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An L^p-L^q-Version of Morgan's Theorem for the Generalized Bessel Transform

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Abstract. The aim of this paper is to prove new quantitative uncertainty principle for the generalized Bessel transform $\mathcal{F}_\Delta$ connected with a generalized Bessel operator Δ on the half line. More precisely we prove An L^p-L^q-version of Morgan's theorem for the generalized Bessel transform.

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1. Introduction

Heisenberg in [9] proved the uncertainty principle. It states that the more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa. This physical idea is illustrated by Heisenberg inequality as mathematical formulations

$$\int_{\mathbb{R}} x^2|f(x)|^2dx \int_{\mathbb{R}} \xi^2|\widehat{f}(\xi)|^2d\xi \geqslant \frac{1}{4}\int_{\mathbb{R}} |f(x)|^2dx.$$

As representation of this, one has Hardy's theorem [1], Morgan's theorem[2]. These theorems have been generalized to many other situations; see, for example, [8, 10].

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In this paper we establish an analogous of L^p-L^q-version of Morgan's theorem for the generalized Bessel transform $\mathcal{F}_\Delta$ associated with the Generalized Bessel operator Δ introduced and studied in [4, 5]. we prove that if $1 \leqslant p, q \leqslant \infty$, $a > 0$, $b > 0$, $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$, then for all measurable function f on $\mathbb{R}$, the conditions

$$e^{a|x|^\gamma} f \in L^p_{\alpha,n}$$

and

$$e^{b|\lambda|^\eta} \mathcal{F}_\Delta(f)(\lambda) \in L^q_{\alpha,n}$$

imply $f = 0$ if

$$(a\gamma)^{\frac{1}{\gamma}} (b\eta)^{\frac{1}{\eta}} > \left(\sin\left(\frac{\pi}{2}(\eta - 1)\right)\right)^{\frac{1}{\eta}}.$$

The structure of the paper is as follows: In section 2, we set some notations and collect some basic results about the Bessel operator and the Bessel transform. In section 3, we give some facts about harmonic analysis related to the second-order singular differential operator on the half line Δ and generalized Bessel transform. In section 4, we state and prove an $L^p - L^q$-version of Morgan's theorem for the generalized Bessel transform. Finally, we conclude the article in Section 5.

2. Preliminaries

In this section, we recapitulate some facts about harmonic analysis related to the Bessel operator $\mathcal{L}_\alpha$. We cite here, as briefly as possible, some properties. For more details we refer to [3, 11].
Throughout this paper we assume that $\alpha > \frac{-1}{2}$.
Defined L^p_α, $1 \leqslant p \leqslant \infty$, as the class of measurable function f on $[0, +\infty[$ for which $\|f\|_{p,\alpha} < \infty$, where

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{\frac{1}{p}} \quad if\ p < \infty$$

and

$$\|f\|_{\infty,\alpha} = \|f\|_\infty = ess\ sup_{x \geqslant 0} |f(x)|.$$

The Bessel operator $\mathcal{L}_\alpha$ is defined as following:

$$\mathcal{L}_\alpha f(x) = \frac{d^2}{dx^2} f(x) + \frac{2\alpha + 1}{x} \frac{d}{dx} f(x).$$

The Fourier transform of ordre α is defined for a function $f \in L^1_\alpha$ by

$$\mathcal{F}_\alpha(f)(\lambda) = \int_0^\infty f(x) j_\alpha(\lambda x) x^{2\alpha+1} dx, \quad \lambda \geqslant 0, \tag{2.1}$$

where

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{z}{2})^{2n}}{n!\,\Gamma(n+\alpha+1)} \quad (z\in\mathbb{C}). \tag{2.2}$$

is the normalized Bessel function of index α.

