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Spectral methods using a finite class of orthogonal polynomials related to inverse Gamma distribution

A. H. Salehi Shayegan [a,*]

[a] *Mathematics Department, Faculty of Basic Science, Khatam-ol-Anbia (PBU) University, Tehran, Iran.*

Abstract. Classical orthogonal polynomials of Jacobi, Laguerre and Hermite are characterized as the infinite sequences of orthogonal polynomials. In this paper, we present a sequence of orthogonal polynomials which is finitely orthogonal with respect to inverse Gamma distribution on infinite interval. General properties of this sequence such as orthogonality relation, Rodrigues type formula, recurrence relations and also some of its applications such as Gauss quadrature, Gauss-Radau quadrature formulas and so on are indicated. In addition, it is well-known that spectral methods for unbounded domains can be essentially classified into four categories; domain truncation, approximation by classical orthogonal systems on unbounded domains, approximation by other non-classical orthogonal systems and mapping. In this paper based on the second category, we propose a spectral method using the finite class of orthogonal polynomial $N_n^{(p)}(x)$ related to inverse Gamma distribution. Error analysis and convergence of the method are thoroughly investigated. At the end, two numerical examples are given for the efficiency and accuracy of the proposed method.

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1. Introduction

Consider the following Sturm-Liouville equation

$$\sigma(x)y_n''(x) + \tau(x)y_n'(x) - \lambda_n y_n(x) = 0, \tag{1}$$

*Corresponding author Email: ahsalehi.kau@gmail.com, ah.salehi@mail.kntu.ac.ir

where $\sigma(x) = ax^2 + bx + c$ and $\tau(x) = dx + e$ are polynomials independent of n and $\lambda_n = n(n-1)a + nd$ is the eigenvalue parameter depending on $n = 0, 1, \cdots$. It is well-known that six special classes of orthogonal polynomials can be derived from the differential equation (1) [1–3]. The first category of orthogonal polynomials, i.e, the Jacobi, Laguerre and Hermite polynomials, are known as the infinite classical orthogonal polynomials and the second one are finite.

It is worth to point out that, for the infinite orthogonal polynomials, the weight functions are related to probability density function of Beta, Gumma and Normal distributions for Jacobi, Laguerre and Hermite polynomials, respectively. Also, three finite classical orthogonal polynomials, i.e., $M_n^{(p,q)}$, $I_n^{(p)}$ and $N_n^{(p)}$ are orthogonal with respect to probability density F distribution, Student's T-distribution and Inverse Gama distribution [1, 4]. Table 1 shows these six classical orthogonal polynomials in details.

Two infinite orthogonal polynomials, namely Laguerre and Hermite polynomials are defined in unbounded domains. So, in the last two decades, a considerable progress has been made in using them in spectral methods for solving PDEs. But, in this paper, we study spectral approximations by orthogonal polynomials $N_n^{(p)}(x)$ with the weight function $w^{(p)} = x^{-p}e^{-\frac{1}{x}}$ on unbounded interval $[0, \infty)$. In general, spectral methods for unbounded domains can be divided into four parts. Domain truncation is the first one, which is based on truncation of unbounded domains into bounded domains and solve PDEs on bounded domains. The second one as is mentioned is approximation by classical orthogonal polynomials such as Laguerre or Hermite polynomials. The third patr is based on approximation by other non-classical orthogonal system or mapped orthogonal systems. Finally, in the forth part, unbounded domains are mapped to bounded domains and used standard spectral methods to solve the mapped PDEs in bounded domains (see [5] and references therein).

The outline of this paper is as follows. In Section 2, we give some properties of orthogonal polynomials $N_n^{(p)}(x)$. In Section 3, we give the analysis of approximations by $N_n^{(p)}(x)$. These results will be useful for error analysis of spectral methods for unbounded domains. In Section 4, we consider spectral-Galerkin methods and provide an error analysis by the obtained results. Finally, we give some numerical examples and show the validation of proposed method.

Table 1. Table of all six classical orthogonal polynomials.

