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Spline Collocation for Nonlinear Fredholm Integral Equations

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Abstract. The collocation method based on cubic B-spline, is developed to approximate the solution of second kind nonlinear Fredholm integral equations. First of all, we collocate the solution by B-spline collocation method then the Newton-Cotes formula use to approximate the integrand. Convergence analysis has been investigated and proved that the quadrature rule is third order convergent. The presented method is tested with four examples, and the errors in the solution are compared with the existing methods $[1, 2, 3, 4]$ to verify the accuracy and convergent nature of proposed methods. The RMS errors in the solutions are tabulated in table 3 which shows that our method can be applied for large values of n, but the maximum n which has been used by the existing methods are only $n = 10$, moreover our method is accurate and stable for different values of n.

Keywords: Nonlinear Fredholm integral equation, Cubic B-spline, Newton-Cotes, Collocation, Convergence analysis.

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1. Introduction

We consider the numerical solution of nonlinear Fredholm integral equations of the second kind in the following general form

$$f(x) = y(x) + \int_a^b k(x, t, f(t))dt, \qquad a \leqslant x \leqslant b. \tag{1}$$

And we assume that the solution is required over a finite interval $[a, b]$ that y and k are continuous on $[a, b]$ and k satisfies a uniform Lipschitz condition in unknown f. Under the above conditions will ensure that there exists a unique continuous solution to the problem (1). The numerical solutions of (1) have been investigated

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by several authors who rely on the approximation of the integral appearing by using various methods and quadratures. Nonlinear phenomena that appear in many applications in scientific fields, such as fluid dynamics, solid state physics, plasma physics and mathematical biology can be modeled by partial differential equations and by integral equations as well. In [5], a comparative study between the modified decomposition method and two of the traditional methods for analytic treatment of nonlinear integral and integro-differential equations has been conducted. In [6], the numerical solution of integral equation by using combination of spline-collocation method and Lagrange interpolation has been derived. Using a global approximation to the solution of a nonlinear integral equation of the Hammerstein is constructed by means of the Sinc basis function in [7]. In [3], a Chebyshev approximation has been used to solve the nonlinear integral equations of Hammerstein type. In [8], a numerical method for solving the nonlinear Volterra-Fredholm integral equations that is based upon Legendre wavelet approximations is presented. In this paper, we will use cubic B-spline collocation to approximate the unknown function and the Newton-Cotes rules to approximate the integrand of the nonlinear Fredholm integral equations of second kind.

2. The method

2.1 *Cubic B-spline method*

To develop the collocation method based on cubic B-spline for the solution of Volterra integral equations, let π be a uniform partition of the interval $[a, b]$ such as $\pi : a = t_0 < t_1 < \cdots < t_{n+2} = b$, where

$$h = (b-a)/(n+2), t_i = a + ih, i = 0, 1, \cdots, n+2. \tag{2}$$

We introduce the spline space

$$S_3(\pi) = \nu \in C^2[a, b]; \nu|_{[t_i, t_{i+1}]} \in P_3,\ i = 0, 1, \cdots, n+2, \tag{3}$$

where P_3 is the class of cubic polynomials. By introducing adjacent knots

$$t_{-2} < t_{-1} < t_0 < \cdots < t_{n+2} < t_{n+3} < t_{n+4}, \tag{4}$$

and the functions $B_i(t), S(t)$ which are defined in the following form

$$B_i(t) = \begin{cases} (t - t_{i-2})^3/h^3, & if\ t \in [t_{i-2}, t_{i-1}] \\ (h^3 + 3h^2(t - t_{i-1}) + 3h(t - t_{i-1})^2 - 3(t - t_{i-1})^3)/h^3, & if\ t \in [t_{i-1}, t_i] \\ (h^3 + 3h^2(t_{i+1} - t) + 3h(t_{i+1} - t)^2 - 3(t_{i+1} - t)^3)/h^3, & if\ \in [t_i, t_{i+1}] \\ (t_{i+2} - t)^3/h^3, & if\ t \in [t_{i+1}, t_{i+2}] \\ 0, & otherwise. \end{cases} \tag{5}$$

$$S(t) = \sum_{i=-2}^{n+2} c_i B_i(t). \tag{6}$$

In the case of second kind integral equation (1), by using cubic B-spline (5) we can approximate the solution and also we can approximate the integrand by Newton- Cotes type methods. When n is even then the Simpson rule can be used and when n is odd we have to use the three-eighth rule,

$$\sum_{i=-2}^{n+2} c_i B_i(x_j) = y(x_j) + h \sum_{i=-1}^{n+1} [w_{i,j} k(x_j, t_i, \sum_{k=-2}^{n+2} c_k B_k(t_i))], j = -1, \ldots, n+1. \quad (7)$$

By solving the system (7) we obtain the vector c_i and also we suppose $c_{-2} = c_{n+2} = 0$ in order to have the cubic B-spline relations, then by substituting c_i in (6) we can obtain an approximate solution for (1).

