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# Penalty Method for an Unilateral Contact Problem with Coulomb's Friction for Locking Materials

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**Abstract.**In this work, we study a unilateral contact problem with non local friction of Coulomb between a locking material and a rigid foundation. In the first step, we present the mathematical model for a static process, we establish the variational formulation in the form of a variational inequality and we prove the existence and uniqueness of the solution. In the second step, using the penalty method we introduce the penalty numerical problem in the form of variational equality where we replace the law behavior and the law contact of Signorini. Then we show the convergence of the continuous penalty solution as the penalty parameter tends to infinity. Then, the analysis of the finite element discretized penalty method is carried out.

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## 1. Introduction

Locking material is material which is deformed under the effect of an external force and the deformation stops once it reaches a certain value "$M$". After that, the material can't be deformed any further whatever the force. As long as the deformation remains bounded, the material is elastic. That's to say once we stop exercising any external force on it, it returns back to its initial physical shape. The variational

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problems encountred in theory of locking materials introduced by Prager and developed on 1985 by F. Demengel and P. Suquet ([3, 4]). In a first step, they are interested in the dual principles governing the equilibrium state of an elastic-locking material. They established the inf-sup equality by means of a penalty method. In a second step, they introduced and discussed the locking limit analysis problem. This problem allows one to determine which displacements can be imposed on a locking body before complete locking. Locking materials are hyperelastic materials for which the strain tensor is constrained to stay in some convex set. We denote by B this convex set (B(x) in the case of a nonhomogeneous material) and we assume that a normality rule holds true in the strain space. The resulting constitutive law reads as follows [11–13]. In this work, we study a problem for a static process of unilateral contact with non local friction of Coulomb between a locking material and a rigid foundation. Here, we consider a mathematical model which describes the contact with non local friction (Coulomb) between a locking material and a rigid foundation, within the framework of small deformations theory. The material's behavior is modeled with a non-linear elastic-locking constitutive law. The contact is described with the Signorini contact conditions. In the first step, we have formulated the mathematical problem as a variational inequality and we show the existence of a unique solution. In the second step, we try to estimate the solution. Because of a nonlinear contact condition and non differentiable behavior law, the penalty method is employed and the convergence analysis of the method in this case of non-linear elastic-locking is established. Using the numerical approximation of the solution, we analyze both the continuous and discrete problems. We limit the analysis to a conformal discretization with piecewise linear finite elements. We show the theoretical convergence of the penalty method.

## 2. Setting of the problem

### 2.1 *The contact problem*

In this section we describe the problem of unilateral contact with Coulomb's friction between a locking body and a rigid foundation.
The physical setting is the following : we consider a locking body which initially occupies an open bounded domain $\Omega \subset R^d$, $d = 2, 3$ with a sufficiently smooth boundary $\partial\Omega = \Gamma$. The body is acted upon by a volume forces of density $f_0$. It is also constrained mechanically on the boundary. To describe these constraints we decompose $\Gamma$ into three mutually disjoint open parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$, on the one hand, and a partition of $\Gamma_D \cup \Gamma_N$ on the other hand, such that $meas(\Gamma_D) > 0$. The body is clamped on $\Gamma_D$ and a surface tractions of density $f_2$ act on $\Gamma_N$. On $\Gamma_C$ the body may come into contact with a rigid obstacle, the so called foundation. The indices $i$, $j$, $k$, $l$ run between 1 and $d$. The summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable, e.g., $u_{i,j} = \partial u_i/\partial x_j$. Everywhere below we use $S^d$ to denote the space of second order symmetric tensors on $R^d$ while " $\cdot$ " and $\|\cdot\|$ will represent the inner product and the Euclidean norm on $R^d$ and $S^d$, that is $\forall u,\, v \in R^d$, $\forall \sigma,\, \tau \in S^d$,

$$u \cdot v = u_i \cdot v_i, \quad \|v\| = (v \cdot v)^{\frac{1}{2}}, \quad \text{and} \quad \sigma \cdot \tau = \sigma_{ij} \cdot \tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}.$$

We denote by $u : \Omega \to R^d$ the displacement field, by $\sigma : \Omega \to S^d$, $\sigma = (\sigma_{ij})$ the stress tensor. We shall adopt the usual notations for normal and tangential components of displacement vector and stress : $v_\nu = v \cdot \nu$, $\quad v_\tau = v - v_\nu \nu$, $\quad \sigma_\nu =$

$(\sigma\nu)\cdot\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$, where $\nu$ denote the outward normal vector on $\Gamma$. Moreover, let $\varepsilon(u) = (\varepsilon_{ij}(u))$ denote the linearized strain tensor given by $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j}+u_{j,i})$. Under the previous assumption, the classical model for this process is the following:

***Problem P.*** Find a displacement field $u : \Omega \to R^d$, a stress field $\sigma : \Omega \to S^d$ such that

$$\sigma(u) \in \mathcal{A}\varepsilon(u) + \partial(I_M(\varepsilon(u))) \qquad \text{in } \Omega, \tag{1}$$

$$\operatorname{Div}\sigma(u) + f_0 = 0 \qquad \text{in } \Omega, \tag{2}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{3}$$

$$\sigma\nu = g \qquad \text{on } \Gamma_N. \tag{4}$$

On the contact surface $\Gamma_C$, we consider

$$\sigma_\nu(u) \leqslant 0, \qquad u_\nu \leqslant 0, \quad \sigma_\nu u_\nu = 0, \qquad \text{on } \Gamma_C, \tag{5}$$

$$\begin{cases} |\sigma_\tau(u)| \leqslant \mu(\|u_\tau\|)|R\sigma_\nu(u)| \\ |\sigma_\tau(u)| < \mu(\|u_\tau\|)|R\sigma_\nu(u)| \Longrightarrow u_\tau = 0. \\ \sigma_\tau(u) = -\mu(\|u_\tau\|)|R\sigma_\nu(u)|\frac{u_\tau}{\|u_\tau\|} \Longrightarrow u_\tau \neq 0 \end{cases} \qquad \text{on } \Gamma_C. \tag{6}$$

With $I_M$ is the indicator function of the set $B = \{\xi \in \mathbb{R}^{d^2} / |\xi| \leqslant M\}$

$$\begin{cases} I_M(\xi) = 0 & \text{if } \xi \in B \\ I_M(\xi) = +\infty & \text{otherwise.} \end{cases}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \overline{\Omega}$. Equation (1) represents the behavior law of the material in which $\mathcal{A}$ denotes the elasticity operator. Equation (2) represents the equilibrium equation for the stress displacement fields. Relations (3) and (4) are the displacement and traction boundary conditions, respectively. The unilateral boundary conditions (5) represent the Signorini law and the (6) represents the Coulomb's friction of the unilateral contact.

## 2.2 *Weak formulation of P*

To present the variational formulation of Problem $P$ we need some additional notation and preliminaries. We start by introducing the spaces

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d,$$

$$\mathcal{H} = \{\tau = (\tau_{ij}) \,|\, \tau_{ij} = \tau_{ji} \in L^2(\Omega)\}, \quad \mathcal{H}_1 = \{\sigma \in \mathcal{H} \,|\, \operatorname{Div}\sigma \in H\}.$$

These are real Hilbert spaces endowed with the inner products

$$(u,v)_H = \int_\Omega u_i v_i \, dx, \quad (u,v)_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},$$

$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, dx, \quad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (\operatorname{Div}\sigma, \operatorname{Div}\tau)_{\mathcal{H}},$$

and the associated norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $v \in H_1$, we also use the notation $v$ to denote the trace $\gamma v$ of $v$ on $\Gamma$. Let $H_\Gamma^{'}$ be the dual of $H_\Gamma$ and let $\langle \cdot, \cdot \rangle_\Gamma$ denote the duality pairing between $H_\Gamma^{'}$ and $H_\Gamma$. For every $\sigma \in \mathcal{H}_1$, $\sigma\nu$ can be defined as the element in $H_\Gamma^{'}$ which satisfies

$$\langle \sigma\nu, \gamma v\rangle_\Gamma = (\sigma, \varepsilon(v))_{\mathcal{H}} + (\mathrm{Div}\,\sigma, v)_H, \quad \forall v \in H_1. \tag{7}$$

Moreover, If $\sigma$ is continuously differentiable on $\overline{\Omega}$, then

$$\langle \sigma\nu, \gamma v\rangle_\Gamma = \int_\Gamma \sigma\nu \cdot v \; da \tag{8}$$

for all $v \in H_1$, where $da$ is the surface measure element. Let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $V^{'}$ and $V$.

