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Mean Value Interpolation on Spheres

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Abstract. In this paper we consider multivariate Lagrange mean-value interpolation problem, where interpolation parameters are integrals over spheres. We have concentric spheres. Indeed, we consider the problem in three variables when it is not correct.

Keywords: Lagrange interpolation, Mean-value interpolation, Multivariate polynomial, Sphere.

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1. Introduction

The interpolation schemes considered so far have one important property in common: the dimension of the interpolation spaces coincides with the number of interpolation conditions. A different approach has been taken by Kergin [3]. He gives a linear projection from the space $C^n(\mathbb{R}^k)$ (continuous differentiable functions on $\mathbb{R}^k$) onto the space of polynomials of degree not exceeding n. He constructed, for given nodes $x_0, \ldots, x_n \in \mathbb{R}^k$, an interpolation polynomial of degree not exceeding n. This interpolant becomes unique because of additional interpolation constraints. Mean Value Multivariate Lagrange Interpolation Problem MVLIP: Let

$$\mathbb{D} = \left\{ \Delta_i \subseteq \mathbb{R}^{\mathrm{k}} : i = 1, \ldots, N \right\}$$

be a collection of N measurable subsets of $\mathbb{R}^{\mathrm{k}}$ with finite nonzero mesures. The existence and uniqueness of a polynomial $p \in \Pi_n^k$ such that $\int_\Delta p(x)dx = c_i, i = 1, \ldots, N$, where c_i's are arbitrary given numbers is called MVLIP.

We denote this mean-value interpolation problem by $(\Pi_n^k, \mathbb{D})^{m.v.}$.

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The problem $(\Pi_n^k, \mathbb{D})^{m.v.}$ is called correct if for any numbers $c_i, i = 1, \ldots, N$ there exists a unique polynomial $p \in \Pi_n^k$ satisfying the above conditions. Now let us explain the general example of mean-value interpolation in arbitrary dimension. More precisely, we have the following result,

THEOREM 1.1 (Kergin)*For any set of* $n+1$ *points* $x_0, \ldots, x_n \in \mathbb{R}^k$ *there is a unique mapping*

$$p : C^n(\mathbb{R}^k) \mapsto \Pi_n^k$$

with the property that for any $f \in C^n(\mathbb{R}^k)$, *any constant coefficient homogeneous differential operator* $q(D)$, $q \in \Pi_n^k$, *and any subset* $J \subseteq \{0, \ldots, k\}$, *with* $\#J = degq + 1$, *there exists a point* x *in the convex hull* $[x(j) : j \in J]$ *such that* $(q\,(D)\,P_f)\,(x) = (q\,(D)\,f)\,(x)$.

Due to the complexity of this condition it is not surprising that the main part of [3] consists of showing the existence of the above interpolation operator which was done in a nonconstructive way. He gives a proof by mean value theorem and the above formula for exsintence and uniqueness of interpolation polynomials. This issue was resolved constructively by Micchelli [8]. It turns out that Kergin interpolation can actually be considered as an extension of the intersection points of hyper planes. Also, Micchelli and Milman [7] give a proof of the existence of projection analogous to Newton form of Hermite interpolation. Indeed, substitution of the univariate Hermite-Genocchi formula

$$f[x_0, \ldots, x_k] = \int_{[x_0,\ldots,x_k]} f^{(k)}, \quad f \in C^k(\mathbb{R}),$$

which expresses the divided difference as a B-spline integral, into the Newton formula allows us to represent the interpolation polynomial $L_k f$ as

$$L_k f(x) = \sum_{j=0}^{k} \int_{[x_0,\ldots,x_k]} f^{(j)} \prod_{n=0}^{j-1} (x - x_n), \quad x \in \mathbb{R}$$

which can be rewritten in a way as

$$L_k f\,(x) = \sum_{j=0}^{k} \int_{[x_0,\ldots,x_k]} D_{x-x_0} \ldots D_{x-x_{j-1}} f, \quad x \in \mathbb{R}.$$

Now, the above formula already gives the Kergin interpolant p with the above properties by simply replacing the qualifier $x \in \mathbb{R}$ by $x \in \mathbb{R}^k$.