3. Error analysis: convergence of the approximate solution

To study the convergence analysis, first we need to recall the following basic theorem in [9].

Remark 1. The most immediate error analysis for spline interpolant S to a given function f defined on an interval $[a, b]$ follows from the second integral relations. Throughout our discussion
$\pi : a = t_0 < t_1 < \ldots < t_{n+2} = b$, is partition in $[a, b]$ and $h = (b - a)/(n + 2)$ is the mesh of our partition.

If $f \in C^4[a, b]$, then $\|D^j(f - s)\|_\infty \leqslant \gamma h^{4-j}, j = 0, 1, 2, 3, 4$ where

$\|f\|_\infty = \max_{0 \leqslant i \leqslant n+2} \sup_{t_{i-1} \leqslant t \leqslant t_i} |f(t)|$ and D^j the $j-th$ derivative (see [10], $P.$112).

The numerical method is said to be convergent if the solution of the approximating set of equations converges to the solution of the exact problem as the step length h tends to zero. Consider the equation

$$f(x) = y(x) + \int_a^b k(x, t, f(t))dt, \qquad a \leqslant x \leqslant b. \quad (8)$$

And suppose that at $x = x_i$, where $z_j = a + jh,\ j = 0, \ldots, n+2,\ x_i = t_i = z_{i+1},\ i = -1, \ldots, n + 1$. We know the quadrature formula at $x = x_i, i = 0, \ldots, n + 2$ is

$$\int_a^b k(x_i, t, f(t))dt = h \sum_{j=-1}^{n+1} w_{ij} k(x_i, t_j, f(t_j)) + E_{i,t}(k(x_i, t, f(t)), i = -1, \ldots, n + 1. \quad (9)$$

By substituting (9) in (8) we have

$$f(x_i) = y(x_i) + h \sum_{j=-1}^{n+1} w_{ij} k(x_i, t_j, f(t_j)) + E_{i,t}(k(x_i, t, f(t)), i = -1, \ldots, n+1. \quad (10)$$

And the corresponding approximating function equations are

$$S(x_i) = y(x_i) + h\sum_{j=-1}^{n+1} w_{ij}k(x_i, t_j, S(t_j)), i = -1, \dots, n+1. \tag{11}$$

Thus we have

$$\begin{aligned} f(x_i) - S(x_i) = h\sum_{j=-1}^{n+1} w_{ij}[k(x_i, t_j, f(t_j)) - k(x_i, t_j, S(t_j))] \\ + E_{i,t}(k(x_i, t, f(t)),\ i = -1, \dots, n+1. \end{aligned} \tag{12}$$

Now if the kernel function k satisfies a Lipschitz condition in its third argument with Lipschitz constant L and $e_i = f(x_i) - S(x_i)$, it follows that

$$|e_i| \leqslant hL\sum_{j=-1}^{n+1} |w_{ij}||e_j| + |E_{i,t}(k(x_i, t, f(t))|,\ i = -1, \dots, n+1. \tag{13}$$

let $w = \max_{i,j}|w_{i,j}|$ and $e = \max_{0\leqslant i\leqslant n+2}|e_i|$, then for sufficiently small h (see [9]).

$$|e_i| \leqslant \ \{(hLw)\sum_{j=-1}^{n+1} |e_j| + |E_{i,t}(k(x_i, t, f(t))|\}/(1 - hLw), i = -1, \dots, n+1. \tag{14}$$

$$\begin{aligned} |e_i| \leqslant \Big[\{|E_{i,t}(k(x_i, t, f(t)))| + hLw(n+3)e\} \\ \exp\{wLih/(1 - hLw)\}\Big]/(1 - hLw), i = -1, \dots, n+1. \end{aligned} \tag{15}$$

Since by assumption both the quadrature error and the function approximate error are zero in the limit, it follows that $lim_h|e_i| = 0$. We may write equivalently $|e_i| = \mathcal{O}(h^{p+1}) + \mathcal{O}(h^{q+1})$, where the error in the quadrature rule is $\mathcal{O}(h^{p+1}) = \mathcal{O}(h^4)$ and the error in the function approximate is $\mathcal{O}(h^{q+1}) = \mathcal{O}(h^5)$.
If we set

$$r = min(p, q) = min(3, 4) \tag{16}$$

then we say the quadrature rule is convergent of the order $r = 3$.

4. Numerical examples

To compare our computed results and justify the accuracy and efficiency of our presented method we consider the following examples which are considered by [1, 2, 3, 4]. The solution of the given examples are obtained for different values of n.

The RMS errors in the solutions,

$$E = ((\sum_{i=-1}^{n+1}[f(x_i) - S(x_i)]^2)/(n+2))^{1/2}, \tag{17}$$

are computed by our purposed method where $f(x)$ is the exact solution and $S(x)$ is the approximated solution of integral equation.

Example 1. Consider the problem

$$f(x) = e^{x+1} - \int_0^1 e^{x-2t} f(t)^3 dt, \qquad 0 \leqslant x \leqslant 1. \tag{18}$$

with the exact solution $f(x) = e^x$.

