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Ranking DMUs on the benchmark line with equal shadow prices

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Abstract. Data envelopment analysis (DEA) with considering the best condition for each decision making unit (DMU) assesses the relative efficiency for it and divides a homogenous group of DMUs into two categories: efficient and inefficient, but traditional DEA models can not rank efficient DMUs. Although some models were introduced for ranking efficient DMUs, Franklin Lio & Hsuan peng (2008), proposed a common weights analysis (CWA) approach for ranking them. These DMUs are ranked according to the efficiency score weighted by the common set of weights and shadow prices. This study shows there are some cases that shadow prices of efficient DMUs are equal, hence this method is not applicable for ranking them. Next, we propose a new method for ranking units with equal shadow prices.

Keywords: Data Envelopment Analysis, Shadow Price, Common Weight Analysis, Benchmark Line.

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1. Introduction

Abstract. Data envelopment analysis (DEA) is a linear programming method for assessing the efficiency and productivity of Decision Making Units (DMU). DEA is widely applied to measure the efficiency of various homogenous units, such as banks, airlines, hospitals, universities, and manufactures. DEA models partitioned DMUs into two groups: efficient, and inefficient. Efficiency score is a positive number and not more than one. Hence, the efficiency score of efficient DMUs is equal to one and this score for inefficient DMUs is less than one. Efficient DMUs have identical efficiency scores, but practically they do not have the equivalent performances. Although, traditional DEA models cannot discriminate between these DMUs, some models are introduced for ranking them.

Andersen et al. (1993) evaluate that a DMUs efficiency possibly exceeds the conventional score 1, by comparing the DMU being evaluated with a linear combination of other DMUs, while excluding the observations of the DMU being eval-

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uated. They try to discriminate between these efficient DMUs, by using different efficiency scores larger than 1. Cook et al. (1992) developed prioritization models to rank only the efficient units in DEA. They divide those with equal scores, on the boundary, by imposing the restrictions on the multipliers (weights) in a DEA analysis. Torgersen et al. achieved a complete ranking of efficient DMUs by measuring their importance as a benchmark for inefficient DMUs.

Franklin Liu & Hsuan peng (2008), proposed the common weights analysis (CWA) methodology for ranking efficient DMUs. In this methodology one linear programming (LP) problem, for all efficient DMUs, is solved for obtaining one common set of weights (CSW). Then these CSWs are used to evaluate the absolute efficiency of each efficient DMUs and rank them. There are some cases that the new efficiency score of efficient DMUs by CSW is equal to 1, as well. In these cases DMUs could be ranked by considering their shadow prices. Indeed, we are looking for the largest shadow price.

In this paper, we show there are some cases that this methodology cannot discriminate between efficient DMUs. A numerical example with 14 DMUs is introduced that after using CWA methodology there are 8 efficient DMUs with equal new efficiency scores and equal shadow prices, as well. Next, we propose a new algorithm for ranking efficient DMUs in DEA. The remaining sections of this paper are as follows: in section 2, we review the concept of DEA framework, reconsider CWA methodology, and introduce the mentioned numerical example. In section 3, we introduce a new algorithm for ranking efficient DMUs and apply this algorithm on the numerical example. Finally in section 4 we make our overall conclusion.

2. BACKGROUND MODELS

DEA developed as one methodology for assessing the comparative efficiency of DMUs. Assume that there are n DMUs ($DMU_j : j = 1, 2, \ldots, n$) which consume the number of m inputs ($x_{ij} : i = 1, 2, \ldots, m$) to produce the number of s outputs ($y_{rj} : r = 1, 2, \ldots, s$). By solving one LP, DEA measures the best practice comparative efficiency of each DMU and hence needs to solve n LPs for evaluating all DMUs. Multiplier model of CCR evaluates the efficiency score of DMU_o, the DMU under consideration, by solving the following LP:

$$
\begin{aligned}
h_o^* = \max_{v_i, u_r} & \; h_o \\
\text{s.t.} \quad & h_j \leqslant 1 \\
& v_i \geqslant \varepsilon \;\; i = 1, 2, \ldots, m \\
& u_r \geqslant \varepsilon \; r = 1, 2, \ldots, s
\end{aligned}
\tag{1}
$$

where

$$h_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}$$

$$h_j = \frac{\sum\limits_{r=1}^{s} u_r y_{rj}}{\sum\limits_{i=1}^{m} v_i x_{ij}}$$

Let (h_o^*, v_o^*, u_o^*) be the optimal solution of this model. Hence, h_o^* is the relative efficiency and (v_o^*, u_o^*) are the optimal weights of inputs and outputs of DMU_o, respectively. The Non-Archimedean infinitesimal, ε, is a positive constant which is used in order to avoid zero weights. If $h_o^* = 1$, then DMU_o is called efficient DMU.