Keeping in mind the boundary condition (5), we introduce the closed subspace of $H_1$ defined by

$$V = \{v \in H_1 \,|\, v = 0 \text{ on } \Gamma_D\},$$

and $K$ be the set of admissible displacements

$$K = \{v \in V \,|\, v_\nu \leqslant 0 \text{ on } \Gamma_C\},$$

and the closed convex

$$K' = \{v \in V \,| \quad , |\varepsilon(v)| \leqslant M \text{ a.e. on } \Omega\}.$$

Since $meas(\Gamma_D) > 0$ and Korn's inequality (see, e.g., [10]) holds, then

$$\|\varepsilon(v)\|_{\mathcal{H}} \geqslant c_k \|v\|_{H_1}, \quad \forall v \in V \tag{9}$$

where $c_k > 0$ is a constant which depends only on $\Omega$ and $\Gamma_D$. Over the space $V$ we consider the inner product given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad \|u\|_V = (u, u)_V^{\frac{1}{2}}, \tag{10}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (9) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on $V$. Therefore $(V, \|\cdot\|_V)$ is a Hilbert space. Moreover, by the Sobolev trace theorem, (9) and (10) there exists a constant $c_0 > 0$ which only depends on the domain $\Omega$, $\Gamma_C$ and $\Gamma_D$ such that

$$\|v\|_{L^2(\Gamma)^d} \leqslant c_0 \|v\|_V, \quad \forall v \in V. \tag{11}$$

As usual, we denote by $(H^s(\Omega))^d, s \in R, d = 1, 2, 3,$ the Sobolev spaces in one, two or three space dimensions. The Sobolev norm of $(H^s(\Omega))^d$ (dual norm if $s < 0$) is denoted by $\|\cdot\|_{s,\Omega}$ and we keep the same notation when $d = 1, 2$ or 3. In the study of problem (1)-(5) we will need to suppose that:

- $(h_1)$ The elasticity operator $\mathcal{A}$ is the fourth order and symmetric;
- $(h_2)$ $\mathcal{A}\xi = (a_{ijkl}\xi_{kl})$;
- $(h_3)$ $a_{ijkl} \in L^\infty(\Omega)$;
- $(h_4)$ there exists $\alpha > 0$ such that $a_{ijkl}(x)\xi_k\xi_l \geqslant \alpha\|\xi\|^2, \;\; \forall \xi \in S^d,$ a.e. $x \in \Omega$.

The coefficient of friction $\mu$ satisfies:

$$\begin{cases} \text{(a) } \mu : \Gamma_C \times \mathbb{R}^+ \to \mathbb{R}^+ : \\ \text{(b) There exists } L_\mu > 0 \text{ such that for all } u, v \in \mathbb{R}^+, \text{ a.e. } x \in \Gamma_C; \\ \quad |\mu(x,u) - \mu(x,v)| \leqslant L_\mu \|u - v\| \\ \text{(c) The mapping } x \to \mu(x,u) \text{ is Lebesgue measurable on } \Gamma_C, \forall u \in \mathbb{R}^+; \\ \text{(d) There exists } \mu^* > 0 \text{ such that} \\ \quad \mu(x,u) \leqslant \mu^* \quad \forall u \in \mathbb{R}^+, \text{ a.e. } x \in \Gamma_C. \end{cases} \tag{12}$$

Next, we use Riesz's representation theorem, consider the elements $f \in V$, given by

$$(f, v)_V = \int_\Omega f_0 \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, da \quad \forall v \in V. \tag{13}$$

We suppose that the mapping $R : H'_{\Gamma_C} \longrightarrow L^\infty(\Gamma_C)$ is linear and continus.
We define the mapping $j : V \times V \longrightarrow \mathbb{R}$ by

$$j(u, v) = \int_{\Gamma_C} \mu(\|u_\tau\|)|R\sigma_\nu(u)|\|v_\tau\| \, da \qquad \forall v \in V. \tag{14}$$

Keeping in mind assumption $(h)$ it follows that the integral in (13) is well-defined. Using Grenn's formula (7) and (8) it is straightforward to see that if $(u, \sigma)$ are sufficiently regular functions satisfying (1)-(6), then

$$(\sigma(u), \varepsilon(v))_{\mathcal{H}} - \langle \sigma_\nu(u), v_\nu \rangle_{\Gamma_C} - \langle \sigma_\tau(u), v_\tau \rangle_{\Gamma_C} = (f, v)_V, \forall v \in V, \tag{15}$$

$$(\mathcal{A}\varepsilon(u), \varepsilon(v))_{\mathcal{H}} + \langle Z(\varepsilon(u)), \varepsilon(v) \rangle - \langle \sigma_\nu(u), v_\nu \rangle_{\Gamma_C} - \langle \sigma_\tau(u), v_\tau \rangle_{\Gamma_C} = (f, v)_V, \forall v \in V, \tag{16}$$

with

$$Z(\varepsilon(u)) \in \partial I_M(\varepsilon(u)).$$

We define

$$a(u, v) = (\mathcal{A}\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \quad \forall u, v \in V.$$

From the previous assumptions, we obtain that $a(.,.)$ is bilinear, symmetric, $V-$elliptic and continuous on $V \times V$.
From convexity of $I_M$ we get :

$$\forall v \in K' \quad \langle Z(\varepsilon(u)), \varepsilon(v - u) \rangle \leqslant I_M(\varepsilon(v)) - I_M(\varepsilon(u)) = 0 \tag{17}$$

and

$$\forall v \in K \quad \langle \sigma_\nu(u), v_\nu \rangle_{\Gamma_C} \geqslant 0. \tag{18}$$

Using (5), (17) and (18) we obtain the weak formulation as a variational inequality.

***Problem PV.*** Find a displacement field $u : \Omega \to R^d$ that:

$$\begin{cases} u \in K \cap K', \\ a(u, v-u) + j(u,v) - j(u,u) \geqslant (f, v-u)_V, \forall v \in K \cap K'. \end{cases} \tag{19}$$

## 3. Setting the penalty problem

### 3.1 *The penalty problem*

Applying the penalty method consists on replacing the behavior law and the Signorini condition. We define:

$$\forall n \in N^*,\, \epsilon > 0 \quad u^\epsilon_{n,\nu} = u^\epsilon_n \nu.$$

In the penalty problem, we replace $I_M(\varepsilon(u^\epsilon_n))$ by $n[(|\varepsilon(u^\epsilon_n)| - M)^+]^2$ and $\sigma_\nu(u^\epsilon_n)$ by $-\frac{1}{\epsilon}[u^\epsilon_{n,\nu}]^+$, with $n \in N^*$ and $\epsilon > 0$ the penalty parameters.

We recall also that for all $a \in \mathbb{R}$, $a^+ = a$ if $a \geqslant 0$ and $a^+ = 0$ if $a \leqslant 0$.
Because $j$ isn't differentiable, we regularize $j$ by $j_\epsilon$ given by:

$$j_\epsilon(u,v) = \int_{\Gamma_C} \mu(\|u_\tau\|)|R\sigma_\nu(u)|\Psi_\epsilon(v)\, da, \quad \forall u, v \in V \tag{20}$$

where

$$\Psi_\epsilon(v) = \sqrt{\|v\|^2 + \epsilon^2} \quad \forall v \in V. \tag{21}$$

We denote :

$$\langle j'_\epsilon(u^\epsilon_n, v), w\rangle = \int_{\Gamma_C} \mu(\|u^\epsilon_{n\tau}\|)|\mathcal{R}\sigma_\nu(u^\epsilon_n)| \frac{v_\tau w_\tau}{\sqrt{\|v_\tau\|^2 + \varepsilon^2}} \quad \forall v, w \in V. \tag{22}$$

We denote

$$\mathcal{R}_n(\tau) = 2n[(|\tau| - M)^+]\frac{\tau}{|\tau|} \quad \forall \tau \in \mathbb{S}^d \tag{23}$$

and

$$\langle [u_\nu]^+, v_\nu\rangle_{\Gamma_C} = \int_{\Gamma_C} [u_\nu]^+ v_\nu da. \tag{24}$$

Using the Green's formula we obtain the variational formulation:

***Problem $PV^\epsilon_n$.*** Find a displacement field $u^\epsilon_n \in V$ such that :

$$a(u^\epsilon_n, v) + (\mathcal{R}_n(\varepsilon(u^\epsilon_n)), \varepsilon(v))_{\mathcal{H}} + \langle \tfrac{1}{\epsilon}[u^\epsilon_{n,\nu}]_+, v_\nu\rangle_{\Gamma_C} + \langle {j_n}'(u^\epsilon_n, u^\epsilon_n), v\rangle = (f, v)_V, \quad \forall v \in V. \tag{25}$$

### 3.2 *Finite element setting*

We suppose that $\Omega$ is a polygonal domain $(d = 2)$ or polyhedral $(d = 3)$. We approche the space $V$ by the finite dimension $V^h$ where $h > 0$ is a spatial discretization parameter

$$V^h = \Big\{ v \in \mathcal{C}(\overline{\Omega})^d; \quad v_{|T_h} \in \mathbb{P}_1(\Omega)^d, \quad v = 0 \text{ sur } \Gamma_D \}$$

with $T = \{T_h\}$ is the triangulations set of $\Omega$ and $\mathbb{P}_1(\Omega)^d$ is the set of polynomials of degree less than 1.
In this subsection we suppose that $\epsilon = \frac{1}{n}$ and $j_n = j_\epsilon$.
The discrete problem of $PV_n$ is given by:

***Problem $PV_n^h$.*** Find the displacement $u_n^h \in V^h$ such that:

$$a(u_n^h, v^h) + (\mathcal{R}_n(\varepsilon(u_n^h)), \varepsilon(v^h))_{\mathcal{H}} + \langle n[u_{n,\nu}^h]_+, v_\nu^h\rangle_{\Gamma_C} + \langle j_n^{'}(u_n^h, u_n^h), v\rangle = (f, v^h)_V \quad \forall v^h \in V^h \tag{26}$$

## 4. Main results and proofs

### 4.1 *Existence and uniqueness of the solution*

THEOREM 4.1 *Assume that* $(h_1) - (h_4)$ *hold. There exists* $L^* > 0$ *such that if* $\frac{L_\mu + \mu^*}{\alpha} < L^*$, *then the Problem* $(PV)$ *has a unique solution* $u$.