THEOREM 1.2 (Micchelli [8]) *The Kergin interpolation polynomial* p *is given as*

$$p_f\,(x) = \sum_{j=0}^{k} \int_{[x_0,\ldots,x_k]} D_{x-x_0} \ldots D_{x-x_{j-1}} f, \quad x \in \mathbb{R}^k,$$

and the error of interpolation takes the form

$$(f - p_f)\,(x) = \int_{[x_0,\ldots,x_k]} D_{x-x_0} \ldots D_{x-x_{j-1}} \ f, x \in \mathbb{R}^k.$$

Another approach in this field was considered by Hakop Hakopian. On that case he introduses the mean-value interpolation problem on lines [1].

2. Materials and Methods

Denote by $\prod_n^3 = \prod_n(\mathbb{R}^3)$ the space of interpolation polynomials in three variables of total degree not exceeding n:

$$\prod_n^3 = \left\{ p\left(x,y,z\right) = \sum_{i+j+k\leq n} a_{ijk}x^i y^j z^k : i,j,k \in \mathbb{Z}_+ \right\}.$$

Set $N = \dim \prod_n^3 = \binom{n+3}{3}$.
Let us fix the set of distinct points

$$\chi_s = \left\{\left(x_1,y_1,z_1\right),\ldots,\left(x_s,y_s,z_s\right)\right\} \subset \mathbb{R}^3$$

as the set of nodes of interpolation.
The classic Lagrange pointwise interpolation problem$(\prod_n^3, \chi_s)$ is to find a unique polynomial $p \in \prod_n^3$ such that

$$(x_l,y_l,z_l) = c_l,\ l = 1,2,\ldots,s, \tag{1}$$

where $c_l,\ l = 1,2,\ldots,s$ are real numbers.

Definition 2.1 *The Lagrange pointwise interpolation problem* $(\prod_n^3, \chi_s)$ *is called correct, if for any real values* $c_l,\ l = 1,2,\ldots,s$ *there exists a unique polynomial* $p \in \prod_n^3$ *satisfying the conditions* (1).

In other words, the Lagrange interpolation problem is to find a unique polynomial

$$p\left(x,y,z\right) = \sum_{i+j+k\leq n} a_{ijk}x^i y^j z^k \in \prod_n^3$$

which the conditions (1) reduce to the following linear system

$$p\left(x_l,y_l,z_l\right) = \sum_{i+j+k\leq n} a_{ijk}\ x_l^i y_l^j z_l^k = c_l\ ,\ l = 1,2,\ldots,s. \tag{2}$$

The correctness of interpolation means that the linear system (2) has a unique solution for arbitrary right hand side values. A necessary condition for this is s=N. Note that in this case the linear system (2) has a unique solution for arbitrary values $c_l^{'}s,\ l = 1,2,\ldots,s,$ if and only if the corresponding homogeneous system has only trivial solution. In other words we have

Proposition 2.2 *The interpolation problem* $(\prod_n^3, \chi_N)$ *is correct if and only if*

$$p \in \prod_n^3,\ p\left(x_l,y_l,z_l\right) = 0,\ l = 1,2,\ldots,N \Rightarrow p = 0.$$

Equivalently: The interpolation problem $(\prod_n^3, \chi_N)$ is not correct if and only if

$$\exists p \in \prod_n^3,\ p \neq 0 \ such\ that\ p(x_l, y_l, z_l) = 0,\ l = 1, 2, \ldots, N \tag{3}$$

In this paper a mean-value interpolation problem is considered where interpolation parameters are integrals over spheres. Here we are going to find a unique polynomial $p \in \prod_n^3$ such that

$$\frac{1}{\mu_3(S_l)} \iiint_{S_l} p(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = c_l,\quad l = 1, 2, \cdots, N, \tag{4}$$

where c_l's are arbitrary given numbers and S_l's are spheres and also $\mu_3(S_l)$ is the area of S_l. We denote this mean-value interpolation problem by$(\prod_n^3, \mathbb{S})^{m.v.}$, where $\mathbb{S}$ is the set of above spheres: $\mathbb{S} = \{\mathrm{S_l} : l = 1,\ 2, \ldots, N\}$.

Same as Definition 1.1 we call the problem $(\prod_n^3, \mathbb{S})^{m.v.}$ correct if, for any number $c_l,\ \mathrm{l} = 1, 2, \ldots, \mathrm{N},$ there exists a unique polynomial $p \in \prod_n^3$ satisfying (4).
Note that for a Lebesgue integrable function f it is convenient to use this interpolation. (See, [1, p. 203])

An example of correct interpolation problem in dimension two is presented in [4, 5]. To extend this problem in dimension three consider the following definition.

