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Application of HPM and HAM to the First Form of Blasius Equation

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Abstract. In this work, the Blasius equation is studied. Homotopy perturbation method (HPM) and homotopy analysis method (HAM) are applied to obtain its solution. Comparison with variational iteration method (VIM) is made to highlight the significant features of employed methods and their capability of handling nonlinear problems. The outcome shows the success of (HPM) and (HAM) for solving nonlinear problems arising in fluid mechanics.

Keywords: Blasius equation, HPM, HAM, VIM.

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1. Introduction

Blasius equation is one of the fundamental and basic equations of fluid dynamics, which described the velocity profile of the fluid in the boundary layer theory on a half infinite interval [3, 4]. Many researchers have investigated analytical and numerical solution methods to handle this problem[5, 6]. Two forms of Blasius equation appear in the fluid mechanic theory, where each is subjected to specific physical conditions. The equation has the forms:

$$\begin{aligned} & u''' + \tfrac{1}{2}u(x)u''(x) = 0 \\ & u(0) = 0, u''(0) = 1, u'(\infty) = 0 \end{aligned} \tag{1}$$

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And:

$$u''' + \frac{1}{2}u(x)u''(x) = 0 \\ u(0) = 0, u'(0) = 0, u'(\infty) = 1 \quad (2)$$

It is well known that the Blasius equation is the mother of all boundary layer equations in fluid mechanics.Obviously, it is difficult to solve these differential equations analytically. Perturbation method is one of the well-known methods to solve nonlinear problems, it is based on the existence of small/ large parameters, the so called perturbation quantity [9]. Many nonlinear problems do not contain such kind of the perturbation quantity, and we can use non perturbation methods such as the Adomains decomposition method [2]. However, both of the perturbation and non-perturbation methods cannot provide us with a simple way to adjust and control the convergence region and rate of given approximate series. In 1992, Liao employed the basic idea of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (HAM) [8]. In this paper, since the two forms of Blasius equations are the same, first formissolved using HPM and HAM and compare is made with exact solution of variational iteration method (VIM) achieved by Abdul-Majid Wazwaz [11].

2. Basic Idea of Perturbation Methods

2.1 *Homotopy Perturbation Method (HPM)*

To illustrate the basic ideas of this method, we consider the following equation:

$$A(u) - f(r) = 0 \quad r \in \Omega, \quad (3)$$

With the boundary condition of:

$$B(u, \frac{\partial \Omega}{\partial n}) = 0, r \in \Gamma \quad (4)$$

Where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Equation (3) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega, \quad (5)$$

Homotopy perturbation structure is shown as follows:

$$H(u, p) = (1 - p)[L(u) - L(u_0)] + p[A(u) - f(r)] = 0. \quad (6)$$

Where,

$$u(r, p) : \Omega \times [0, 1] \to R \quad (7)$$

In equation (6), $p \in [0, 1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of equation (6) can be written as a power series in p, as following.

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots \quad (8)$$

And the best approximation for solution is:

$$u = lim_{p\to 1} = u_0 + u_1 + u_2 + u_3 + \cdots \tag{9}$$

Interested readers may refer to work of Ganji and Sadighi [10], for a detailed discussion on the above convergence and the principle of the HPM.

2.2 *Homotopy Analysis Method (HAM)*

In this paper, we apply the homotopy analysis method to discuss the problem. To show the basic idea, let us consider the following differential equation:

$$N[u(\tau)] = 0.$$

Where N is a nonlinear operation, denotes independent variable, is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way, by means of generalizing the traditional homotopy method, Liao constructs the so called zero order deformation equation:

$$(1-p)L[\Phi(\tau,p) - u_0(\tau)] = p\overline{h}H(\tau)N[\Phi(\tau,p)]. \tag{10}$$

Where $P \in [0,1]$ in the embedding parameter, $\overline{h} \neq 0$ is a non-zero auxiliary parameter, $H(\tau) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau), \Phi(\tau,p)$ is an unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when P=0 and P=1 it holds

$$\Phi(\tau,0) = u_0(\tau), \Phi(\tau,1) = u(\tau).$$

respectively. Thus as p increases from zero to one, the solution $\Phi(\tau,p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$, expanding $\Phi(\tau,p)$ in Tailor series with respect to p, one has:

$$\Phi(\tau,p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau)p^m. \tag{11}$$