*Proof*
The mapping $a(.,.)$ is bilinear, symmetric, continuous and $V-$elliptic and we know that $K \cap K'$ is a closed convex and not empty.
The functional $j$ satisfies: in one hand $\forall u \in V$, $j(u,.)$ is convex and l.s.c on $V$ and in the second hand $\forall u_1, u_2 \in V$ and we have

$$\big|j(u_1,u_2) - j(u_1,u_1) + j(u_2,u_1) - j(u_2,u_2)\big| =$$
$$\Big|\int_{\Gamma_C} (\mu(\|u_{1\tau}\|)|R\sigma_\nu(u_1)| - \mu(\|u_{2\tau}\|)|R\sigma_\nu(u_2)|)(\|u_{2\tau}| - |u_{1\tau}\|)\, da\Big| =$$
$$\Big|\int_{\Gamma_C} (\mu(\|u_{1\tau}\|) - \mu(\|u_{2\tau}\|))|R\sigma_\nu(u_1)| + \mu(\|u_2\|)(|R\sigma_\nu(u_1)| - |R\sigma_\nu(u_2)|)(\|u_{2\tau}\| - \|u_{1\tau}\|)\, da\Big| \leqslant$$
$$\int_{\Gamma_C} L_\mu(\|u_{2\tau}\| - \|u_{1\tau}\|)^2 + \mu^*\big|R\sigma_\nu(u_1) - R\sigma_\nu(u_2)\big|\big|\|u_{2\tau}\| - \|u_{1\tau}\|\big|\, da.$$

Applying the continuity of $R$ and (11), we get

$$\big|j(u_1,u_2) - j(u_1,u_1) + j(u_2,u_1) - j(u_2,u_2)\big| \leqslant c_0^2(L_\mu + c_1\mu*)\|u_2 - u_1\|_V^2$$

where $c_1$ is a positive constant depending of $R$ and $\sigma_\nu$. Then there exists $c_2 > 0$ such that

$$\big|j(u_1,u_2) - j(u_1,u_1) + j(u_2,u_1) - j(u_2,u_2)\big| \leqslant c_2(L_\mu + \mu*)\|u_2 - u_1\|_V^2.$$

Let $L^* = \frac{1}{c_2}$. If $\frac{L_\mu+\mu*}{\alpha} < L^*$ then $c_2(L_\mu + \mu*) < \alpha$. In this case the problem has a unique solution. ∎

### 4.2 *Existence and uniqueness of penalty problem*

THEOREM 4.2 *Assume that* $(h_1) - (h_4)$ *hold. For any* $n \in N^*$ *and* $\epsilon > 0$

1) *the penalty problem* $(PV_n^\epsilon)$ *has at least one solution.*
2) *there exists* $L^* > 0$ *such that if* $\frac{L_\mu+\mu^*}{\alpha} < L^*$. *Then the Problem* $(PV_n^\epsilon)$ *has a unique solution* $u_n^\epsilon$.

To prove this theorem,
we simplify the problem by defining the following operators $T_n$, $S^\epsilon$ and $B_n^\epsilon$ by:

$$\begin{cases} (T_n v, w)_V = (\mathcal{R}_n(\varepsilon(v)), \varepsilon(w))_{\mathcal{H}} \quad \forall v, w \in V. \\ (S^\epsilon v, w)_V = \frac{1}{\epsilon}\langle [v_\nu]^+, w_\nu\rangle_{\Gamma_C} = \frac{1}{\epsilon}\int_{\Gamma_C} v_\nu^+ w_\nu da \quad \forall v, w \in V. \\ (B_n^\epsilon v, w)_V = a(v,w) + (T_n v, w)_V + (S^\epsilon v, w)_V \quad \forall v, w \in V. \end{cases}$$

The problem $PV_n^\epsilon$ is written as follow:

***Problem*** $\boldsymbol{PV_n^\epsilon}$**.** Find the displacement $u_n^\epsilon : \Omega \to \mathbb{R}^d$ such that:

$$(B_n^\epsilon u_n^\epsilon, v)_V + \langle j_\epsilon^{'}(u_n^\epsilon, u_n^\epsilon), v\rangle = (f, v)_V \quad \forall v \in V. \tag{27}$$

The following lemmas will be useful

LEMMA 4.3

$$\begin{aligned} &(1)\ \mathrm{T_n}\ \textit{is the Lipshitz continuous}: \\ &(T_n u_1 - T_n u_2, v)_V \leqslant 4n\, \|u_1 - u_2\|_V\, \|v\|_V \quad \forall u_1, u_2, v \in V \\ &(2)\ \mathrm{T_n}\ \textit{is monotone}\ : \\ &(T_n u_1 - T_n u_2, u_1 - u_2)_V \geqslant 0 \quad \forall u_1, u_2 \in V \end{aligned} \tag{28}$$

*Proof*

1)

$$\begin{aligned} R = (T_n u_1 - T_n u_2, v)_V &= 2n\int_\Omega \left( [|\varepsilon(u_1)| - M]^+ \frac{\varepsilon(u_1)}{|\varepsilon(u_1)|} - [|\varepsilon(u_2)| - M]^+ \frac{\varepsilon(u_2)}{|\varepsilon(u_2)|} \right) \varepsilon(v)\ da \\ &= 2n\int_\Omega \left( \frac{[|\varepsilon(u_1)|-M]^+ \varepsilon(u_1)|\varepsilon(u_2)| - [|\varepsilon(u_2)|-M]^+ \varepsilon(u_2)|\varepsilon(u_1)|}{|\varepsilon(u_2)||\varepsilon(u_1)|} \right) \varepsilon(v)\ da. \end{aligned}$$

There are three cases:
- If $|\varepsilon(u_1)| \leqslant M$, $|\varepsilon(u_2)| \leqslant M$ then $R = 0$

- If $|\varepsilon(u_1)| > M, \quad |\varepsilon(u_2)| \leqslant M$ hence

$$\begin{aligned}
|R| &= 2n\left|\int_\Omega (|\varepsilon(u_1)| - M)\frac{\varepsilon(u_1)}{|\varepsilon(u_1)|}\varepsilon(v)\ da\right| \\
&\leqslant 2n\int_\Omega (|\varepsilon(u_1)| - M)\frac{|\varepsilon(u_1)|}{|\varepsilon(u_1)|}|\varepsilon(v)|\ da \\
&= 2n\int_\Omega (|\varepsilon(u_1)| - M)|\varepsilon(v)|\ da \\
&\leqslant 2n\int_\Omega (|\varepsilon(u_1)| - |\varepsilon(u_2)|)|\varepsilon(v)|\ da \\
&\leqslant 2n\int_\Omega |\varepsilon(u_1) - \varepsilon(u_2)||\varepsilon(v)|\ da \\
&\leqslant 2n\left\|u_1 - u_2\right\|_V \left\|v\right\|_V.
\end{aligned}$$

- If $|\varepsilon(u_1)| > M$ and $|\varepsilon(u_2)| > M$ then

$$\begin{aligned}
|R| &= 2n\left|\int_\Omega \left((|\varepsilon(u_1)| - M)\frac{\varepsilon(u_1)}{|\varepsilon(u_1)|} - (|\varepsilon(u_2)| - M)\frac{\varepsilon(u_2)}{|\varepsilon(u_2)|}\right)\varepsilon(v)\ da\right|. \\
&= 2n\left|\int_\Omega \left((\varepsilon(u_1) - \varepsilon(u_2)) - M(\frac{\varepsilon(u_1)}{|\varepsilon(u_1)|} - \frac{\varepsilon(u_2)}{|\varepsilon(u_2)|})\right)(\varepsilon(v))\right| \\
&\leqslant 2n\int_\Omega |(\varepsilon(u_1) - \varepsilon(u_2)||(\varepsilon(v)| + 2n\left|\int_\Omega -M(\frac{\varepsilon(u_1)}{|\varepsilon(u_1)|} - \frac{\varepsilon(u_2)}{|\varepsilon(u_2)|})(\varepsilon(v))\right| \\
&\leqslant 2n\left\|u_1 - u_2\right\|_V \left\|v\right\|_V + 2n\int_\Omega M\left|\frac{\varepsilon(u_1)}{|\varepsilon(u_1)|} - \frac{\varepsilon(u_2)}{|\varepsilon(u_2)|}\right||\varepsilon(v)| \\
&\leqslant 2n\left\|u_1 - u_2\right\|_V \left\|v\right\|_V + 2n\int_\Omega |\varepsilon(u_1) - \varepsilon(u_2)|\,|\varepsilon(v)| \\
&\leqslant 4n\left\|u_1 - u_2\right\|_V \left\|v\right\|_V.
\end{aligned}$$