Type	Polynomial	$\sigma(x)$	$\tau(x)$	Weight function	Interval
Infinite	Jacobi	$1-x^2$	$-(\alpha+\beta+2)x+(\beta-\alpha)$	$(1-x)^\alpha(1+x)^\beta; \alpha, \beta > -1$	$[-1, 1]$
Infinite	Laguerre	x	$\alpha+1-x$	$x^\alpha e^{-x}; \alpha > -1$	$[0, \infty)$
Infinite	Hermite	1	$-2x$	e^{-x^2}	$(-\infty, \infty)$
Finite	$M_n^{(p,q)}$	x^2+x	$(2-p)x+(1+q)$	$x^q(1+x)^{-(p+q)}$	$[0, \infty)$
Finite	$I_n^{(p)}$	x^2+1	$(3-2p)x$	$(1+x^2)^{-(p-\frac{1}{2})}$	$(-\infty, \infty)$
Finite	$N_n^{(p)}$	x^2	$(2-p)x+1$	$x^{-p}e^{-\frac{1}{x}}$	$[0, \infty)$

2. Orthogonal polynomials related to inverse Gamma distribution

2.1 *Orthogonality and its consequences*

Consider the following differential equation

$$x^2y'' + ((2-p)x+1)\,y' - n(n+1-p)y = 0. \tag{2}$$

By applying the Frobenius method an explicit polynomial solution of (2) can be obtained as follows

$$N_n^{(p)}(x) = (-1)^n \sum_{k=0}^{n} k! \binom{p-(n+1)}{k} \binom{n}{n-k} (-x)^k.$$

These polynomials are finitely orthogonal with respect to the weight function $w^{(p)}(x) = x^{-p}e^{-\frac{1}{x}}$ on the half line $\mathbb{R}_+ := [0, \infty)$, i.e.,

$$\int_0^\infty x^{-p}e^{-\frac{1}{x}} N_n^{(p)}(x) N_m^{(p)}(x)dx = 0 \quad \Leftrightarrow \quad \begin{cases} m \neq n\,, \;\; p > 2\ell + 1, \\ \ell = \max\{m, n\}, \end{cases}$$

and

$$\gamma_n^{(p)} := \int_0^\infty x^{-p}e^{-\frac{1}{x}} \left(N_n^{(p)}(x)\right)^2 dx = \frac{n!(p-1-n)!}{p-1-2n}. \tag{3}$$

The Rodrigues' formula for this class of functions takes the form

$$N_n^{(p)}(x) = (-1)^n x^p e^{\frac{1}{x}} \frac{d^n}{dx^n} \left(x^{-p+2n} e^{-\frac{1}{x}}\right).$$

For example, if $n = 0, 1, 2$ and 3, we have

$$\begin{aligned}
&N_0^{(p)}(x) = 1, \\
&N_1^{(p)}(x) = (p-2)x - 1, \\
&N_2^{(p)}(x) = (p-4)(p-3)x^2 - 2(p-3)x + 1, \\
&N_3^{(p)}(x) = (p-6)(p-5)(p-4)x^3 - 3(p-5)(p-4)x^2 + 3(p-4)x - 1.
\end{aligned}$$

The leading coefficient $k_n^{(p)}$ of $N_n^{(p)}(x)$ is

$$k_0^{(p)} = 1, \qquad k_n^{(p)} = \prod_{i=1}^{n} (p - (n+i)). \tag{4}$$

Since the explicit formula of the polynomials exist, one can get the three-term recurrence formula that generates the $N_n^{(p)}(x)$ as follows

$$\begin{aligned}
N_{n+1}^{(p)}(x) = &\left(\frac{(p-(2n+2))(p-(2n+1))}{p-(n+1)} x - \frac{p(p-(2n+1))}{(p-(n+1))(p-2n)} \right) N_n^{(p)}(x) \\
&- \frac{n(p-(2n+2))}{(p-(n+1))(p-2n)} N_{n-1}^{(p)}(x).
\end{aligned} \tag{5}$$