Proposition 2.1

(i) If both f and $\mathcal{F}_\alpha$ are in L^1_α then

$$f(x) = \frac{1}{4^\alpha(\Gamma(\alpha+1))^2}\int_0^\infty \mathcal{F}_\alpha(f)(\lambda)j_\alpha(\lambda x)\lambda^{2\alpha+1}d\lambda, \quad for\ almost\ all\ x\geqslant 0$$

(ii) For every $f\in L^1_\alpha\bigcap L^2_\alpha$ we have

$$\int_0^\infty |f(x)|^2x^{2\alpha+1}dx = \frac{1}{4^\alpha(\Gamma(\alpha+1))^2}\int_0^\infty |\mathcal{F}_\alpha(f)(\lambda)|^2\lambda^{2\alpha+1}d\lambda.$$

3. Harmonic Analysis Associated with Δ

In this section we provide some facts about harmonic analysis related to the second-order singular differential operator on the half line Δ. We cite here, as briefly as possible, some properties. For more details we refer to [4, 5].
Consider the second-order singular differential operator on the half line

$$\Delta f(x) = \frac{d^2}{dx^2}f(x) + \frac{2\alpha+1}{x}\frac{d}{dx}f(x) - \frac{4n(\alpha+n)}{x^2}f(x)$$

where $\alpha > \frac{-1}{2}$ and $n = 0,1,2,....$ For $n = 0$ we regain the Bessel operator $\mathcal{L}_\alpha$.
Let $\mathcal{M}$ be the map defined by

$$\mathcal{M}f(x) = x^{2n}f(x).$$

Let $L^p_{\alpha,n}, 1\leqslant p\leqslant\infty$, be the class of measurable functions f on $[0,\infty[$ for which

$$\|f\|_{p,\alpha,n} = \|\mathcal{M}^{-1}f\|_{p,\alpha+2n} < \infty.$$

and

$$d\mu_\alpha(x) = x^{2\alpha+1}dx$$

. For $\lambda\in\mathbb{C}$ and $x\in\mathbb{R}$, put

$$\varphi_\lambda(x) = x^{2n}j_{\alpha+2n}(\lambda x), \tag{3.1}$$

where $j_{\alpha+2n}$ is the normalized Bessel function of index $\alpha+2n$ given by (3.1).

Proposition 3.1

- φ_λ possesses the Laplace integral representation

$$\varphi_\lambda(x) = a_{\alpha+2n}x^{2n}\int_0^1 cos(\lambda tx)(1-t^2)^{\alpha+2n-\frac{1}{2}}dt, \tag{3.2}$$

where

$$a_{\alpha+2n} = \frac{2\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+\frac{1}{2})}.$$

- φ_λ satisfies the differential equation

$$\Delta\varphi_\lambda = -\lambda^2\varphi_\lambda$$

- For all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$,

$$|\varphi_\lambda(x)| \leqslant x^{2n} e^{|Im\lambda||x|}. \tag{3.3}$$

Definition 3.2 The generalized Bessel transform is defined for a function $f \in L^1_{\alpha,n}$ by

$$\mathcal{F}_\Delta(f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x)x^{2\alpha+1}dx, \quad \lambda \geqslant 0. \tag{3.4}$$

Remark 3.3

- By (2.1) and (3.2) observe that

$$\mathcal{F}_\Delta = \mathcal{F}_{\alpha+2n} \circ \mathcal{M}^{-1}, \tag{3.5}$$

where $\mathcal{F}_{\alpha+2n}$ is the Fourier transform of order $\alpha + 2n$ given by (2.1).

Theorem 3.4

(i) For every $f \in L^1_{\alpha,n} \bigcap L^2_{\alpha,n}$ we have the Plancherel formula

$$\int_0^\infty \mid f(x) \mid^2 x^{2\alpha+1}dx = \frac{1}{4^{\alpha+2n}(\Gamma(\alpha+2n+1))^2}\int_0^\infty \mid \mathcal{F}_\Delta(f)(\lambda) \mid^2 \lambda^{2(\alpha+2n)+1}d\lambda.$$

(ii) The generalized Fourier transform $\mathcal{F}_\Delta$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2([0,\infty[, \frac{1}{4^{\alpha+2n}(\Gamma(\alpha+2n+1))^2}\lambda^{2(\alpha+2n)+1}d\lambda)$.