Where:

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \Phi(\tau,p)}{\partial p^m} |_{p=0} \tag{12}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\overline{h}$ and the auxiliary function are so properly chosen, the series (11) convergence at p =1, one has:

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau).$$

which must be one of solutions of original nonlinear equation, as proved by Liao. As $\overline{h} = -1$ and $H(tau) = 1$, equation(10) becomes:

$$(1-p)L[\Phi(\tau,p) - u_0(\tau)] + pN[\Phi(\tau,p)] = 0.$$

which is used mostly in the homotopy perturbation method, whereas the solution obtained directly, without using Tailor series. According to the definition (12) the governing equation can be deduced from the zero- order deformation equation (10). Define the vector $u_n = \{u_0(\tau), u_1(\tau), \cdots, u_n(\tau)\}$ Differentiating equation (10) m times respect to embedding parameter P and then setting P=0 and finally dividing then by $m!$, we have the so called mth order deformation equation:

$$L[u_m(\tau) - X_m u_{m-1}(\tau)] = \overline{h} H(\tau) R_m(\overrightarrow{u_{m-1}}) \tag{13}$$

Where:

$$R_m(\overrightarrow{u_{m-1}}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\Phi(\tau, p)}{\partial p^{m-1}}|_{p=0} \tag{14}$$

$$X_m = \begin{cases} 0 & m \leqslant 1, \\ 1 & m > 1. \end{cases}$$

It should be emphasized that $u_m(\tau)$ for $m \geqslant 1$ is governed by the linear equation (11) with the linear boundary conditions that came from original problem, which can be easily solved by symbolic computation software such as Maple. Interested readers may refer to work of Abbasbandy [1], for a detailed discussion on the principle of the HAM.

3. Results and Discussion

3.1 *Application of HPM to the First Form of Blasius Equation*

We consider the first form of Blasius equation and construct the following homotopy for equation (1)

$$(1-p)(\frac{d^3}{dx^3}u(x)) + p(\frac{d^3}{dx^3}u(x) + 0.5u(x)(\frac{d^2}{dx^2}u(x)) = 0. \tag{15}$$

Substituting U from equation (8) into equation (15) and rearranging based on powers of p terms , we have:

$$p^0 : \frac{d^3}{dx^3}u_0(x) = 0, \tag{16}$$

$$p^1 : \frac{d^3}{dx^3}u_2(x) + 0.5u_0(x)(\frac{d^2}{dx^2}u_0(x)) = 0, \tag{17}$$

$$p^2 : \frac{d^3}{dx^3}u_2(x) + 0.5u_0(x)(\frac{d^2}{dx^2}u_1(x) + 0.5u_1(x)(\frac{d^2}{dx^2 u_0(x)})) = 0. \tag{18}$$

Solving equation (16) - (18) subject to initial condition given by equation (1), we have:

$$u_0 = \frac{1}{2}Ax^2 + x$$

$$u_1 = -\frac{1}{240}A^2x^5 - \frac{1}{48}Ax^4,$$

$$u_2 = \frac{11}{161280}A^3x^8 + \frac{11}{20160}A^2x^7 + \frac{1}{960}Ax^6.$$

And when $p \to 1$, then we can obtain:

$$u = u_0 + u_1 + u_2 = \frac{11}{161280}A^3x^8 + \frac{11}{20160}A^2x^7 + \frac{1}{960}Ax^6 - \frac{1}{240}A^2x^5 - \frac{1}{48}Ax^4 + \frac{1}{2}Ax^2 + x.$$

This is exactly the same as that obtained by Wazwaz [11] through VIM. Figure.1 presents a comparison between VIM and HPM solution.