Also, it is not difficult to verify that

$$\frac{d}{dx} N_n^{(p)}(x) = n(p-(n+1)) N_{n-1}^{(p-2)}(x). \tag{6}$$

This means that the finite set $\left\{\frac{d}{dx}N_n^{(p>2\ell+1)}(x)\right\}_{n=1}^{n=\ell}$ is also orthogonal with respect to $w^{(p-2)}(x)$ (see [6]). Moreover, replacing (6) in (2) yields

$$x^2\frac{d}{dx}N_n^{(p)}(x)+(1-px)N_n^{(p)}(x)=N_{n+1}^{(p+2)}(x).$$

2.2 *Gauss quadrature and Gauss-Radau quadrature formulas*

Among all integration rules with $n+1$ points, it is well-known that $(n+1)$-point Gauss quadrature rule

$$\int_a^b f(x)w(x)dx=\sum_{j=0}^{n} f(x_j)w_j+E_n[f], \tag{7}$$

has the highest possible precision degree and is analytically exact for polynomials of degree at most $2n+1$, where nodes x_j are zeros of an orthogonal polynomial, w_j's are corresponding weights and $E_n[f]$ is the quadrature error. If $f(x)\in C^{n+1}[a,b]$, we have

$$E_n[f]=\frac{1}{(n+1)!}\int_a^b f^{(n+1)}(\xi_x)\prod_{i=0}^{n}(x-x_i)dx, \qquad \xi_x\in[a,b]. \tag{8}$$

In next theorem, we obtain the $(n+1)$-point Gauss quadrature rule associated with $N_{n+1}^{(p)}(x)$.

Theorem 2.1 *(Gauss quadrature) Let $\{x_j\}_{j=0}^n$ be the set of zeros of $N_{n+1}^{(p)}(x)$, $(p>2n+3)$. Then there exists a unique set of quadrature weights $\{w_j\}_{j=0}^n$ such that*

$$\int_0^\infty q(x)w^{(p)}(x)dx=\sum_{j=0}^{n} q(x_j)w_j, \qquad \forall q\in P_{2n+1},$$

where the quadrature weights are all positive and given by

$$w_j=\frac{(p-(2n+2))\,n!(p-(n+2))!}{N_n^{(p)}(x_j)N_n^{(p-2)}(x_j)(n+1)(p-(n+2))}. \tag{9}$$

Proof In view of Theorem 3.5 in [5], it suffices to derive the second formula for the quadrature weights in (9). Let L_j be the Lagrange basis polynomials related to $\{x_j\}_{j=0}^n$, i.e.,

$$L_j(x)=\prod_{i=0,\,i\neq j}^{n}\frac{x-x_i}{x_j-x_i},$$

so taking $f(x)=L_j(x)$ in (7) and using (8), we get the quadrature weights

$$w_j=\int_0^\infty L_j(x)w^{(p)}(x)dx, \qquad 0\leqslant j\leqslant n. \tag{10}$$

Applying Christoff-Darboux formula, we have

$$w_j = \int_0^\infty L_j(x) w^{(p)}(x) dx = \frac{k_{n+1}^{(p)}}{k_n^{(p)}} \frac{\left\| N_n^{(p)} \right\|_{w^{(p)}}^2}{N_n^{(p)}(x_j) \frac{d}{dx} N_{n+1}^{(p)}(x_j)}, \qquad 0 \leqslant j \leqslant n.$$

We note that from (6), we have

$$\frac{d}{dx} N_{n+1}^{(p)}(x_j) = (n+1)(p-(n+2)) N_n^{(p-2)}(x_j). \tag{11}$$

Hence, using (3), (4) and (11) yields

$$\begin{aligned} w_j &= \frac{k_{n+1}^{(p)}}{k_n^{(p)}} \frac{\left\| N_n^{(p)} \right\|_{w^{(p)}}^2}{N_n^{(p)}(x_j) \frac{d}{dx} N_{n+1}^{(p)}(x_j)} \\ &= \frac{\prod_{i=1}^{n+1} (p-(n+1+i))}{\prod_{i=1}^{n} (p-(n+i))} \frac{\frac{n!(p-(n+1))!}{p-(2n+1)}}{N_n^{(p)}(x_j) \frac{d}{dx} N_{n+1}^{(p)}(x_j)} \\ &= \frac{(p-(2n+1))(p-(2n+2))}{(p-(n+1))} \frac{n!(p-(n+1))!}{N_n^{(p)}(x_j) N_n^{(p-2)}(x_j)(n+1)(p-(n+2))(p-(2n+1))} \\ &= \frac{(p-(2n+2))\, n!(p-(n+2))!}{N_n^{(p)}(x_j) N_n^{(p-2)}(x_j)(n+1)(p-(n+2))}. \end{aligned}$$