For all cases we find:

$$|R| \leqslant 4n\left\|u_1 - u_2\right\|_V \left\|v\right\|_V.$$

2) The mapping $\phi : \mathbb{S}^d \longrightarrow \mathbb{R};\ \xi \longmapsto n\left((|\xi| - M)^+\right)^2$ is convex function and continuously differentiable and

$$\forall \xi, \psi \in \mathbb{S}^d; \quad \langle \phi_n'(\xi), \psi\rangle = 2n\left((|\xi| - M)^+\right)\frac{\xi}{|\xi|}.\psi.$$

Hence the mapping $F : V \longrightarrow \mathbb{R};\ u \longmapsto \phi(\varepsilon(u))$ is also convex function and continuously differentiable
and

$$\forall (u, v) \in V^2; \quad \langle \phi_n'(\varepsilon(u)), \varepsilon(v)\rangle = 2n\left((|\varepsilon(u)| - M)^+\right)\frac{\varepsilon(u)}{|\varepsilon(u)|}\varepsilon(v).$$

From convexity property of $\phi_n$ we obtain that $\phi_n'$ is monotone, then $\langle \phi_n'(\varepsilon(u)) - \phi_n'(\varepsilon(v)), \varepsilon(u) - \varepsilon(v)\rangle \geqslant 0$.

■

LEMMA 4.4

$$\begin{array}{l}(1)\,\mathrm{S}^\epsilon \;\; \textit{is the Lipshitz} \;: \\ |(S^\epsilon u - S^\epsilon v, w)_V| \leqslant \frac{c_0}{\epsilon}\|u-v\|_V\|w\|_V \quad \forall u,v,w \in V. \\ (2)\,\mathrm{S}^\epsilon \;\; \textit{is monotone} \;:\; \langle \mathrm{S^\epsilon v - S^\epsilon u, v - u}\rangle \geqslant 0 \quad \forall \mathrm{v, w} \in \mathrm{V}.\end{array} \tag{29}$$

*Proof*

1)

$$\begin{aligned}\left|(S^\epsilon v - S^\epsilon u, w)_V\right| &= \frac{1}{\epsilon}\left|\langle [v_\nu]^+ - [(u)_\nu]^+, w_\nu\rangle_{\Gamma_C}\right| \\ &= \frac{1}{\epsilon}\Big|\int_{\Gamma_C}[v_\nu]^+ - [u_\nu]^+)w_\nu\Big| \\ &\leqslant \frac{1}{\epsilon}\left(\int_{\Gamma_C}\left|[v_\nu]^+ - [u_\nu]^+)\right|\left|w_\nu\right|\right),\end{aligned}$$

using the inequality $\left|[v_\nu]^+ - [u_\nu]^+\right| \leqslant \left|v_\nu - u_\nu\right|$ we obtain:

$$\left|(S^\epsilon v - S^\epsilon u, w)_V\right| \leqslant \frac{c_0}{\epsilon}\|v-u\|_V\|w\|_V.$$

2) We remark that for all $a, b \in \mathbb{R}$

$$\begin{aligned}([a]^+ - [b]^+)(a-b) &= a[a]^+ + b[b]^+ - a[b]^+ - b[a]^+ \\ &\geqslant ([a]^+)^2 + ([b]^+)^2 - 2[a]^+[b]^+ \\ &= ([a]^+ - [b]^+)^2 \geqslant 0\end{aligned}$$

■

Let $\kappa$ be a real positive number, we set

$$G = \left\{h \in L^2(\Gamma_C); \langle h, u_n^\epsilon\rangle \geqslant 0 \text{ and } \|h\|_{L^2(\Gamma_C)} \leqslant \kappa\right\}$$

where is $G$ is a closed convex.
For $h \in G$, we define the function $\widetilde{j}_h$ on $\mathcal{K}$ by:

$$\widetilde{j}_h(v) = \int_{\Gamma_C} h v_\tau \, da. \tag{30}$$

From the Riesz's representation theorem and the fact that $V$ is a real Hilbert space, there exists $f^h \in V$ such that $(f^h, v)_V = (f, v)_V - \widetilde{j}_h(v) \quad \forall v \in V$.
For any $h \in G$ we associate the following intermediate problem :

***Problem*** $\boldsymbol{PV_n^{h\epsilon}}$. Find a displacement field $u_n^{h\epsilon} : \Omega \to R^d$ such that:

$$a(u_n^{h\epsilon}, v) + (\mathcal{R}_n(\varepsilon(u_n^{h\epsilon})), \varepsilon(v))_{\mathcal{H}} + \frac{1}{\epsilon}\langle [u_{n\nu}^{h\epsilon}]_+, v_\nu\rangle_{\Gamma_C} = (f^h, v)_V \forall v \in V. \tag{31}$$

Lemma 4.5

*i) For any* $h \in G$, *the problem* $PV_n^{h\epsilon}$ *has a unique solution* $u_n^{h\epsilon}$ .
*ii) The function* $h \in G \longrightarrow u_n^{h\epsilon}$ *is the Lipshitz.*
*iii)* $\exists c > 0, \quad \|u_n^{h\epsilon}\|_V \leqslant c\|f^h\|_V$.

*Proof*

i) We show that $B_n^\epsilon$ is strongly monotone and the Lipshitz continuous:

- $\forall u, v \in V$ we have
$(B_n^\epsilon v - B_n^\epsilon u, v-u)_V = a(v-u, v-u) + (T_n v - T_n u, v-u)_V + (S^\epsilon v - S^\epsilon u, v-u)_V$.
Using lemma (4.3) and lemma (4.4) we obtain

$$(B_n^\epsilon v - B_n^\epsilon u, v-u)_V \geqslant a(v-u, v-u).$$

From the ellipticity of $a(.,.)$ we deduce that $B_n^\epsilon$ is strongly monotone.

- We show next that $B_n^\epsilon$ is the Lipshitz continuous.
From the Lipshitz continuity of $T_n$, $S^\epsilon$ and $a(.,.)$ is bilinear continuous, we deduce that $B_n^\epsilon$ is the Lipshitz continuous.
Thus, $PV_n^{h\epsilon}$ has a unique solution $u_n^{h\epsilon}$.

ii) We show now that the function $h \longrightarrow u_n^{h\epsilon}$ is the Lipshitz
Let $h_1, h_2$ two elements of $G$,

$$\begin{aligned}(B_n^\epsilon u_n^{h_1\epsilon}, v - u_n^{h_1\epsilon})_V =&(f^{h_1}, v - u_n^{h_1\epsilon}) \quad \forall v \in V\\ (B_n^\epsilon u_n^{h_2}, v - u_n^{h_2\epsilon})_V =&(f^{h_2}, v - u_n^{h_2\epsilon}) \quad \forall v \in V.\end{aligned}$$

We replace in the first inequality $v$ by $u_n^{h_2}$ and in the second $v$ by $u_n^{h_1}$ we obtain

$$\begin{aligned}(B_n^\epsilon u_n^{h_1\epsilon}, u_n^{h_2\epsilon} - u_n^{h_1\epsilon})_V =&(f_1^h, u_n^{h_2\epsilon} - u_n^{h_1\epsilon})\\ (B_n^\epsilon u_n^{h_2\epsilon}, u_n^{h_1\epsilon} - u_n^{h_2\epsilon})_V =&(f_2^h, u_n^{h_1\epsilon} - u_n^{h_2\epsilon}),\end{aligned}$$

we sum the last equalities:

$$(B_n^\epsilon u_n^{h_1} - B_n^\epsilon u_n^{h_2}, u_n^{h_1} - u_n^{h_2})_V = (f^{h_1} - f^{h_2}, u_n^{h_1\epsilon} - u_n^{h_2\epsilon})_V.$$

Using $PV_n^{h_1\epsilon}$ and $PV_n^{h_2\epsilon}$, we obtain:

$$\begin{aligned}(B_n^\epsilon u_n^{h_1} - B_n^\epsilon u_n^{h_2}, u_n^{h_1} - u_n^{h_2})_V =&\int_{\Gamma_C} (h_2 - h_1)(u_{n\tau}^{h_1\epsilon} - u_{n\tau}^{h_2\epsilon})da\\ \leqslant&\|h_1 - h_2\|_{L^2(\Gamma_C)}\|u_n^{h_1\epsilon} - u_n^{h_2\epsilon}\|_{L^2(\Gamma_C)}\\ \leqslant&c_0\|h_1 - h_2\|_{L^2(\Gamma_C)}\|u_n^{h_1\epsilon} - u_n^{h_2\epsilon}\|_V.\end{aligned}$$