Definition 2.3 *The centroid of the region S is called the point with the coordinates:*

$$x^* = \frac{\iiint_S x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\mu_3(S)},\ y^* = \frac{\iiint_S y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\mu_3(S)},\ z^* = \frac{\iiint_S z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\mu_3(S)},$$

where $\mu_3(S)$ is an area of the region S.

Statement The problem in $\mathbb{R}^d$ with d+1 arbitrary regions and n=1 is correct if and only if the centroids of the regions are not laying on a hyperplane.

Lemma 2.4 *The problem in $\mathbb{R}^3$ with four arbitrary regions and n=1 is correct if and only if the centroids of the regions are not laying on a hyperplane.*

Proof Let us show this problem. Let $p \in \prod_n^3$ and $\iiint_{S_l} p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0, l = 1, \cdots, 4$. Let also the centroids are not collinear. Then

$$\iiint_{S_l} p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0 \Rightarrow p(x^*, y^*, z^*) = 0, l = 1, \cdots, 4 \Rightarrow p = 0$$

Thus the problem is correct. This result follows from the statement: if $S$is bounded region with non-zero area and $p \in \prod_1^3,\quad p(x, y, z) = Ax + By + Cz + D$ then $\iiint_{S_l} p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mu_3(S).\ (Ax^\star + By^\star + Cz^\star + D)$ if and only if there is a hyperplane with equation $p = 0$ passes through the centroid of S. ■

Another special case is considered in [6]. To introduce it let us get a set Δ be a measurable set in $\mathbb{R}^d$ with finite non-zero measure. The following set we call $\lambda-$shift of Δ

$$\Delta^\lambda = \left\{y + \lambda :\quad y \in \mathbb{R}^d\right\}.$$

Let us fix a set of $N_d = \dim \prod_n^d = \binom{n+d}{d}$ distinct nodes $\Lambda = \{\lambda_l : l = 1, \ldots, N_d\} \subset \mathbb{R}^d$. The following set we call the set of $\Lambda-$ shifts of Δ

$$\Lambda(\Delta) := \{\Delta^\lambda : \lambda \in \Lambda\}$$

THEOREM 2.5 *Suppose that* $\mu_d() \neq 0$. *Then the mean-value interpolation* $(\prod_n^d, \Lambda\,())^{m.v.}$ *is correct if and only if the Lagrange pointwise interpolation problem* $(\prod_n^d, \Lambda)$*is correct.*

Consider an arbitrary set of N distinct balls of same radius$r$: $\mathbb{B} := \{B_{a_l,r} : l = 1, \ldots, N_d\}$. Let $A = \{a_l : l = 1, \ldots, N_d\}$ be the set of centers of the balls.

THEOREM 2.6 *The mean-value interpolation* $(\prod_n^d, \mathbb{B})$ *is correct if and only if the Lagrange pointwise interpolation* $(\prod_n^d, A)$ *is correct.*

In the next section we consider the multivariate mean-value interpolation for polynomials of arbitrary degree with regions which are concentric spheres. We conclude that in this case the problem is not correct.

3. Results and Discussion

Let us consider a mean-value interpolation with polynomials of arbitrary degree and the regions are the above spheres, i.e, the mean-value interpolation problem$(\prod_n^3, \mathbb{S})^{m.v.}$. Denote by $[x]$ the greatest integer less than or equal to x.

THEOREM 3.1 *Suppose that among regions of interpolation problem* $(\prod_n^3, \mathbb{S})^{m.v.}$ *there are* $[\frac{n}{2}] + 2$ *concentric spheres, where* $n \geq 1$. *Then the mean-value interpolation problem* $(\prod_n^3, \mathbb{S})^{m.v.}$ *is not correct.*

For proof of this theorem we need the following

LEMMA 3.2 *Assume that Theorem 3.1 is valid for* $n = 2t$ *then it is valid for* $n = 2t + 1$, *too.*

Proof We assume that the parameters related to the concentric spheres are linearly dependent in the case n = 2t, i.e.,

$$\sum_{l=1}^{t+2} c_l \iiint_{s_l} p \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = 0, \ \forall p \in \prod_{2t}^{3},$$