■

Theorem 2.2 *(Gauss-Radau quadrature) Let $x_0 = 0$ and $\{x_j\}_{j=1}^n$ be the zeros of*

$$q_n(x) = \frac{1}{x} \left(N_{n+1}^{(p)}(x) + N_n^{(p)}(x) \right), \qquad p > 2n+3.$$

Then there exists a unique set of quadrature weights $\{w_j\}_{j=0}^n$ such that

$$\int_0^\infty q(x) w^{(p)}(x) dx = \sum_{j=0}^n q(x_j) w_j, \quad \forall q \in P_{2n}.$$

Moreover, the quadrature weights are all positive and can be expressed as

$$\begin{aligned} w_0 &= \frac{1}{(-1)^{n+2}(p-(2n+2))} \int_0^\infty q_n(x) w^{(p)}(x) dx, \\ w_j &= \frac{(p-(2n+1))(p-(2n+2))}{(p-(n+1))} \frac{\| q_{n-1} \|_{\tilde{w}^{(p)}}^2}{q_{n-1}(x_j) {q'}_n(x_j)}, \qquad 1 \leqslant j \leqslant n, \end{aligned} \tag{12}$$

where $\tilde{w}^{(p)}(x) = x w^{(p)}(x)$.

Proof Using (7), (8) and (10) for any $q \in P_n$ we have

$$\int_0^\infty q(x)w^{(p)}(x)dx = \sum_{j=0}^n q(x_j) \int_0^\infty L_j(x)w^{(p)}(x)dx = \sum_{j=0}^n q(x_j)w_j.$$

Next, for any $q \in P_{2n}$, we write

$$q(x) = x\ r(x)q_n(x) + s(x), \quad r \in P_{n-1}, s \in P_n,$$

which implies

$$\begin{aligned}
&\int_0^\infty q(x)w^{(p)}(x)dx = \int_0^\infty x\ r(x)q_n(x)w^{(p)}(x)dx + \int_0^\infty s(x)w^{(p)}(x)dx \\
&= \int_0^\infty r(x)N_{n+1}^{(p)}(x)w^{(p)}(x)dx + \int_0^\infty r(x)N_n^{(p)}(x)w^{(p)}(x)dx + \int_0^\infty s(x)w^{(p)}(x)dx \\
&= \int_0^\infty s(x)w^{(p)}(x)dx = \sum_{j=0}^n s(x_j)w_j = \sum_{j=0}^n q(x_j)w_j. \qquad \forall q \in P_{2n}.
\end{aligned} \tag{13}$$

As is observed

$$\int_0^\infty x\ r(x)q_n(x)w^{(p)}(x)dx = 0.$$

This means that $\{q_n : n \geqslant 0\}$ defines a sequence of polynomials orthogonal with respect to the weight function $\tilde{w}^{(p)}(x) = xw^{(p)}(x)$ where the leading coefficient of q_n is $k_{n+1}^{(p)}$.

Taking $q(x) = L_k^2(x) \in P_{2n}$ in (13), we conclude that $w_k > 0$ for $0 \leqslant k \leqslant n$.

Again form (10), we derive

$$w_j = \int_0^\infty L_j(x)w^{(p)}(x)dx = \int_0^\infty \frac{xq_n(x)}{(q_n(x_j) + x_j q'_n(x_j))(x - x_j)} w^{(p)}(x)dx, \qquad 0 \leqslant j \leqslant n.$$

For $j = 0$, we have

$$\begin{aligned}
w_0 &= \frac{1}{q_n(0)} \int_0^\infty q_n(x)w^{(p)}(x)dx \\
&= \frac{1}{(-1)^{n+2}(n+1)(p-(n+2)) + (-1)^{n+1}n(p-(n+1))} \int_0^\infty q_n(x)w^{(p)}(x)dx \\
&= \frac{1}{(-1)^{n+2}(p-(2n+2))} \int_0^\infty q_n(x)w^{(p)}(x)dx.
\end{aligned}$$