We know that

$$\begin{aligned}\alpha\|u_n^{h_1\epsilon} - u_n^{h_2\epsilon}\|_V^2 \leqslant&a(u_n^{h_1\epsilon} - u_n^{h_2\epsilon}, u_n^{h_1\epsilon} - u_n^{h_2\epsilon})\\ \leqslant&(B_n^\epsilon u_n^{h_1\epsilon} - B_n^\epsilon u_n^{h_2\epsilon}, u_n^{h_1\epsilon} - u_n^{h_2\epsilon})_V,\end{aligned}$$

then

$$\|u_n^{h_1\epsilon} - u_n^{h_2\epsilon}\|_V \leqslant \frac{c_0}{\alpha}\|h_1 - h_2\|_{L^2(\Gamma_C)} \tag{32}$$

iii) We show now that $\exists c > 0, \quad \|u_n^{h\epsilon}\|_V \leqslant c\|f^h\|_V$.
Using $(B_n^\epsilon u_n^{h\epsilon}, u_n^h)_V = (f^h, u_n^{h\epsilon})$, we obtain

$$\begin{aligned}\alpha\|u_n^{h\epsilon}\|_V^2 \leqslant& a(u_n^{h\epsilon}, u_n^{h\epsilon})\\ \leqslant&(B_n^\epsilon u_n^h, u_n^{h\epsilon})_V\\ \leqslant&\|f^h\|_V\|u_n^{h\epsilon}\|_V,\end{aligned}$$

thus $\|u_n^{h\epsilon}\|_V \leqslant \frac{1}{\alpha}\|f^h\|_V$

■

LEMMA 4.6 *The mapping* $h \to u_n^{h\epsilon}$ *is weakly continuous.*

*Proof*

Let $(h_k)_k$ be a sequence converging weakly in $L^2(\Gamma_C)$ to $h$.
Using (32) we get $\|u_n^{h_k\epsilon}\|_V \leqslant \frac{c_0}{\alpha}\|h_k\|_{L^2(\Gamma_C)}$, then $(u_n^{h_k\epsilon})_k$ is bounded in $V$.

It implies that it exists a subsequence $(u_n^{h_k\epsilon})_k$ converging weakly to $\widetilde{u}_n^\epsilon$.
We have in one hand:

$$(f^{h_k}, v - u_n^{h_k\epsilon}) = (f, v - u_n^{h_k\epsilon})_V - \widetilde{j}_{h_k}(v - u_n^{h_k\epsilon}) \tag{33}$$

$$|\widetilde{j}_{h_k}(u_n^{h_k\epsilon}) - \widetilde{j}_{h_k}(\widetilde{u}_n^\epsilon)| \leqslant \|h_k\|_{L^2(\Gamma_C)}\|u_n^{h_k\epsilon} - \widetilde{u}_n^\epsilon\|_{L^2(\Gamma_C)}. \tag{34}$$

In other hand, the Sobolev's trace $\gamma : V \to L^2(\Gamma_C)$ is compact.

Then $(u_n^{h_k\epsilon})_k \longrightarrow \widetilde{u}_n^\epsilon$ as $k \longrightarrow \infty$ and

$$(f^{h_k}, v - u_n^{h_k\epsilon})_k \longrightarrow (f, v - \widetilde{u}_n^\epsilon). \tag{35}$$

We take $v = B_n^\epsilon u_n^{h_k\epsilon}$ in the equality $(B_n^\epsilon u_n^{h_k\epsilon}, v)_V = (f^{h_k}, v)_V$, we find that $\|B_n^\epsilon u_n^{h_k\epsilon}\| \leqslant \|f^{h_k}\|_V$,
it implies that there exists $L > 0$ verifying

$$\|B_n^\epsilon u_n^{h_k\epsilon}\| \leqslant L, \quad \forall k \in N.$$

From

$$(B_n^\epsilon u_n^{h_k\epsilon}, u_n^{h_k\epsilon} - \widetilde{u}_n^\epsilon)_V = (B_n^\epsilon u_n^{h_k\epsilon}, u_n^{h_k\epsilon} - v)_V + (B_n^\epsilon u_n^{h_k\epsilon}, v - \widetilde{u}_n^\epsilon)_V$$

we obtain that

$$(B_n^\epsilon u_n^{h_k\epsilon}, u_n^{h_k\epsilon} - \widetilde{u}_n^\epsilon)_V \leqslant (f^{h_k}, u_n^{h_k\epsilon} - v)_V + L\|v - \widetilde{u}_n^\epsilon\|_V$$

It results that

$$\limsup_{k\to+\infty}(B_n^\epsilon u_n^{h_k\epsilon}, u_n^{h_k\epsilon} - \widetilde{u}_n^\epsilon)_V \leqslant (f, \widetilde{u}_n^\epsilon - v)_V + L\|v - \widetilde{u}_n^\epsilon\|_V$$

Taking $v = \widetilde{u}_n^\epsilon$ we obtain that $\limsup_{k\to\infty}(B_n^\epsilon u_n^{h_k\epsilon}, u_n^{h_k\epsilon} - \widetilde{u}_n^\epsilon)_V \leqslant 0$ and then

$$(B_n^\epsilon \widetilde{u}_n^\epsilon, \widetilde{u}_n^\epsilon - v)_V \leqslant \liminf_{k\to\infty}(B_n^\epsilon u_n^{h_k\epsilon}, u_n^{h_k\epsilon} - v)_V. \tag{36}$$

We combine (35) and (36) to get

$$\begin{cases} \widetilde{u}_n^\epsilon \in V \\ (B_n^\epsilon \widetilde{u}_n^\epsilon, \widetilde{u}_n^\epsilon - v)_V \leqslant (f, \widetilde{u}_n^\epsilon - v) \quad \forall v \in V. \end{cases} \tag{37}$$

For any $\widetilde{v} \in V$ we change $v$ by $\widetilde{v} = \widetilde{u}_n^\epsilon + v$ and $\widetilde{v} = \widetilde{u}_n^\epsilon - v$ in (37) to obtain that $\widetilde{u}_n^\epsilon$ is a solution of $PV_n^\epsilon$.
We deduce that $\widetilde{u}_n^\epsilon = u_n^{h\epsilon}$ and $(u_n^{h_k\epsilon})_k \rightharpoonup u_n^{h\epsilon}$ as $k \longrightarrow +\infty$ and finally, the function $h \to u_n^{h\epsilon}$ is weakly continuous on $L^2(\Gamma_C)$.

■

LEMMA 4.7 *The mapping* $\Lambda : h \longmapsto \mu(|u_{n\tau}^{h\epsilon}|)|R\sigma_\nu(u_n^{h\epsilon})|\frac{u_{n\tau}^{h\epsilon}}{\sqrt{\|u_{n\tau}^{h\epsilon}\|^2+\epsilon^2}}$ *has a fixed point.*

*Proof*

Let $h \in G$, we recall that $\langle h, u_n^{h\epsilon}\rangle \geqslant 0$ and $\|h\|_{L^2(\Gamma_C)} \leqslant \kappa$.
Using $\langle h, u_n^{h\epsilon}\rangle \geqslant 0$ we obtain that $(f^h, u_n^{h\epsilon})_V \leqslant (f, u_n^{h\epsilon})_V$, and applying (4.3) we get $\|u_n^{h\epsilon}\|_V \leqslant \frac{1}{\alpha}\|f\|_V$.
Now we have $\|\Lambda h\|_{L^2(\Gamma_C)} \leqslant \mu^*\|\|u_{n\tau}^{h\epsilon}\||R\sigma_\nu(u_n^{h\epsilon})|\frac{u_{n\tau}^{h\epsilon}}{\sqrt{\|u_{n\tau}^{h\epsilon}\|^2+\epsilon^2}}\|_{L^2(\Gamma_C)}$.
We know that $R$ is linear continuous, then there exists $c > 0$ such that

$$\|\Lambda h\|_{L^2(\Gamma_C)} \leqslant c_0\mu^*\frac{c}{\alpha}\|f\|_V.$$

If we choose $\kappa = c_0\mu^*\frac{c}{\alpha}\|f\|_V$ then $\Lambda : G \longrightarrow G$.
Because $G$ is a closed convex not empty of $L^2(\Gamma_C)$ and $\Lambda$ is weakly continuous and using the Schauder's theorem of fixed point, we deduce that $\Lambda$ has a fixed point.

■

Now, we are ready to prove theorem (4.2).

1) Existence:
   Let $g^*$ be a fixed point of $\Lambda$. Using $\Lambda g^* = g^*$ we deduce that $u_n^{g^*\epsilon}$ is a solution of $PV_n^\epsilon$.