As noted by Love11 et al. (1994) model (1) can lead to a large number of DMUs having DEA scores of unity. To avoid this possibility, they employed a DMU ranking procedure developed by Andersen and Petersen (1993). That is, by excluding the constraint for the DMU_o, $h_o < l$, in model (1), we obtain

$$\begin{aligned} &\max_{v_i, u_r} h_o \\ &\text{s.t.} \quad h_j \leqslant 1 \ j \neq 0 \\ &\qquad\ \ v_i \geqslant \varepsilon \ \ i = 1, 2, \ldots, m \\ &\qquad\ \ u_r \geqslant \varepsilon \ r = 1, 2, \ldots, s \end{aligned} \tag{2}$$

For the sake of computation, we may use the following linear programming problem which is equivalent to model (2) by duality

$$\begin{aligned} \nu_o^* = &\min_{\nu_o, \lambda_j} \nu_o \\ &\text{s.t.} \ \sum_{j \neq o} \lambda_j x_{ij} \leqslant \nu_o x_{io}, \ i = 1, \ldots, m \\ &\qquad \sum_{j \neq o} \lambda_j y_{rj} \geqslant y_{ro}, \quad r = 1, \ldots, s \\ &\qquad \lambda_j \geqslant 0 \qquad\qquad\quad j \neq o. \end{aligned} \tag{3}$$

Then the optimal values to model (3) (or model (2)) ν_o^* can be either less than, or equal to, or greater than one. Now we are able to rank the DMUs according to their aggregated output to aggregated input ratios given by ν_o^*. Note that $h_o^* = \nu_o^*$ if $h_o^* < 1$ in model (1).

In more recent literature there have been various approaches put forward for dealing with ranking efficient DMUs. Let V_i', $(i = 1, 2, \ldots, m)$ and U_r', $(r = 1, 2, \ldots, s)$ be two arbitrary common weights vectors for the ith input and rth output of DMU_j, $j \in E = \{j | h_j^*, j = 1, 2, \ldots, n\}$. In Figure 1 the vertical and horizontal axis are set to be virtual input $(\sum_{i=1}^{m} V_i' x_{ij},\ j \in E)$ and virtual output $(\sum_{r=1}^{s} U_r' y_{rj},\ j \in E)$, respectively. Since the maximum efficiency score is one, the benchmark line is the straight line that passes through the origin with slope one. $\triangle_j^{I'}$ and $\triangle_j^{O'}$ are the virtual gaps between DMU_j and its projection point on the benchmark line. CWA considers only efficient DMUs and determines an optimal CSW, such that theses DMUs could be as close as possible to their projection points on the benchmark line.

We can apply the following LP:

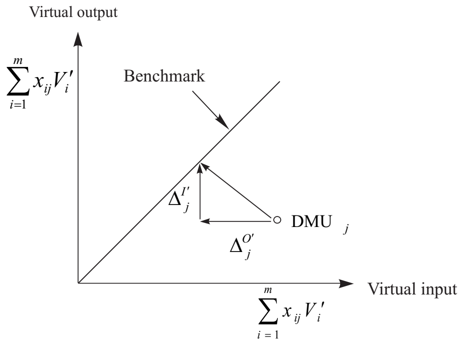

Figure 1.

$$
\begin{aligned}
\triangle^* = \min & \sum_{j\in E} \triangle_j \\
\text{s.t.} & \sum_{r=1}^{s} U_r y_{rj} - \sum_{i=1}^{m} V_i x_{ij} + \triangle_j = 0 \; j\in E \\
& V_i \geqslant \varepsilon \qquad\qquad i = 1,2,\dots,m \\
& U_r \geqslant \varepsilon \qquad\qquad r = 1,2,\dots,s
\end{aligned}
\tag{4}
$$

where $\triangle_j = \triangle_j^I + \triangle_j^O$. Model (4) is equivalent to the following model:

$$
\begin{aligned}
-\triangle^* = \max & \sum_{r=1}^{s} Y_r U_r - \textstyle\sum_{i=1}^{m} X_i V_i \\
\text{s.t.} & \sum_{r=1}^{s} U_{rj} y_{rj} - \sum_{i=1}^{m} V_{ij} x_{ij} \leqslant 0 \; j\in E \\
& V_i \geqslant \varepsilon \qquad\qquad i = 1,2,\dots,m \\
& U_r \geqslant \varepsilon \qquad\qquad r = 1,2,\dots,s
\end{aligned}
\tag{5}
$$

where $Y_r = \sum_{j\in E} y_{rj}$ and $X_i = \sum_{j\in E} x_{ij}$. Now, consider the dual form of Model (5) as follows:

$$
\begin{aligned}
\max & \; \varepsilon\Big(\sum_{r=1}^{s} P_r + \sum_{i=1}^{m} Qi\Big) \\
\text{s.t.} & \sum_{j\in E} \pi_j y_{rj} - P_r = Y_r \; r = 1,\dots,s \\
& \sum_{j\in E} \pi_j x_{ij} - Q_i = X_i \; i = 1,\dots,m \\
& \pi_j \geqslant 0 \qquad\qquad j \in E \\
& P_r \geqslant 0 \qquad\qquad r = 1,2,\dots,s \\
& Q_i \geqslant 0 \qquad\qquad i = 1,2,\dots,m
\end{aligned}
\tag{6}
$$

The CWA-efficiency score of each DMU_j, $j \in E$ is equal to $\zeta_j^* = \frac{\sum_{r=1}^{s} U_r^* y_{rj}}{\sum_{i=1}^{m} V_i^* x_{ij}}$, where V^* and U^* are the optimal CSW obtained by solving Model (5). Therefore DMU_i has a higher rank than DMU_j, if one of the following conditions is satisfied:

1. $\zeta_i^* > \zeta_j^*$.
2. $\zeta_i^* = \zeta_j^* < 1 \quad \& \quad \triangle_i^* < \triangle_j^*$.
3. $\zeta_i^* = \zeta_j^* = 1 \quad \& \quad \pi_i^* > \pi_j^*$.