In addition for $1 \leqslant j \leqslant n$, since $\{q_n\}$ are orthogonal with respect to $\tilde{w}^{(p)}$, so the integral part turns out to be the weight of the Gauss quadrature associated with

n nodes being the zeros of $q_n(x)$. Hence, using Theorem 2.1 we can conclude that

$$
\begin{aligned}
w_j &= \frac{1}{x_j}\frac{k_{n+1}^{(p)}}{k_n^{(p)}}\frac{\|q_{n-1}\|^2_{\tilde{w}^{(p)}}}{q_{n-1}(x_j)q'_n(x_j)} \\
&= \frac{(p-(2n+1))(p-(2n+2))}{(p-(n+1))}\frac{\|q_{n-1}\|^2_{\tilde{w}^{(p)}}}{q_{n-1}(x_j)q'_n(x_j)}, \qquad 1 \leqslant j \leqslant n.
\end{aligned}
$$

■

2.3 *Computation of nodes and weights*

In this section an efficient algorithm, which is called Eigenvalue Method, is given for computing zeros of orthogonal polynomials.

Theorem 2.3 *[5] The zeros $\{x_j\}_{j=0}^n$ of the orthogonal polynomial p_{n+1} are eigenvalues of the following symmetric tridiagonal matrix*

$$
A_{n+1} = \begin{bmatrix} \alpha_0 & \beta_1 & & & \\ \beta_1 & \alpha_1 & \beta_2 & & \\ & \ddots & \ddots & & \\ & & \beta_{n-1} & \alpha_{n-1} & \beta_n \\ & & & \beta_n & \alpha_n \end{bmatrix},
$$

where

$$
\alpha_j = \frac{b_j}{a_j}, \quad j \geqslant 0; \qquad \beta_j = \frac{1}{a_{j-1}}\sqrt{\frac{a_{j-1}c_j}{a_j}}, \quad j \geqslant 1,
$$

with $\{a_j, b_j, c_j\}$ being the coefficients of the three-term recurrence relation, namely,

$$
p_{j+1}(x) = (a_j x - b_j)p_j(x) - c_j p_{j-1}(x), \quad j \geqslant 1,
$$

with $p_{-1} := 0$.

Hence, in order to obtain the zeros of $N_n^{(p)}(x)$ where $p > 2n+1$, we use Theorem 2.3 and three-term recurrence relation (5), that is

$$
\begin{aligned}
\alpha_j &= \frac{p}{(p-(2j+2))(p-2j)}, \\
\beta_j &= \frac{p-j}{(p-2j)(p-(2j-1))}\sqrt{\frac{j(p-(2j-1))}{(p-j)(p-(2j+1))}}.
\end{aligned}
$$

According to Gauss integration theory, by having zeros of orthogonal polynomials, one can approximate the integral $\int_0^\infty f(x)w^{(p)}(x)dx$ with precision degree $2n+1$ provided to be convergence. Due to this, the following results are given in tables 2, 3 and 4 for $n = 1, 2$ and 3, respectively.

Also, using Gauss-Radau integration formula, by having zeros of $q_n(x)$, one can approximate the integral $\int_0^\infty f(x)w^{(p)}(x)dx$ with precision degree $2n$ provided to be convergence. Due to this the following results are given in tables 5, 6 and 7 for $n = 1, 2$ and 3, respectively.

Table 2. 2-point integration formula $n = 1$.

p	(x_0, w_0)	(x_1, w_1)
6	$(0.211325, 22.3923)$	$(0.788675, 1.6077)$
7	$(0.166667, 108)$	$(0.5, 12)$
8	$(0.138197, 628.328)$	$(0.361803, 91.6718)$
9	$(0.11835, 4283.63)$	$(0.28165, 756.367)$

Table 3. 3-point integration formula $n = 2$.

p	(x_0, w_0)	(x_1, w_1)	(x_2, w_2)
8	$(0.128886, 565.215)$	$(0.302535, 154.362)$	$(1.06858, 0.42252)$
9	$(0.109039, 3682.09)$	$(0.231933, 1349.56)$	$(0.659028, 8.3458)$
10	$(0.0948473, 27668.4)$	$(0.188128, 12523.9)$	$(0.467024, 127.661)$
11	$(0.0841378, 235701)$	$(0.158272, 125374)$	$(0.35759, 1805.15)$