2) Uniqueness:
   Let $u_{1,n}^\epsilon$ and $u_{2,n}^\epsilon$ be two solutions of $PV_n^\epsilon$.
   We have for any $v \in V$

$$\begin{cases} (B_n^\epsilon u_{1,n}^\epsilon, v)_V + \langle j_\epsilon'(u_{1,n}^\epsilon, u_{1,n}^\epsilon), v\rangle = (f, v)_V \\ (B_n^\epsilon u_{2,n}^\epsilon, v)_V + \langle j_\epsilon'(u_{2,n}^\epsilon, u_{2,n}^\epsilon), v\rangle = (f, v)_V. \end{cases}$$

Then

$$(B_n^\epsilon u_{1,n}^\epsilon - B_n^\epsilon u_{2,n}^\epsilon, v)_V = \langle j_\epsilon'(u_{2,n}^\epsilon, u_{2,n}^\epsilon), v\rangle - \langle j_\epsilon'(u_{1,n}^\epsilon, u_{1,n}^\epsilon), v\rangle. \tag{38}$$

Taking $v = u^{\epsilon}_{1,n} - u^{\epsilon}_{2,n}$ in (39) we obtain

$$(B^{\epsilon}_n u^{\epsilon}_{1,n} - B^{\epsilon}_n u^{\epsilon}_{2,n}, u^{\epsilon}_{1,n} - u^{\epsilon}_{2,n})_V = \langle j^{'}_{\epsilon}(u^{\epsilon}_{2,n}, u^{\epsilon}_{2,n}), u^{\epsilon}_{1,n} - u^{\epsilon}_{2,n}\rangle - \langle j^{'}_{\epsilon}(u^{\epsilon}_{1,n}, u^{\epsilon}_{1,n}), u^{\epsilon}_{1,n} - u^{\epsilon}_{2,n}\rangle.$$

Since $j_{\epsilon}(u,.)$ is convex we obtain

$$j_{\epsilon}(u,v) - j_{\epsilon}(u,u) \geqslant \langle j^{'}_{\epsilon}(u,u), v-u\rangle \quad \forall u, v \in V. \tag{39}$$

Thus

$$\begin{aligned}
(B^{\epsilon}_n u^{\epsilon}_{1,n} - B^{\epsilon}_n u^{\epsilon}_{2,n}, u^{\epsilon}_{1,n} - u^{\epsilon}_{2,n})_V &\leqslant j_{\epsilon}(u^{\epsilon}_{1,n}, u^{\epsilon}_{2,n}) - j_{\epsilon}(u^{\epsilon}_{1,n}, u^{\epsilon}_{1,n}) + j_{\epsilon}(u^{\epsilon}_{2,n}, u^{\epsilon}_{1,n}) - j_n(u^{\epsilon}_{2,n}, u^{\epsilon}_{2,n}) \\
&= \int_{\Gamma_C} \mu(\|u^{\epsilon}_{1,n\tau}\|)(|R\sigma_\nu(u^{\epsilon}_{1,n})| - |R\sigma_\nu(u^{\epsilon}_{2,n})|)(\sqrt{\|u^{\epsilon}_{1,n\tau}\|^2 + \epsilon^2} - \sqrt{\|u^{\epsilon}_{2,n\tau}\|^2 + \epsilon^2}) \\
&\quad + \int_{\Gamma_C} |R\sigma_\nu(u^{\epsilon}_{2,n})|(\mu(\|u^{\epsilon}_{1,n\tau}\| - \mu(\|u^{\epsilon}_{2,n\tau}\|)(\sqrt{\|u^{\epsilon}_{1,n\tau}\|^2 + \epsilon^2} - \sqrt{\|u^{\epsilon}_{2,n\tau}\|^2 + \epsilon^2}) \\
&\leqslant (\mu^* c_1 c_0^2 + L_\mu \|R\sigma_\nu(u^{\epsilon}_{2,n})\|_{L^\infty(\Gamma_C)} c_0^2)\|u^{\epsilon}_{1,n} - u^{\epsilon}_{2,n}\|^2_V
\end{aligned}$$

where $c_1$ is a positive constant.
Then, there exist a positive constant $C$ such that

$$\alpha\|u^{\epsilon}_{1,n} - u^{\epsilon}_{2,n}\|^2_V \leqslant C(L_\mu + \mu^*)\|u^{\epsilon}_{1,n} - u^{\epsilon}_{2,n}\|^2_V.$$

Let $L^* = \frac{1}{C}$. If $\frac{L_\mu + \mu^*}{\alpha} < L^*$ then $u^{\epsilon}_{1,n} = u^{\epsilon}_{2,n}$.

### 4.3 *Convergence of penalty method*

In this subsection, we suppose that $\epsilon = \frac{1}{n}$, $B^{\epsilon}_n = B_n$, $S_n = S^{\epsilon}$ and $j_\epsilon = j_n$.
The problem $PV^{\epsilon}_n$ becomes:

***Problem*** $\boldsymbol{PV_n}$. Find the displacement $u_n : \Omega \to \mathbb{R}^d$ such that:

$$(B_n u_n, v)_V + \langle j^{'}_n(u_n, u_n), v\rangle = (f, v)_V \quad \forall v \in V \tag{40}$$

The following lemma will be useful

Lemma 4.8 *Let $(f_n)_n$ a sequence of positives functions which converge to $f$ weakly in $L^2(\Omega)$, and $L > 0$.*
*For $n \in N$, we define $A_n = \{x \in \Omega / f_n(x) \geqslant L\}$ a measurable set .*
*If $meas(A_n)_n \longrightarrow 0$, then $f \in L^\infty(\Omega)$ and $\|f\|_{L^\infty} \leqslant L$.*

*Proof* We consider $\varphi \in \mathcal{C}^\infty_0(\Omega)$ and $\epsilon > 0$.
For $n$ large enough we have

$$\Big|\int_{\Omega\setminus A_n} \varphi(f_n - f)\Big| \leqslant \epsilon.$$

$$\Big|\int_{\Omega\setminus A_n} \varphi f\Big| \leqslant \epsilon + \Big|\int_{\Omega\setminus A_n} \varphi f_n\Big| \leqslant \epsilon + L\int_{\Omega\setminus A_n} |\varphi|$$

From $meas(A_n) \longrightarrow 0$, then $\varphi f \chi_{A_n} \longrightarrow 0$ a.e in $\Omega$ and $|\varphi f \chi_{A_n}| \leqslant |f\varphi| \in L^1(\Omega)$.

Using the dominate convergence theorem, we obtain

$$|\int_{A_n} \varphi f| \longrightarrow 0.$$

When $\epsilon \longrightarrow 0$, we find

$$|\int_{\Omega} \varphi f| \leqslant L \int_{\Omega} |\varphi|$$

We deduce that $f \in L^\infty(\Omega)$ and $\|f\|_{L^\infty} \leqslant L$. ■

THEOREM 4.9 *Assume that $(h_1) - (h_4)$ hold and there exists $L^* > 0$ such that $\frac{L_\mu + \mu^*}{\alpha} < L^*$.*
*Let $u_n$ be the solution of $PV_n$. Then the sequence $(u_n)$ converges strongly to $u$.*

*Proof*

a) We show that $(u_n)$ converges weakly:
We have

$$(B_n u_n, u_n)_V + \langle j_n^{'}(u_n, u_n), u_n \rangle = (f, u_n)_V.$$

From $\langle j_n^{'}(u_n, u_n), u_n \rangle \geqslant 0$, the $V-$ellipticity of $B_n$ and Cauchy-shwartz inequality we get $\alpha \|u_n\|_V^2 \leqslant \|f\|_V \|u_n\|_V$ and $\|u_n\|_V \leqslant \frac{1}{\alpha}\|f\|_V$.
Then the sequence $(u_n)$ is bounded in $V$ and there exists a subsequence denoted $(u_n)$ converging weakly to $\widetilde{u} \in V$.

b) We show that $\widetilde{u} \in K$:
Since the Sobolev's trace on $L^2(\Gamma_C)$ is compact, we find that

$$(u_n) \longrightarrow \widetilde{u} \text{ strongly on } L^2(\Gamma_C) \tag{41}$$

and

$$\lim_{+\infty} \|u_{n,\nu}^+\|_{L^2(\Gamma_C)} = \|\widetilde{u}_\nu^+\|_{L^2(\Gamma_C)}.$$

We remark that $(T_n u_n, u_n)_V \leqslant (B_n u_n, u_n)_V \leqslant (f, u_n)_V \leqslant \|f\|_V \|u_n\|_V$, then

$$\int_{\Gamma_C} u_{n,\nu}^+ u_{n,\nu} \leqslant \frac{1}{n}\frac{1}{\alpha}\|f\|_V^2$$

and

$$n\|u_n\|_{L^2(\Gamma_C)}^2 \leqslant \frac{1}{\alpha}\|f\|_V^2. \tag{42}$$

Thus

$$\lim_{+\infty} \|u_{n,\nu}^+\|_{L^2(\Gamma_C)} = 0.$$

It results that $\|\widetilde{u}_\nu^+\|_{L^2(\Gamma_C)} = 0$ and $\widetilde{u}_\nu^+ = 0$ a.e on $\Gamma_C$ and $\widetilde{u}_\nu \leqslant 0$ on $\Gamma_C$. Then $\widetilde{u} \in K$.

c) By Lemma(4.8) we have $\widetilde{u} \in K'$.

