A Homotopy Perturbation Algorithm and Taylor Series Expansion Method to Solve a System of Second Kind Fredholm Integral Equations

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Abstract. In this paper, we will compare a Homotopy perturbation algorithm and Taylor series expansin method for a system of second kind Fredholm integral equations. An application of He's homotopy perturbation method is applied to solve the system of Fredholm integral equations. Taylor series expansin method reduce the system of integral equations to a linear system of ordinary differential equation.

Keywords: HPM, Taylor Series, Integral Equation.

1. Introduction and Preliminary Notes

Many new methods, such as variational method [5, 6], variational iteration method [7], and others [8,9] are proposed to eliminate the shortcomings arising in the small parameter assumption. A review of recently developed nonlinear analysis method can be found in [10]. In recent year, the application of the HPM in nonlinear problems has been undertaken by scientists and engineers, since this method is used to continuously deform a simple problem that is easy to solve into a difficult problem under study. the HPM proposed first by He in 1998 was further developed and improved by He. Recently, the application of homotopy perturbation method theory have appeared in the works of many scientists, which shows that the method has become a powerful mathematical tool [11].
In this paper, we use a modified Taylor series expansion method for solving Fredhom integral equations system of second kind. This method first presented in [4] for solving Fredholm integral equations of second kind and then in [3] for solving

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Voltra integral equations of second kind.
Consider the second kind Fredholm integral equations system of the form

$$F(s) = G(s) + \int_0^1 K(s,t)F(t)\ dt, \qquad 0 \leqslant s \leqslant 1, \qquad (1)$$

where

$$F(s) = [f_1(s), f_2(s), ..., f_n(s)]^T,$$

$$G(s) = [g_1(s), g_2(s), ..., g_n(s)]^T,$$

$$K(s,t) = [k_{ij}(s,t)], \qquad i,j = 1, 2, ..., n.$$

In Eq.(1) the function K and G are given, and F is the solution to be determined [1,2]. We assume that [1] has a unique solution.

1.1   *Homotopy perturbation Method*

To convey an idea of the HPM, we consider a general equation of type:

$$L(u) = 0, \qquad (2)$$

where L is an integral or differential operator. We difine a convex homotopy $H(u,p)$ by

$$H(u,p) = (1-p)F(u) + pL(u), \qquad (3)$$

where $F(u)$ is a functional operator whit unknown solution v_0, which can be obtained easily. It is clear that, for:

$$H(u,p) = 0, \qquad (4)$$

from which we have

$$H(u,0) = F(u) \quad , \quad H(u,1) = L(u).$$

This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(u,1)$. The embedding parameter p monotonously from zero to a unit as the trivial problem $F(u) = 0$ continuously deforms to original problem $L(u) = 0$). the embedding parameter $p \in [0,1]$ can be considered as an expanding parameter [12] to obtain:

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + \cdots. \qquad (5)$$

When $p \rightarrow 1$,(4) corresponds to (2) becomes the approximate solution of (2), i.e.

$$f = \lim_{p \rightarrow 1} u = \sum_{i=0}^{\infty} u_i. \tag{6}$$

It is well known that the series (6) is convergent for most cases, and also that the rate of convergence depends on $L(u)$ (see [13].
Concerning the system of Fredholm integral equation of (1), the solution would be taken in the following form:

$$f_1(s) = \sum_{i=0}^{\infty} p^i f_{1i} = f_{10} + p f_{11} + p^2 f_{12} + \cdots, \tag{7}$$

$$f_2(s) = \sum_{i=0}^{\infty} p^i f_{2i} = f_{20} + p f_{21} + p^2 f_{22} + \cdots,$$

1.2 Taylor series expansion Method

Consider the ith equation of (1):

$$f_i(s) = g_i(s) + \int_0^1 \sum_{j=1}^{n} k_{ij}(s,t) f_j(t) dt, \qquad i = 1, 2, ..., n. \tag{8}$$

A Taylor series expansion can be made for the solution $f_j(t)$ in the integral Eq.(8):

$$f_j(t) = f_j(s) + f_j'(s)(t-s) + ... + \frac{1}{m!} f_j^{(m)}(s)(t-s)m + E(t), \tag{9}$$

where $E(t)$ denotes the error between $f_j(t)$ and its Taylor series expansion (9).

$$E(t) = \frac{1}{(m+1)!} f_j^{(m+1)}(s)(t-s)^{(m+1)} + ...$$

If we use the first m term of Taylor series expansion (9) (as an approximate for $f_j(t)$ in (8) and neglige the $\int_0^1 \sum_{j=1}^{n} k_{ij}(s,t) E(t) dt$, then substituting (9) for $f_j(t)$ in the integral in Eq.(8), we have

$$f_i(s) \simeq g_i(s) + \int_0^1 \sum_{j=1}^{n} k_{ij}(s,t) \sum_{r=0}^{m} \frac{1}{r!}(t-s)^r f_j^{(r)}(s) dt, \qquad i = 1, 2, ..., n, \tag{10}$$