Table 4. 4-point integration formula $n = 3$.

p	(x_0, w_0)	(x_1, w_1)	(x_2, w_2)	(x_3, w_3)
10	$(0.0912917, 24899.2)$	$(0.174484, 15065.4)$	$(0.388858, 355.273)$	$(1.34537, 0.056014)$
11	$(0.0802695, 202812)$	$(0.144871, 154334)$	$(0.29304, 5730.89)$	$(0.815153, 3.03374)$
12	$(0.0718343, 1.8562 \times 10^6)$	$(0.124101, 1.68624 \times 10^6)$	$(0.234433, 86258.6)$	$(0.569631, 95.5948)$
13	$(0.0651411, 1.88643 \times 10^7)$	$(0.108685, 1.97628 \times 10^7)$	$(0.194982, 1.2874 \times 10^6)$	$(0.431191, 2371.49)$

Table 5. 2-point Gauss-Radau integration formula $n = 1$.

p	(x_0, w_0)	(x_1, w_1)
6	$(0, 6)$	$(0.333333, 18)$
7	$(0, 24)$	$(0.25, 96)$
8	$(0, 120)$	$(0.2, 600)$
9	$(0, 720)$	$(0.166667, 4320)$

Table 6. 3-point Gauss-Radau integration formula $n = 2$.

p	(x_0, w_0)	(x_1, w_1)	(x_2, w_2)
8	$(0, 48)$	$(0.166667, 648)$	$(0.5, 24)$
9	$(0, 240)$	$(0.138197, 4546.63)$	$(0.361803, 253.375)$
10	$(0, 1440)$	$(0.11835, 36194.5)$	$(0.28165, 2685.49)$
11	$(0, 10080)$	$(0.103673, 323080)$	$(0.229661, 29719.5)$

Table 7. 4-point Gauss-Radau integration formula $n = 3$.

p	(x_0, w_0)	(x_1, w_1)	(x_2, w_2)	(x_3, w_3)
10	$(0, 720)$	$(0.109039, 33768.6)$	$(0.231933, 5818.74)$	$(0.659028, 12.6638)$
11	$(0, 4320)$	$(0.0948473, 291716)$	$(0.188128, 66571.1)$	$(0.467024, 273.35)$
12	$(0, 30240)$	$(0.0841378, 2.80137 \times 10^6)$	$(0.158272, 792141)$	$(0.35759, 5048.09)$
13	$(0, 241920)$	$(0.0757383, 2.96432 \times 10^7)$	$(0.136607, 9.94381 \times 10^6)$	$(0.287654, 87867.9)$

2.4 *Interpolation and discrete transforms*

Let $\{x_j, w_j\}_{j=0}^{n}$ be a set of Gauss or Gauss-Radau quadrature nodes and weights. Define the associated discrete inner product and discrete norm as

$$\langle u, v \rangle_{n,w^{(p)}} := \sum_{j=0}^{n} u(x_j) v(x_j) w_j, \qquad \|u\|_{n,w^{(p)}} := \sqrt{\langle u, u \rangle_{n,w^{(p)}}}.$$

Note that $\langle .,. \rangle_{n,w^{(p)}}$ is an approximation to the continuous inner product $(.,.)_{w^{(p)}}$ and the exactness of Gauss-type quadrature formulas implies

$$\langle u, v \rangle_{n,w^{(p)}} = (u, v)_{w^{(p)}}, \qquad \forall u, v \in P_{2n+\sigma},$$

where $\sigma = 1$ and 0 for the Gauss and Gauss-Radau quadratures, respectively.

For any $u \in C[0,\infty)$, the interpolation operator $I_n : C[0,\infty) \to P_n$ is defined such that

$$(I_n u)(x_j) = u(x_j), \qquad 0 \leqslant j \leqslant n,$$

which can be expressed by

$$(I_n u)(x) = \sum_{j=0}^{n} \tilde{u}_j N_j^{(p)}(x) \in P_n.$$

Given the physical values $\{u(x_j)\}_{j=0}^n$, the coefficients $\{\tilde{u}_j\}_{j=0}^n$ can be determined by

$$\tilde{u}_j = \frac{1}{\gamma_j^{(p)}} \sum_{i=0}^{n} u(x_i) N_j^{(p)}(x_i) w_i \qquad 0 \leqslant j \leqslant n,$$

where $\gamma_j^{(p)}$ is defined in (3).