The decision variable π_j^* in Model (6) is also the shadow price of Model (5), and hence represents the total virtual gap scale to the benchmark line that can be reduced while we release the upper bound of efficiency 1 for DMUs.

Now we introduce a numerical example that, for some cases, none of these conditions are hold and hence CWA cannot rank these efficient DMUs.

3. Numerical Example

Consider a data set of a numerical example which shown in Table (1). This example contains 14 DMUs with 2 inputs and 1 output and, as it is shown in Table (1), there are 8 CCR-efficient DMUs.

Table 1.

DMUs	x_1	x_2	y_1	CCR-efficiency
1	1.692	0.564	0.564	1.000
2	1.068	1.068	0.534	1.000
3	0.984	2.952	0.984	1.000
4	0.456	2.280	0.456	0.667
5	5.136	3.852	0.856	0.381
6	3.738	3.204	0.534	0.308
7	2.363	2.025	0.675	0.615
8	1.424	2.848	0.712	0.667
9	0.890	1.780	0.356	0.533
10	0.181	0.399	0.145	1.000
11	1.455	0.873	0.582	1.000
12	2.366	0.806	0.793	1.000
13	0.158	3.462	0.905	1.000
14	0.334	2.234	0.642	1.000

Table (2) gives the detailed ranking information assessed by adopting CWA and A&P methodologies for 8 CCR-efficient DMUs:

Table 2.

Efficient DMUs	$\triangle_j^*$	π_j^*	ζ_j^*	A&P Rank	CWA Rank
1	0	4.814	1	2	1
2	0	0	1	3	3
3	0	0	1	3	3
10	0	0	1	3	3
11	0	0	1	3	3
12	0	0	1	3	3
13	0	2.890	1	1	2
14	0	0	1	3	3

There are 6 A&P and CWA-efficient DMUs with equal shadow prices. Hence, A&P and CWA- methodologies cannot rank them.

4. A NEW ALGORITHM FOR RANKING EFFICIENT DMUS

Now, we introduce a new algorithm for ranking efficient DMUs based on a new model. This model finds CSW in the worst condition such that only one efficient DMU lies on the benchmark line.

$$
\begin{array}{lll}
\max & \sum\limits_{r=1}^{s} U_r - \sum\limits_{i=1}^{m} V_i & \\
\text{s.t.} & \sum\limits_{r=1}^{s} U_r y_{rj} - \sum\limits_{i=1}^{m} V_i x_{ij} + \triangle_j = 0 & j \in E \\
& m d_j \leqslant \triangle_j \leqslant M d_j & j \in E \\
& \sum\limits_{j \in E} d_j = |E| - 1 & \\
& d_j \in \{0,1\} & j \in E \\
& V_i \geqslant \varepsilon & i = 1,2,\ldots,m \\
& U_r \geqslant \varepsilon & r = 1,2,\ldots,s \\
& \triangle_j \geqslant 0 & j \in E
\end{array}
\tag{7}
$$

where m and M are two positive parameters, and $\triangle_j$ is the virtual gap of DMU_j from the benchmark line. d_j is a binary variable and $\sum_{j\in E} d_j = |E| - 1$ forces only one of these binary variables must be equal to zero. Also, if $d_j = 0$, then $\triangle_j = 0$; and otherwise $\triangle_j > 0$. On the other hand, the CCR-efficient DMU_j has the highest rank iff $\triangle_j = 0$. Considering the objective function, Model (7) finds CSW with lowest value of output multiple and highest value of input multiple-worst condition.

Model (7) just finds a DMU with highest rank and for ranking all efficient DMUs we propose the following algorithm:

step 0: Let $E = \{j|\ DMU_j \quad \text{is CCR-efficient}\}$.
step 1: Solve Model (7) and suppose $\triangle_p = 0$.
step 2: Let $E = E - \{p\}$.
step 3: If $|E| = 1$, then stop; otherwise go to step 1.

Table (3) shows the result of applying the above algorithm on the data set in Table (1):

Table 3.

DMUs	13	14	1	3	12	10	11	2
Rank	1	2	3	4	5	6	7	8

Therefore this algorithm is applicable in this case.

5. CONCLUSIONS

In this study we used CWA methodology for ranking efficient DMUs. This methodology ranks efficient DMUs considering the efficiency score weighted by the CSWs and shadow prices. The disability of this methodology is shown by a numerical example that shadow prices of efficient DMUs are equal. Next, we proposed a new algorithm for ranking efficient DMUs with equal shadow prices.

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