$$f_i(s) \simeq g_i(s) + \sum_{j=1}^{n} \sum_{r=0}^{m} \frac{1}{r!} f_j^{(r)}(s) \int_0^1 k_{ij}(s,t)(t-s)^r dt, \qquad i = 1, 2, ..., n, \tag{11}$$

$$f_i(s) - \sum_{j=1}^{n} \sum_{r=0}^{m} \frac{1}{r!} f_j^{(r)}(s) \left[\int_0^1 k_{ij}(s,t)(t-s)^r dt \right] \simeq g_i(s), \qquad i = 1, 2, ..., n, \tag{12}$$

if the integrals in Eq.(12) can be solved analytically, then the bracketed quantities are functions of s alone. So Eqs. (12) becomes a linear system of ordinary differential equations that can be solved. However, this requires the manufacture of an

appropriate number of boundary conditions. Now we present a method to manu-
facturing boundary conditions in easy way. In order to manufacturing boundary
conditions, we first differentiate both sides of (8) to get that for $0 < s < 1$ and
$i = 1, 2, ..., n$:

$$f_i'(s) = g_i'(s) + \int_0^1 \sum_{j=1}^n k'_{ij_s}(s,t) f_j(t) dt, \tag{13}$$

$$\vdots$$

$$f_i^{(m)}(s) = g_i^{(m)}(s) + \int_0^1 \sum_{j=1}^n k_{ij_s}^{(m)}(s,t) f_j(t) dt, \tag{14}$$

where $k_{ij_s}^{(m)}(s,t) = \partial k_{ij}(s,t)/\partial s^m$. Substitute $f_j(s)$ for $f_j(t)$ in the integral equations
(13) and (14) to obtain that for $0 < s < 1$ and $i = 1, 2, ..., n$:

$$f_i'(s) - \Big[\int_0^1 \sum_{j=1}^n k'_{ij_s}(s,t) dt \Big] f_j(s) \simeq g_i'(s), \tag{15}$$

$$\vdots$$

$$f_i^{(m)}(s) - \Big[\int_0^1 \sum_{j=1}^n k_{ij_s}^{(m)}(s,t) dt \Big] f_j(s) \simeq g_i^{(m)}(s). \tag{16}$$

Now Eq.(12) combined together with (15) and (16) become a mth order linear
system of algebraic equations that can be solved analytically or numerically. For
further illustration we now use this method with $n = 2$ and $m = 1$ for solving
a second kind Fredholm integral equation system. Consider the following second
kind Fredholm integral equation system:

$$\begin{cases} f_1(s) &= g_1(s) + \int_0^1 k_{11}(s,t) f_1(t) + k_{12}(s,t) f_2(t) dt, \\ f_2(s) &= g_2(s) + \int_0^1 k_{21}(s,t) f_1(t) + k_{22}(s,t) f_2(t) dt, \end{cases} \tag{17}$$

a Taylor series expansion can be made for $f_1(t)$ and $f_2(t)$ as follows:

$$f_1(t) = f_1(s) + (t-s) f_1'(s) + \frac{1}{2!}(t-s)^2 f_1''(s) + E_1(t),$$

$$f_2(t) = f_2(s) + (t-s) f_2'(s) + \frac{1}{2!}(t-s)^2 f_2''(s) + E_2(t).$$

Substituting $f_1(t)$ and $f_2(t)$ in Eqs.(17) gives

$$\begin{cases} f_1(s) &\simeq g_1(s) + \int_0^1 k_{11}(s,t) \Big\{ f_1(s) + (t-s) f_1'(s) + \frac{1}{2!}(t-s)^2 f_1''(s) \Big\} dt \\ &+ \int_0^1 k_{21}(s,t) \Big\{ f_2(s) + (t-s) f_2'(s) + \frac{1}{2!}(t-s)^2 f_2''(s) \Big\} dt, \\ f_2(s) &\simeq g_2(s) + \int_0^1 k_{21}(s,t) \Big\{ f_1(s) + (t-s) f_1'(s) + \frac{1}{2!}(t-s)^2 f_1''(s) \Big\} dt \\ &+ \int_0^1 k_{22}(s,t) \Big\{ f_2(s) + (t-s) f_2'(s) + \frac{1}{2!}(t-s)^2 f_2''(s) \Big\} dt. \end{cases} \tag{18}$$

Eq.(18) becomes a linear system of ordinary differential equation and can be solved after producing boundary conditions. For this we first differentiate both sides of Eqs.(17) to get:

$$\begin{cases} f_1'(s) & = g_1'(s) + \int_0^1 k_{11}'(s,t)f_1(t)dt + \int_0^1 k_{12}'(s,t)f_2(t)dt, \\ f_2'(s) & = g_2'(s) + \int_0^1 k_{21}'(s,t)f_1(t)dt + \int_0^1 k_{22}'(s,t)f_2(t)dt. \end{cases} \tag{19}$$

Substitute $f_j(s)$ for $f_j(t)$ in the integrals in Eqs.(19) gives

$$\begin{cases} f_1'(s) & = g_1'(s) + \{\int_0^1 k_{11}'(s,t)dt\}f_1(s) + \{\int_0^1 k_{12}'(s,t)dt\}f_2(s), \\ f_2'(s) & = g_2'(s) + \{\int_0^1 k_{21}'(s,t)dt\}f_1(s) + \{\int_0^1 k_{22}'(s,t)dt\}f_2(s), \end{cases} \tag{20}$$

now Eq.(18) combined with Eqs.(20) become a linear two order system of algebraic equations that can be solved easily.