4. An L^p-L^q-Version of Morgan's Theorem for F_Δ

We start by getting the following lemma of Phragmen-Lindlöf type using the same technique as in [6, 7]. We need this lemma to prove the main result of this paper.

Lemma 4.1 Suppose that $\rho \in]1,2[$, $q \in [1,\infty]$, $\sigma > 0$ and $B > \sigma \sin\left(\frac{\pi}{2}(\rho-1)\right)$. If g is an entire function on $\mathbb{C}$ verifiying:

$$|g(x+iy)| \leqslant C.e^{\sigma|y|^\rho} \tag{4.1}$$

and

$$e^{B|x|^\rho} g/_{\mathbb{R}} \in L^q_{\alpha,n} \tag{4.2}$$

for all $x, y \in \mathbb{R}$ then $g = 0$.

Theorem 4.2 Let $1 \leqslant p, q \leqslant \infty$, $a > 0$, $\quad b > 0$, $\quad \gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$, then for all measurable function f on $\mathbb{R}$, the conditions

$$e^{a|x|^\gamma} f \in L^p_{\alpha,n} \tag{4.3}$$

and

$$e^{b|\lambda|^\eta} \mathcal{F}_\Delta(f)(\lambda) \in L^q_{\alpha,n} \tag{4.4}$$

imply $f = 0$ if

$$(a\gamma)^{\frac{1}{\gamma}} (b\eta)^{\frac{1}{\eta}} > \left(\sin\left(\frac{\pi}{2}(\eta-1)\right)\right)^{\frac{1}{\eta}}. \tag{4.5}$$

Proof The function

$$\mathcal{F}_\Delta(f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x) x^{2\alpha+1} dx, \quad \lambda \geqslant 0.$$

is well defined, entirely on $\mathbb{C}$, and satisfies the condition

$$\begin{aligned}
|\mathcal{F}_\Delta(f)(\lambda)| &= |\int_0^\infty f(x)\varphi_\lambda(x) d\mu_\alpha(x)|, \\
&\leqslant \int_0^\infty |f(x)||x|^{2n} e^{|x||\zeta|} d\mu_\alpha(x), \\
&= \int_0^\infty |\mathcal{M}_n^{-1} f(x)| e^{|x||\zeta|} d\mu_{\alpha+2n}(x), \quad \forall \lambda = \xi + i\zeta \in \mathbb{C}
\end{aligned}$$

Applying Hölder inequality, we get

$$\begin{aligned}
|\mathcal{F}_\Delta(f)(\lambda)| &\leqslant |\left(\int_0^\infty \left(\mathcal{M}_n^{-1} f(x)|e^{a|x|^\gamma}\right)^p d\mu_{\alpha+2n}(x)\right)^{\frac{1}{p}} \left(\int_0^\infty \left(|e^{-a|x|^\gamma} e^{|x||\zeta|}\right)^{p'} d\mu_{\alpha+2n}(x)\right)^{\frac{1}{p'}}, \\
&\leqslant C \left(\int_0^\infty \left(|e^{-a|x|^\gamma} e^{|x||\zeta|}\right)^{p'} d\mu_{\alpha+2n}(x)\right)^{\frac{1}{p'}}.
\end{aligned}$$

where p' is the conjugate exponent of p.
Let $C \in I =](b\eta)^{\frac{-1}{\eta}} \sin\left(\frac{\pi}{2}(\eta-1)\right)^{\frac{1}{\eta}}, (a\gamma)^{\frac{1}{\gamma}}[$.
Applying the convex inequality

$$|ty| \leqslant (\frac{1}{\gamma})|t|^\gamma + (\frac{1}{\eta})|y|^\eta \tag{4.6}$$

to the positive numbers $C|x|$ and $\frac{|\zeta|}{C}$, we obtain

$$|x||\zeta| \leqslant (\frac{C^\gamma}{\gamma})|x|^\gamma + (\frac{1}{\eta C^\eta})|\zeta|^\eta \tag{4.7}$$

and the following relation holds

$$\int_0^\infty e^{-ap'|x|^\gamma} e^{p'|x||\zeta|} d\mu_{\alpha+2n}(x) \leqslant e^{\frac{p'|\zeta|^\eta}{\eta C^\eta}} \int_0^\infty e^{-p'(a-\frac{C^\gamma}{\gamma})|x|^\gamma} d\mu_{\alpha+2n}(x). \tag{4.8}$$