2.5 *Differentiation in the physical space*

Let $\{L_j\}_{j=0}^n$ be the Lagrange basis polynomials associated with the Gauss or Gauss-Radau points $\{x_j\}_{j=0}^n$. Clearly for any $u \in P_n$, we have

$$u(x) = \sum_{j=0}^{n} u(x_j) L_j(x).$$

Hence, differentiating it m times leads to

$$u^{(m)}(x_k) = \sum_{j=0}^{n} u(x_j) L_j^{(m)}(x_k), \qquad 0 \leqslant k \leqslant n.$$

These derivative values can be evaluated by the general formula

$$\mathbf{u}^{(m)} = D^m \mathbf{u}, \qquad (D^m = DD \dots D, \quad m \geqslant 1),$$

in which

$$D = (d_{kj})_{0 \leqslant j,k \leqslant n} = \left(L'_j(x_k)\right)_{0 \leqslant j,k \leqslant n},$$
$$\mathbf{u}^{(m)} = \left(u^{(m)}(x_0), u^{(m)}(x_1), \dots, u^{(m)}(x_n)\right)^T, \quad \mathbf{u}^{(0)} = \mathbf{u}.$$

Hence, it suffices to evaluate the first-order differentiation matrix D.

Theorem 2.4 *The entries of D are determined for Gauss points by*

$$d_{kj} = {L'}_j(x_k) = \begin{cases} \frac{N_n^{(p-2)}(x_k)}{N_n^{(p-2)}(x_j)} \frac{1}{(x_k - x_j)}, & k \neq j, \\ \frac{n(p-(n+3))N_{n-1}^{(p-4)}(x_k)}{2N_n^{(p-2)}(x_k)} & k = j, \end{cases}$$

and for Gauss-Radau points by

$$d_{kj} = L'_j(x_k) =$$
$$\begin{cases} \frac{(n+1)(p-(n+2))N_n^{(p-2)}(x_k)+n(p-(n+1))N_{n-1}^{(p-2)}(x_k)}{(n+1)(p-(n+2))N_n^{(p-2)}(x_j)+n(p-(n+1))N_{n-1}^{(p-2)}(x_j)} \frac{1}{(x_k - x_j)}, & k \neq j, \\ \frac{(n+1)(p-(n+2))n(p-(n+3))N_{n-1}^{(p-4)}(x_k)+n(p-(n+1))(n-1)(p-(n+2))N_{n-2}^{(p-4)}(x_k)}{2(n+1)(p-(n+2))N_n^{(p-2)}(x_k)+n(p-(n+1))N_{n-1}^{(p-2)}(x_k)}, & k = j. \end{cases}$$

Proof The Lagrange basis polynomials can be expressed by

$$L_j(x) = \frac{N_{n+1}^{(p)}(x)}{\frac{d}{dx}\left(N_{n+1}^{(p)}(x)\right)|_{x=x_j}\ (x - x_j)}, \qquad p > 2n+3, \quad 0 \leqslant j \leqslant n.$$

We have

$$\begin{aligned} d_{kj} = {L'}_j(x_k) &= \frac{\frac{d}{dx}\left(N_{n+1}^{(p)}(x)\right)|_{x=x_k}}{\frac{d}{dx}\left(N_{n+1}^{(p)}(x)\right)|_{x=x_j}} \frac{1}{(x_k - x_j)} \\ &= \frac{(n+1)(p-(n+2))N_n^{(p-2)}(x_k)}{(n+1)(p-(n+2))N_n^{(p-2)}(x_j)} \frac{1}{(x_k - x_j)} \\ &= \frac{N_n^{(p-2)}(x_k)}{N_n^{(p-2)}(x_j)} \frac{1}{(x_k - x_j)}. \qquad \forall k \neq j. \end{aligned}$$