We take $f_n = \varepsilon(u_n)$ and $K = M + \eta$ with $0 < \eta < 1$ and $A_n = \{x \in \Omega / |\varepsilon(u_n)| > M + \eta\}$. We show that $meas(A_n) \longrightarrow 0$.

We have $\eta^2 |A_n| \leqslant \int_{A_n} (|\varepsilon(u_n)| - M)^2 \leqslant \int_\Omega [(|\varepsilon(u_n)| - M)^+]^2 \leqslant \frac{C}{n} \longrightarrow 0$. Applying (4.8), we get $\|\varepsilon(\widetilde{u})\|_{[L^\infty(\Omega)]^{d^2}} \leqslant M + \eta$ and $|\varepsilon(\widetilde{u})| \in L^\infty(\Omega)$.

Finally, $\|\varepsilon(\widetilde{u})\|_{[L^\infty(\Omega)]^{d^2}} \leqslant M$.

d) Let $n \in N^*$ and $v \in K \cap K'$, we show that $(T_n u_n, v - u_n)_V \leqslant 0$.

- If $|\varepsilon(u_n)| \leqslant M$, then $\mathcal{R}_n(\varepsilon(u_n) = 0$, then $(T_n u_n, v - u_n)_V = 0$.
- If $|\varepsilon(u_n)| \geqslant M$, so using the Cauchy-Schwartz inequality we obtain

$$
\begin{aligned}
(T_n u_n, v - u_n)_V &= 2n \int_\Omega ((|\varepsilon(u_n)| - M)^+) \frac{\varepsilon(u_n)}{|\varepsilon(u_n)|} (\varepsilon(v) - \varepsilon(u_n)) \\
&= 2n \int_\Omega \frac{(|\varepsilon(u_n)| - M)}{|\varepsilon(u_n)|} \varepsilon(u_n)(\varepsilon(v) - \varepsilon(u_n)) \\
. &\leqslant 2n \int_\Omega \frac{(|\varepsilon(u_n)| - M)}{|\varepsilon(u_n)|} |\varepsilon(u_n))||\varepsilon(v)| - |\varepsilon(u_n)|^2 \\
&= 2n \int_\Omega \frac{(|\varepsilon(u_n)| - M)}{|\varepsilon(u_n)|} |\varepsilon(u_n)|(|\varepsilon(v)| - |\varepsilon(u_n)|) \\
&\leqslant 2n \int_\Omega \frac{(|\varepsilon(u_n)| - M)}{|\varepsilon(u_n)|} |\varepsilon(u_n)|(M - |\varepsilon(u_n)|) \leqslant 0.
\end{aligned}
$$

We deduce that

$$
\forall v \in K \cap K' \quad (T_n u_n, v - u_n)_V \leqslant 0 \tag{43}
$$

e) We show that $(S_n u_n, v - u_n)_V \leqslant 0 \quad \forall v \in K \cap K'$

$$
(S_n u_n, v - u_n)_V = n \int_{\Gamma_C} u_{n,\nu}^+ (v_\nu - u_{n,\nu}) = n \int_{\Gamma_C} (u_{n,\nu}^+ v_\nu - u_{n,\nu}^+ u_{n,\nu}).
$$

Since $u_{n,\nu}^+ v_\nu \leqslant 0$ and $u_{n,\nu}^+ u_{n,\nu} \geqslant 0$ we obtain $(S_n u_n, v - u_n)_V \leqslant 0$. Then

$$
\forall v \in K \cap K' \quad (S_n u_n, v - u_n)_V \leqslant 0 \tag{44}
$$

From (43) and (44) we find that

$$
a(u_n, v - u_n) + \langle j_n'(u_n, u_n), v - u_n \rangle \geqslant (f, v - u_n)_V \quad \forall v \in K \cap K'.
$$

Using (39) we deduce that

$$
a(u_n, v - u_n) + j_n(u_n, v) - j_n(u_n, u_n) \geqslant (f, v - u_n)_V \quad \forall v \in K \cap K'.
$$

We apply (41) to obtain

$$
a(\widetilde{u}, v - \widetilde{u}) + j(\widetilde{u}, v) - j(\widetilde{u}, \widetilde{u}) \geqslant (f, v - \widetilde{u})_V \quad \forall v \in K \cap K'.
$$

Thus $\widetilde{u} = u$ and $\widetilde{u}$ is unique and all subsequences of $(u_n)$ converge weakly to $u$.

f) We show that $(u_n) \longrightarrow u$ as $n \longrightarrow +\infty$.
We have

$$\begin{aligned}\alpha\|u_n - u\|_V^2 &\leqslant a(u_n - u, u_n - u)\\ &= a(u_n, u_n - u) - a(u, u_n - u)\\ &= -a(u_n, u - u_n) - a(u, u_n - u)\\ &= (T_n u_n, u - u_n)_V + (S_n u_n, u - u_n)_V + \langle j_n'(u_n, u_n), u - u_n\rangle - (f, u - u_n)\\ &\quad - a(u, u_n - u)\\ &\leqslant j(u, u) - j(u, u_n) - a(u, u_n - u),\end{aligned}$$

and then, $(u_n) \longrightarrow u$ strongly as $n \longrightarrow +\infty$ by (41).

■

### 4.4 *Theorem3.4*

In this subsection, we treat the case of the finite element discretized penalty method, we suppose also that $\epsilon = \frac{1}{n}$.

THEOREM 4.10 *Assume that $(h_1) - (h_4)$ hold. For any $n \in N^*$ we suppose that $h = \frac{1}{n^2}$. Then*

1. *there exists $L^* > 0$ such that $\frac{L_\mu + \mu^*}{\alpha} < L^*$ then the problem $PV_n^h$ has a unique solution $u_n^h$ in $V^h$.*
2. *the sequence $(u_n^h) \longrightarrow u$ strongly in $V$ as $h \longrightarrow 0$.*

*Proof* We apply theorem (4.2) to deduce that the problem (26) has a unique solution $u_n^h$ in $V^h$. Let $U = \{v \in \mathcal{C}^\infty(\Omega)^d;\quad v = 0 \text{ on one neighborhood of } \Gamma_D\}$ and $R^h : U \longrightarrow V^h$ such that for $v \in U$, $R^h v$ is the linear interpolate of $v$.
We recall that $U$ is dense in $V$ and $\forall v \in U$

$$\begin{cases} \|R^h v - v\|_V \leqslant Ch\|v\|_{H^2(\Omega)^d} \\ \|R^h v - v\|_{L^2(\Gamma)} \leqslant Ch\|v\|_V \\ (R^h v)_h \longrightarrow v \text{ strongly in } V \text{ as } h \longrightarrow 0. \end{cases}$$

We take $v = u_n^h$ in (26) and using

$$\begin{cases} (T_n u_n^h, u_n^h)_V \geqslant 0, \\ (S_n u_n^h, u_n^h)_V \geqslant 0, \\ \langle j_n'(u_n^h, u_n^h), u_n^h\rangle \geqslant 0, \end{cases}$$

we find

$$a(u_n^h, u_n^h) \leqslant (f, u_n^h)_V.$$

Using the $V-$ellipticity of $a(.,.)$ we find

$$\|u_n^h\|_V \leqslant \frac{\|f\|_V}{\alpha}.$$

From $\begin{cases} a(u_n^h, u_n^h) \geqslant 0, \\ (S_n u_n^h, u_n^h)_V \geqslant 0, \\ \langle j_n'(u_n^h, u_n^h), u_n^h \rangle \geqslant 0, \end{cases}$ we get

$$(T_n u_n^h, u_n^h)_V \leqslant (f, u_n^h)_V \leqslant \|f\|_V \|u_n^h\|_V \leqslant \frac{\|f\|_V^2}{\alpha}.$$

Then $2n \int_\Omega (|\varepsilon(u_n^h)| - M)^+ |\varepsilon(u_n^h)| \leqslant \frac{\|f\|_V^2}{\alpha}$.
It follows

$$2n \int_\Omega (|\varepsilon(u_n^h)| - M)^2 \leqslant \frac{\|f\|_V^2}{\alpha}.$$

Then

$$\|T_n u_n^h\|_V \leqslant \sqrt{2n} \frac{\|f\|_V}{\sqrt{\alpha}} \qquad \forall h > 0. \tag{45}$$