Since $C \in I$, then $a > \frac{C^\gamma}{\gamma}$, and thus the integral

$$\int_0^\infty e^{-p'(a-\frac{C^\gamma}{\gamma})|x|^\gamma} d\mu_{\alpha+2n}(x)$$

is finite. Moreover

$$|\mathcal{F}_\Delta(f)(\lambda)| \leqslant Const.e^{\frac{p'|\zeta|^\eta}{\eta C^\eta}}, \quad forall \lambda \in \mathbb{C}. \tag{4.9}$$

By virtue of relations(4.3), (4.4), (4.9) and Lemma 4.1 , we obtain that $\mathcal{F}_\Delta f = 0$. Then $f = 0$ by Theorem 3.4. ■

5. Conclusion

In this paper, using a generalized Bessel transform associated with the generalized Bessel operator, we obtained an L^p-L^q-version of Morgan's. We proved that if $1 \leqslant p, q \leqslant \infty$, $a > 0$, $b > 0$, $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$, then for all measurable function f on $\mathbb{R}$, the conditions

$$e^{a|x|^\gamma} f \in L^p_{\alpha,n}$$

and

$$e^{b|\lambda|^\eta} \mathcal{F}_\Delta(f)(\lambda) \in L^q_{\alpha,n}$$

imply $f = 0$ if

$$(a\gamma)^{\frac{1}{\gamma}} (b\eta)^{\frac{1}{\eta}} > \left(\sin\left(\frac{\pi}{2}(\eta-1)\right)\right)^{\frac{1}{\eta}}.$$

The demonstration of this result is based on the lemma of Phragmen-Lindlöf type.

References

[1] G. H. Hardy, "A theorem concerning Fourier transforms," Journal of the London Mathematical Society, vol. 8, pp. 227-231, 1933.

[2] G. W. Morgan, "A note on Fourier transforms," Journal of the London Mathematical Society, vol. 9, pp. 187-192, 1934.

[3] K. Trimèche, Generalized Harmonic Analysis and Wavelet Packets, Gordon and Breach Science Publishers, 2001.

[4] R.F. Al Subaie and M.A. Mourou, Transmutation operators associated with a Bessel type operator on the half line and certain of their applications, Tamsui Oxford Journal of Information and Mathematical Sciences 29(3) (2013) 329-349, Aletheia University.

[5] R.F. Al Subaie , M.A. Mourou (2013). The Continuous Wavelet Transform for A Bessel Type Operator on the Half Line. Mathematics and Statistics, 1 , 196 - 203. doi: 10.13189/ms.2013.010404.

[6] S. Ayadi, An L^p-L^q-version of Morgan's theorem for the dunkl transform, Math. Sci. Res. J. 8(11) (2004) 318-327.

[7] S. Ben Farah and K. Mokni, Uncertainty principle and the L^p-L^q-version of Morgan's theorem on some groups, Russ. J. Math. Phys. 10(3) (2003) 1-16.

[8] S. C. Bagchi and S. K. Ray, "Uncertainty principles like Hardy's theorem on some Lie groups," Journal of the Australian Mathematical Society. Series A, vol. 65, no. 3, pp. 289-302, 1998.

[9] W. Heisenberg, Über den anschaulichen inhalt der quantentheoretischen kinematik und machanik, Z. Phys. 43 (1927) 172-198.

[10] A. Abouelaz, R. Daher and El. Loualid, An L^p-L^q-Version of morgan's theorem for the generalized Dunkl transform, to appear in FACTA UNIVERSITATIS (NITS) Ser. Math. Inform.

[11] M. El Hammaa and R. Daherb, $(\delta, \gamma, 2)$-Bessel Lipschitz Functions in the Space $L_{2,\alpha}(\mathbb{R}^+)$, International Journal of Mathematical Modelling and Computations Vol. 04, No. 01, (2014) 17-23.