Now, applying inverse Elzaki transform on Equation (46) gives

$$\begin{aligned}
\nu_0 &= x + \frac{1}{2}(e^t + e^{-t}),\\
\nu_1 &= dirac(t,1)(\sin x - x + \frac{x^3}{6}) - \frac{x^3}{6},\\
\nu_2 &= dirac(t)\frac{x^5}{120} - dirac(t,1)(x - \sin x - \frac{x^3}{6} + \frac{x^5}{120})\\
&+ dirac(t,2)(x - \sin x - \frac{x^3}{6} + \frac{x^5}{120}) - \frac{x^3}{6} + \frac{x^5}{120},\\
&\vdots
\end{aligned} \tag{47}$$

Therefor, the solution of Equation (39) can be obtained as

$$u(x,t) = \nu_0 + \nu_1 + \nu_2 + \ldots .$$

Table 3 and Figure 3 approve that the introduced method can overcome deficiencies that arise in HPM and can be used to solve different problems.

Table 3. The relative errors for HPM, and $EHPM$ at $x = 0.5$ for Equation (39).

t	RE_{HPM}	RE_{EHPM}
1.0000e+000	6.9267e-001	7.6377e-007
2.0000e+000	1.1163e+000	3.6418e-007
3.0000e+000	7.6185e-001	1.4646e-007
4.0000e+000	3.9071e-001	5.5590e-008
5.0000e+000	2.9072e-001	2.0682e-008
6.0000e+000	3.8784e-001	7.6398e-009
7.0000e+000	5.5288e-001	2.8148e-009
8.0000e+000	7.0707e-001	1.0361e-009
9.0000e+000	8.2237e-001	3.8122e-010
1.0000e+001	8.9838e-001	1.4025e-010

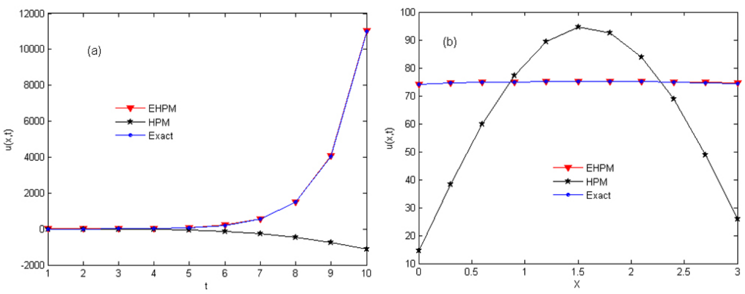

Figure 3. Comparison between the results of EHPM, HPM and the exact solution of Equation (39). at a) $x = 5$ and b) $t = 5$.

6. Conclusion

In this paper, the EHPM and HPM were used to obtain the analytic solutions of partial differential equations. The comparison between the numerical results obtained by EHPM and HPM were made and it was found that EHPM is more effective than HPM because of higher level of accuracy and less amount of computational work. Hence, it may be concluded that this method is a powerful and an efficient technique in finding the solutions for wide classes of problems. The computations associated with the examples in this paper were performed using Matlab 7.

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