From continuity of $a(.,.)$ there exists $m_a > 0$ such that

$$|a(w,v)| \leqslant m_a \|w\|_V \|v\|_V, \quad \forall v, w \in V.$$

Hence

$$a(u_n^h, v) \leqslant m_a \frac{\|f\|_V}{\alpha} \|v\|_V, \quad \forall v \in V. \tag{46}$$

There exists a subsequence of $(u_n^h)$ denoted also $(u_n^h)$ which converges weakly to $u^*$.
We follow the same steps with (4.9) to show that $(u_n^h) \longrightarrow u^*$ on $L^2(\Gamma_C)$ and $u^* \in K \cap K'$.
For all $v \in U$, we have

$$a(u_n^h, u_n^h - R^h v) + (T_n u_n^h, u_n^h - R^h v)_V + (S_n u_n^h, u_n^h - R^h v)_V + (j_n'(u_n^h, u_n^h), u_n^h - R^h v)_V = (f, u_n^h - R^h v)_V.$$

and

$$(T_n u_n^h, u_n^h - R^h v)_V = (T_n u_n^h, u_n^h - v)_V + (T_n u_n^h, v - R^h v)_V.$$

From (43) and (44) we have for all $v \in K \cap K' \cap U$

$$\begin{cases} (T_n u_n^h, v - u_n^h)_V \leqslant 0 \\ (S_n u_n^h, v - u_n^h)_V \leqslant 0. \end{cases} \tag{47}$$

Then

$$\begin{cases} (T_n u_n^h, u_n^h - R^h v)_V \geqslant (T_n u_n^h, v - R^h v)_V \\ (S_n u_n^h, u_n^h - R^h v)_V \geqslant (T_n u_n^h, v - R^h v)_V. \end{cases}$$

From one side we have

$$a(u_n^h, u_n^h - u^*) = a(u_n^h, u_n^h - R^h v) + a(u_n^h, R^h v - u^*) \tag{48}$$

then

$$\begin{aligned} a(u_n^h, u_n^h - u^*) &= -(T_n u_n^h, u_n^h - R^h v)_V - (S_n u_n^h, u_n^h - R^h v)_V \\ &\quad - \langle j_n'(u_n^h, u_n^h), u_n^h - R^h v\rangle + (f, u_n^h - R^h v)_V + a(u_n^h, R^h v - u^*) \\ &\leqslant -(T_n u_n^h, v - R^h v)_V - (S_n u_n^h, v - R^h v)_V \\ &\quad - \langle j_n'(u_n^h, u_n^h), u_n^h - R^h v\rangle + (f, u_n^h - R^h v)_V + a(u_n^h, R^h v - u^*). \end{aligned}$$

Then

$$\begin{aligned} a(u_n^h, u_n^h - u^*) &\leqslant -(T_n u_n^h, v - R^h v)_V - (S_n u_n^h, v - R^h v)_V \\ &\quad - \langle j_n'(u_n^h, u_n^h), u_n^h - R^h v\rangle + (f, u_n^h - R^h v)_V + a(u_n^h, R^h v - u^*). \end{aligned} \tag{49}$$

For the other side, we use (46) to obtain

$$\begin{aligned} \limsup_{h\to 0} a(u_n^h, R^h v - u^*) &\leqslant \limsup_{h\to 0} m_a \frac{\|f\|_V}{\alpha} \|R^h v - u^*\|_V \\ &= m_a \frac{\|f\|_V}{\alpha} \|v - u^*\|_V \end{aligned}$$

and

$$\begin{cases} \lim\limits_{h\to 0} j_n'(u_n^h, u_n^h), u_n^h - R^h v\rangle = j(u^*, u^*) - j(u^*, v) \\ \lim\limits_{h\to 0} (f, u_n^h - R^h v)_V = (f, u^* - v)_V \\ |(T_n u_n^h, v - R^h v)_V| \leqslant \sqrt{2n} \frac{\|f\|_V}{\sqrt{\alpha}} \|v - R^h v\|_V \\ \qquad \leqslant Ch\sqrt{2n} \frac{\|f\|_V}{\sqrt{\alpha}} \|v\|_{H^2(\Omega)^d}. \end{cases} \tag{50}$$

Using $h = \frac{1}{n^2}$ in (50) we get $\lim\limits_{h\to 0}(T_n u_n^h, v - R^h v)_V = 0$. Then

$$|(S_n u_n^h, v - R^h v)_V| = |\int_{\Gamma_C} n u_n^{h+} (v - R^h v)_\nu| \leqslant Cnh \|u_n^h\|_{L^2(\Gamma_C)} \|v\|_V.$$

Using (42) we find

$$|(S_n u_n^h, v - R^h v)_V| \leqslant C c_0 h n \frac{1}{\alpha} \|f\|_V \|v\|_V.$$

Because $h = \frac{1}{n^2}$, we deduce that

$$\lim_{h\to 0} (S_n u_n^h, v - R^h v)_V = 0 \tag{51}$$

From (49),(50) and (51) we deduce that

$$\limsup_{h\to 0} a(u_n^h, u_n^h - u^*) \leqslant j(u^*, v) - j(u^*, u^*) + (f, u^* - v)_V + m_a \frac{\|f\|_V}{\alpha}\|v - u^*\|_V \quad \forall v \in K\cap K'\cap U. \tag{52}$$

Since $K \cap K' \cap U$ is dense in $K \cap K'$ then (53) is also valid in $K \cap K'$.
We take $v = u^*$ in (53) we get

$$\limsup_{h\to 0} a(u_n^h, u_n^h - u^*) \leqslant 0 \tag{53}$$

and for any $v \in K \cap K'$ we have

$$\begin{aligned} a(u_n^h, v - u_n^h) &= a(u_n^h, v - u^*) + a(u_n^h, u^* - u_n^h) \\ &= (T_n u_n^h, v - u^*)_V + (S_n u_n^h, v - u^*)_V + \langle j_n'(u_n^h, u_n^h), v - u^*\rangle + a(u_n^h, u^* - u_n^h), \end{aligned}$$

that leads by (53), after passing to the limit, to

$$a(u^*, v - u^*) \geqslant -j(u^*, v) + j(u^*, u^*) + (f, v - u^*)_V \quad \forall v \in K \cap K',$$

then $u^*$ is a solution of a variational inequality. Finally, we deduce that $u^*$ is unique and $u^* = u$.
Now, we show now that $(u_n^h) \longrightarrow u$ strongly.
Let $v \in K \cap K' \cap U$, we have

$$\begin{aligned} \alpha\|u_n^h - u\|_V^2 &\leqslant a(u_n^h - u, u_n^h - u) \\ &= a(u_n^h, u_n^h - u) - a(u, u_n^h - u) \\ &= a(u_n^h, u_n^h - R^h v) + a(u_n^h, R^h v - u) - a(u, u_n^h - u) \\ &= (f, u_n^h - R^h v)_V - (T_n u_n^h, u_n^h - R^h v)_V - (S_n u_n^h, u_n^h - R^h v)_V \\ &\quad - \langle j_n'(u_n^h, u_n^h), u_n^h - R^h v\rangle - a(u, u_n^h - u) \\ &= (f, u_n^h - R^h v)_V + (T_n u_n^h, v - u_n^h)_V - (T_n u_n^h, v - R^h v)_V \\ &\quad + (S_n u_n^h, v - u_n^h)_V - (S_n u_n^h, v - R^h v)_V - \langle j_n'(u_n^h, u_n^h), u_n^h - R^h v\rangle - a(u, u_n^h - u). \end{aligned}$$

Using (47) we find that

$$\begin{aligned} \alpha\|u_n^h - u\|_V^2 \leqslant\ & (f, u_n^h - R^h v)_V - (T_n u_n^h, v - R^h v)_V \\ & - (S_n u_n^h, v - R^h v)_V - \langle j_n'(u_n^h, u_n^h), u_n^h - R^h v\rangle - a(u, u_n^h - u). \end{aligned}$$

Now, we pass to the limit using (50), we obtain

$$\lim_{h\to 0} \alpha\|u_n^h - u\|_V^2 \leqslant (f, u - v)_V - j(u, u) + j(u, v) \quad \forall v \in K \cap K' \cap U. \tag{54}$$

Since $K \cap K' \cap U$ is dense in $K \cap K'$ we deduce that (54) is valid in $K \cap K'$.
If we take $v = u$ we find that $(u_n^h) \longrightarrow u$ strongly. ∎

## 5. Conclusion

In this paper, we consider a classical penalty method applied to the unilateral contact problem with Coulomb friction for locking material. We obtain various estimates depending on the mesh $h$ and on the penalty parameters $\epsilon\,;\, n$. We show that the theorical convergence of the penalty method gives the best results when $\epsilon = \frac{1}{n}$ and $h = \frac{1}{n^2}$. We note that any choice $\epsilon = \frac{C}{n}$ and $h = \frac{C}{n^s}$ with $C$ a positive constant and $s > 1$ would give the same theoretical convergence.
The next work we are interested in will be to study the theoretical and the numerical penalty method in order to find the error estimate for the unilateral contact problem.

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