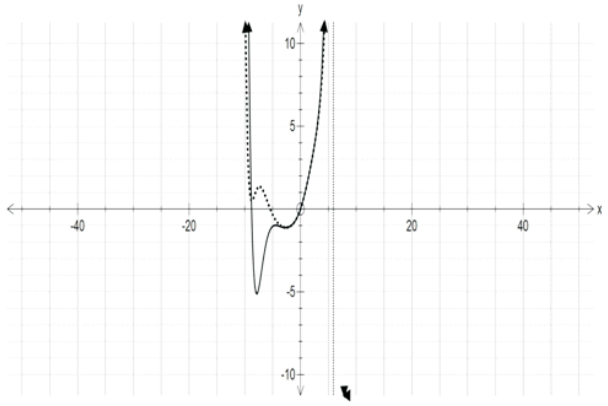

Figure 1. The obtained solution by HPM in comparison with VIM, continuous line: HPM, dashed line: VIM

It can be concluded that the curves are satisfactorily feet. Since Blasius equation cannot be easily solved by other methods, therefore, the hemotopy perturbation method is an acceptable solution for first form of Blasius equation.

3.2 ***Application of HAM to First Form of Blasius Equation***

Now we consider the following differential equation for first form of Blasius equation:

$$N = \frac{\partial^3}{\partial x^3}\Phi(x,p) + 0.5\Phi(x,p)(\frac{\partial^2}{\partial x^2}\Phi(x,p))$$

$$L = \frac{\partial^3}{\partial x^3}\Phi(x,p) - \frac{d^3}{dx^3}u_0(x)$$

Where N is a nonlinear operator and L is an auxiliary linear operator, respectively. Choosing m =2 and initial guess:

$$u_0 = x + 0.5Ax^2.$$

And rearranging equation(10) subjected to initial condition, we have:

$$u_1 = \frac{1}{240} hA^2x^5 + \frac{1}{48} hAx^4$$

$$u_2 = \frac{1}{480} hA(\frac{11}{360} hA^2x^8 + \frac{11}{42} hAx^7 + \frac{1}{2} hx^6 + \frac{1}{60}(120A + 120hA)x^5 + \frac{1}{24}(240h + 240)x^4).$$

Hence, the 2th order approximation of $u(x)$ can be generally express by:

$$u(x) = u_0 + u_1 + u_2$$

$$u(x) = \frac{1}{480} hA(\frac{11}{360} hA^2x^8 + \frac{11}{42} hAx^7 + \frac{1}{60}(120A + 120hA)x^5 + \frac{1}{24}(240h + 240)x^4) + \frac{1}{240} hA^2x^5 + \frac{1}{48} hAx^4 + 0.5Ax^2 + x. \tag{19}$$

Choosing $A = 0.5227030796$, $\overline{h} = 0.001$ and plot equation (19) we have Figure.2 as follows. In Figure.2, the exact solution obtained by Wazwaz [11] using VIM is compared with the HAM solution.

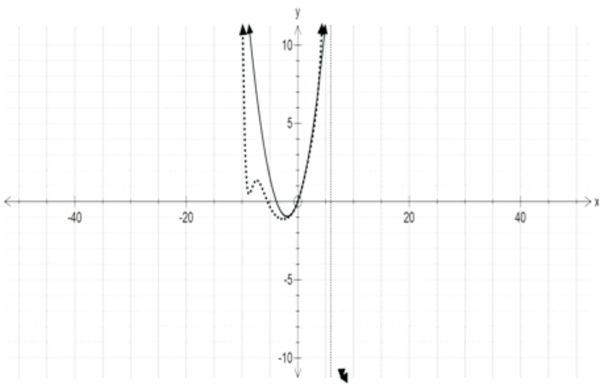

Figure 2. The obtained solution by HAM in comparison with VIM, continuous line: HAM, dashed line: VIM

4. Conclusion

In this paper, the homotopyperturbation method (HPM) and the homotopy analysis method (HAM) have been applied to achieve the solution of Blasius equation which is one of the basic equations of fluid dynamics. It has been shown that HPM and HAM provide very accurate numerical solutions for non-liner problems in comparison with variational iteration method (VIM). The numerical results we achieved here justify the advantage of these methods and it can be concluded that hemotopy methods can successfully solve nonlinear problems arising in fluid mechanics.

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