Where not all S_l's are zero. Without loss of generality, assume that the center of concentric spheres is the origin. Now for any $p \in \ \prod_{2k+1}^3$we have

$$\sum_{l=1}^{t+2} c_l \iiint_{s_l} p(x,y,z) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \sum_{l=1}^{t+2} c_l \iiint_{s_l} a_{2t+1,0,0} x^{2t+1} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z + \cdots$$
$$+ \sum_{l=1}^{t+2} c_l \iiint_{s_l} a_{0,0,2t+1} z^{2t+1} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z + \sum_{l=1}^{t+2} c_l \iiint_{s_l} q(x,y,z) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z$$

where $q \in \prod_{2t}^{3}$. Hence

$$\sum_{l=1}^{t+2} c_l \iiint_{s_l} p(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0 + \cdots + 0 + \sum_{l=1}^{t+2} c_l \iiint_{s_l} q(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Therefore we have same linear dependence for $\prod_{2k+1}^{3}$:

$$\sum_{l=1}^{t+2} c_l \iiint_{s_l} p(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \sum_{l=1}^{t+2} c_l \iiint_{s_l} q(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0$$

■

Proof [of Theorem 3.1] First consider the case $n = 1$. This case follows from Lemma 2.4, since the center of two concentric spheres and the centers of other spheres are on a hyperplane.
In view of Lemma 3.2 it is sufficient to assume that $n = 2t$, where $n > 1$.

To prove the mean-value interpolation is not correct, according to proposition 1.1 it is enough to show

$$\exists p \in \prod_{n}^{3} \backslash \{0\}, \text{such that } \iiint_{s_l} p(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0,\ l = 1, \cdots, N \tag{5}$$

where S_l's, $l = 1, \ldots, t+2$ are the concentric spheres with center $(0,0,\ 0)$ and radius r_l and $N-(t+2)$ other regions are arbitrary given. If the degree of monomials x or y or z are odd the integral of monomials on concentric spheres will equal to zero. Thus we can consider only the even monomials.
One can compute the integral over concentric spheres

$$\begin{aligned}\iiint_{s_l} p(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= r_l^3 \iiint_{s:x^2+y^2+z^2\leqslant 1} p(r_l x, r_l y, r_l z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= r_l^3 \iiint_{s} \sum_{i+j+k\leqslant n} a_{ijk} x^i y^j z^k\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= r_l^3 \iiint_{s} \sum_{i+j+k\leqslant n} a_{ijk} r_l^{i+j+k} \iiint x^i y^j z^k\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\end{aligned}$$

Hence,

$$\begin{aligned}\iiint_{s_l} p(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r_l^3 \frac{4}{3}\pi[a_{0,0,0} r_l^0 + r_l^2 \frac{1}{5}(a_{2,0,0} + a_{0,2,0} + a_{0,0,2})] + \cdots \\ + r_l^{2t+2}[a_{2t,0,0} \iiint_{s_l} x^{2t}(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ + a_{2t-2,2,0} \iiint_{s_l} x^{2t-2} y^2 (x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \cdots \\ a_{0,0,2t} \iiint_{s_l} z^{2t}(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z]\end{aligned} \tag{6}$$

Thus in view of the relation
$\iiint_{s_l} p(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \sum_{i+j+k\leqslant n} a_{ijk} \iiint x^i y^j z^k\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0, l = 1, \cdots, N$ and

formula (6) we have the following homogeneous linear system:

$$a_{0,0,0} = 0,$$
$$a_{2,0,0} + a_{0,2,0} + a_{0,0,2} = 0,$$
$$.,$$
$$.,$$
$$a_{2t-2,0,0} \iiint_{s_l} x^{2t-2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \cdots + a_{0,0,2t-2} \iiint_{s_l} z^{2t-2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0,$$
$$a_{2t,0,0} \iiint_{s_l} x^{2t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + a_{2t-2,0,0} \iiint_{s_l} x^{2t-2}y^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \cdots$$
$$+ a_{0,0,2t} \iiint_{s_l} z^{2t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z]$$

Next we add to this system the homogeneous conditions over the remaining we add to this system the homogeneous conditions from (5) over the other arbitrary regions:

$$\iiint_{s_l} p(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0,\ l = t + 3, \ldots, N.$$

Now the resulting system has $N\ -1$ equations and N unknowns which are the coefficients of p. Therefore it has a non-trivial solution. It is easily seen that the polynomial with these coefficients satisfy (5). Hence the problem is not correct. ■

4. Conclusion

In this research one can conclude that the corresponding interpolation polynomials are not unique. Namely, if some the regions are concentric spheres on the space then the interpolation problem is not poised.

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