2.   Numerical Example

Example 2.1 Consider the following Fredholm system of integral equation:

$$f_1(s) = \frac{s}{18} + \frac{17}{36} + \int_0^1 \frac{s+t}{3}(f_1(t) + f_2(t))dt, \tag{21}$$

$$f_2(s) = s^2 - \frac{19}{12}s + 1 + \int_0^1 st(f_1(t) + f_2(t))dt,$$

with the exact solution $f_1(s) = s + 1$ and $f_2(s) = s^2 + 1$ (see [14]).

Homotopy perturbation algorithm:

We may choose a convex homotopy such that as $H(f_1, f_2, p)$ with components

$$H_1(f_1, f_2, p) = f_1(s) - g_1(s) - p\int_0^1 \frac{s+t}{3}(f_1(t) + f_2(t))dt = 0, \tag{22}$$

$$H_2(f_1, f_2, p) = f_2(s) - g_2(s) - p\int_0^1 st(f_1(t) + f_2(t))dt = 0.$$

Substituting (7) into (22) , and equating the terms with identical powers of p, we

have

p^0 :

$$f_{10}(s) = g_1(s) \Rightarrow f_{10}(s) = \frac{s}{18} + \frac{17}{36} \simeq 0.0556s + 0.4722,$$

$$f_{20}(s) = g_2(s) \Rightarrow f_{20}(s) = s^2 - \frac{19}{12}s + 1 \simeq s^2 - 1.5833s + 1,$$

p^1 :

$$f_{11}(s) = \int_0^1 \frac{s+t}{3}(f_{10}(t) + f_{20}(t))dt \Rightarrow f_{11}(s) \simeq 0.3472s + 0.1590,$$

$$f_{21}(s) = \int_0^1 st(f_{10}(t) + f_{20}(t))dt \Rightarrow f_{21}(s) \simeq 0.4769s,$$

p^2 :

$$f_{12}(s) = \int_0^1 \frac{s+t}{3}(f_{11}(t) + f_{21}(t))dt \Rightarrow f_{12}(s) \simeq 0.1903s + 0.1181,$$

$$f_{22}(s) = \int_0^1 st(f_{11}(t) + f_{21}(t))dt \Rightarrow f_{22}(s) \simeq 0.3542s,$$

p^3 :

$$f_{13}(s) = \int_0^1 \frac{s+t}{3}(f_{12}(t) + f_{22}(t))dt \Rightarrow f_{13}(s) \simeq 0.1301s + 0.0802,$$

$$f_{23}(s) = \int_0^1 st(f_{12}(t) + f_{22}(t))dt \Rightarrow f_{23}(s) \simeq 0.2405s,$$

$$\vdots$$

Therefore, the approximate solution of example 3.1 can be readily obtained by

$$f_1(s) = \sum_{n=0}^{\infty} f_{1n}(s) \quad , \quad f_2(s) = \sum_{n=0}^{\infty} f_{2n}(s). \tag{23}$$

In practice, all terms of series (23) cannot be determind and so we use an approximation of the solution by the following truncated series:

$$\varphi_{1,m}(s) = \sum_{n=0}^{m-1} f_{1n}(s) \quad , \quad \varphi_{2,m}(s) = \sum_{n=0}^{m-1} f_{2n}(s). \tag{24}$$

The solution with eleven terms are given as

$$\varphi_{1,11}(s) \simeq 0.9813s + 0.9885,$$

$$\varphi_{2,11}(s) \simeq s^2 - 0.0345s + 1.$$

Taylor series expansion method:

In this example we have

$$k_{11}(s,t) = k_{21}(s,t) = \frac{s+t}{3} \quad , \quad k_{12}(s,t) = k_{22}(s,t) = st$$

where

$$k'_{11}(s,t) = k'_{21}(s,t) = \frac{1}{3} \quad , \quad k'_{12}(s,t) = k'_{22}(s,t) = t. \tag{25}$$

Substituting (25) into Eq.(20) and combined with Eq.(18) we have

$$f_1(s) \simeq -\frac{0.00277778(110808 - 42312s + 21569s^2 + 30570s^3 - 52530s^4 + 21600s^5)}{-369 + 601s - 510s^2 + 150s^3},$$

$$f_2(s) \simeq -\frac{0.00277778(132840 - 282456s + 542911s^2 - 465450s^3 + 182130s^4 - 21600s^5)}{-369 + 601s - 510s^2 + 150s^3}.$$

3. Conclusion

In this paper, we use an application of homotopy perturbation method and Taylor series expansion method for solving the system of second kind Fredholm integral equations. Numerical result shows that both methods are useful and powerful.

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