Example 2. Consider the problem

$$f(x) = e \times x + 1 - \int_0^1 (x+t)e^{f(t)} dt, \qquad 0 \leqslant x \leqslant 1. \tag{19}$$

with the exact solution $f(x) = x$.

Example 3. Consider the problem

$$f(x) = \sin(\pi x/2) - 2xln3 + \int_0^1 (4xt + \pi x \sin(\pi t))(1/(f(t)^2 + t^2 + 1))dt, 0 \leqslant x \leqslant 1 \tag{20}$$

with the exact solution $f(x) = \sin(\pi x/2)$.

Example 4. Consider the problem

$$f(x) = 1 - 5x/12 + \int_0^1 xtf(t)^2 dt, \qquad 0 \leqslant x \leqslant 1 \tag{21}$$

with the exact solution $f(x) = 1 + x/3$.

We solved these examples by our presented method (7). We solved example 1 with $h = 1/10$ to compare our results with results in [2, 4].The absolute errors in particular points are tabulated in table 1 which shows that our method is more accurate in comparison with [2, 4]. And we solved example 2 with $n = 3$. The absolute errors in particular points are tabulated in table 2 which shows that our method in comparison with method in [1, 3] is more accurate. And we solved examples $3, 4$ with $h = 1/10$. The absolute errors in particular points are tabulated in table 3. And also we solved problems $1, 2, 3, 4$ with different values of $n = 10, 30, 50, 100, 150, 200$. The RMS errors in the solutions are tabulated in table 4 which shows that our method can be applied for large values of n, but the maximum n which has been used by the existing methods are only $n = 10$, moreover our method is accurate and stable for different values of n.

Table 1. The errors $|E|$ in solution of examples 1 at particular points for $h = 1/10$.

x	Method 0f [2]	Method of [2]	Method of [4]	Our Method
0			1.80(-5)	3.04(-7)
0.1	5.92(-3)	2.04(-3)	1.30(-5)	3.40(-7)
0.2	1.15(-3)	3.29(-3)	1.20(-5)	3.79(-7)
0.3	1.00(-2)	8.69(-3)	1.01(-5)	4.24(-7)
0.4	1.98(-2)	1.69(-2)	2.34(-5)	4.74(-7)
0.5	1.92(-1)	1.86(-2)	1.89(-5)	5.29(-7)
0.6	9.77(-3)	1.17(-2)	2.80(-5)	5.92(-7)
0.7	1.81(-3)	2.92(-3)	1.81(-5)	6.61(-7)
0.8	1.58(-2)	8.08(-3)	1.90(-5)	7.39(-7)
0.9	3.27(-2)	2.16(-2)	3.03(-5)	3.81(-7)
1			4.92(-5)	8.26(-7)

Table 2. The errors $|E|$ in solution of examples 2 at particular points for $n = 3$.

x	Method of [3]	Method of [1]	Our Method
0	0.10(-5)	0.2(-2)	0.72(-4)
0.2	0.32(-3)	0.1(-1)	0.62(-4)
0.4	0.25(-3)	0.2(-1)	0.52(-4)
0.6	0.20(-3)	0.1(-1)	0.42(-4)
0.8	0.17(-3)	0	0.32(-4)
1	0.17(-3)	0.1(-3)	0.22(-4)

Table 3. The errors $||E||$ in solution of example 3 and 4 at particular points of $h = 1/10$.

x	Example 3	Example 4
0	0	0
0.1	1.35697(-5)	0
0.2	2.71394(-5)	2.22045(-16)
0.3	4.07091(-5)	2.22045(-16)
0.4	5.42788(-5)	2.22045(-16)
0.5	6.78485(-5)	2.22045(-16)
0.6	8.14182(-5)	4.44089(-16)
0.7	9.49879(-5)	2.22045(-16)
0.8	1.08558(-4)	4.44089(-16)
0.9	1.22127(-4)	4.44089(-16)
1	1.35697(-4)	6.66134(-16)

Table 4. The RMS errors for examples 1, 2, 3, and 4 for different values of n.

n	10	30	50	100	150	200
Example 1	6.9(-14)	4.1(-17)	1.1(-18)	6.7(-21)	3.4(-22)	4.0(-23)
Example 2	3.5(-12)	2.1(-15)	5.5(-17)	3.5(-19)	1.7(-20)	2.0(-21)
Example 3	5.2(-09)	2.7(-12)	6.9(-14)	4.3(-16)	2.1(-17)	2.5(-18)
Example 4	1.6(-31)	1.1(-31)	5.2(-31)	2.1(-31)	5.0(-31)	1.9(-31)

5. Conclusions

We have shown that the approximations to Fredholm integral equations of the second kind can be obtained by using certain simple numerical quadrature rule and collocation spline. Our computed results by the suggested method are compared with the methods in [1, 2, 3, 4] and also we verified that the presented method can be applied with large number of n. Our method is stable because when h is decreasing the error in the solution for our method is also decreasing.

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