For $k = j$, we get

$$d_{kk} = \lim_{x \to x_k} {L'}_k(x) = \frac{1}{\frac{d}{dx}\left(N_{n+1}^{(p)}(x)\right)|_{x=x_k}} \lim_{x \to x_k} \frac{\frac{d}{dx}\left(N_{n+1}^{(p)}(x)\right)(x - x_k) - N_{n+1}^{(p)}(x)}{(x - x_k)^2}$$
$$= \frac{\frac{d^2}{dx^2}\left(N_{n+1}^{(p)}(x)\right)|_{x=x_k}}{2\frac{d}{dx}\left(N_{n+1}^{(p)}(x)\right)|_{x=x_k}} = \frac{(n+1)(p-(n+2))n(p-(n+3))N_{n-1}^{(p-4)}(x_k)}{2(n+1)(p-(n+2))N_n^{(p-2)}(x_k)}$$
$$= \frac{n(p-(n+3))N_{n-1}^{(p-4)}(x_k)}{2N_n^{(p-2)}(x_k)}.$$

The Lagrange basis polynomials for Gauss-Radau points will be

$$L_j(x) = \frac{N_{n+1}^{(p)}(x) + N_n^{(p)}(x)}{\frac{d}{dx}\left(N_{n+1}^{(p)}(x) + N_n^{(p)}(x)\right)|_{x=x_j}\ (x - x_j)}, \qquad p > 2n+3, \quad 0 \leqslant j \leqslant n.$$

Similar to above, we derive

$$d_{kj} = \frac{(n+1)(p-(n+2))N_n^{(p-2)}(x_k) + n(p-(n+1))N_{n-1}^{(p-2)}(x_k)}{(n+1)(p-(n+2))N_n^{(p-2)}(x_j) + n(p-(n+1))N_{n-1}^{(p-2)}(x_j)} \frac{1}{(x_k - x_j)}, \qquad \forall k \neq j,$$

and

$$d_{kk} = \frac{(n+1)(p-(n+2))n(p-(n+3))N_{n-1}^{(p-4)}(x_k)}{2(n+1)(p-(n+2))N_n^{(p-2)}(x_k) + n(p-(n+1))N_{n-1}^{(p-2)}(x_k)} + \frac{n(p-(n+1))(n-1)(p-(n+2))N_{n-2}^{(p-4)}(x_k)}{2(n+1)(p-(n+2))N_n^{(p-2)}(x_k) + n(p-(n+1))N_{n-1}^{(p-2)}(x_k)}.$$

■

2.6 *Differentiation in the frequency space*

In addition to the differentiation in physical space, we have another differentiation, and that is the differentiation in frequency space. In this differentiation, one tries to give the expansion coefficients of the derivatives of a function based on the expansion coeffcients of the main function. More precisely, given $u \in P_n$, instead of using the Lagrange basis polynomials, we expand u in terms of the orthogonal polynomials

$$u(x) = \sum_{j=0}^{n} \tilde{u}_j N_j^{(p)}(x), \qquad \text{with} \qquad \tilde{u}_j = \frac{1}{\gamma_j^{(p)}} \int_0^\infty u(x) N_j^{(p)}(x) w^{(p)}(x) dx,$$

and

$$u'(x) = \sum_{j=1}^{n} \tilde{u}_j \frac{d}{dx}\left(N_j^{(p)}(x)\right) = \sum_{j=0}^{n} \tilde{u}_j^{(1)} N_j^{(p)}(x) \in P_{n-1} \text{ with } \tilde{u}_n^{(1)} = 0. \tag{14}$$

In order to express $\{\tilde{u}_j^{(1)}\}_{j=0}^n$ in terms of $\{\tilde{u}_j\}_{j=0}^n$, we know that $\left\{\frac{d}{dx}\left(N_j^{(p)}(x)\right)\right\}$ are also orthogonal [6–9]. Indeed, this property holds for the classical orthogonal polynomials. To do show, consider equation (2). Then by differentiating it with respect to x and writing $z(x) = y'(x)$, we obtain

$$x^2 z'' + ((4-p)x - 1) z' - (n-1)(n+2-p)z = 0. \tag{15}$$

Again by applying the Frobenius method, we get the explicit solution

$$\frac{d}{dx}\left(N_n^{(p)}(x)\right) = (-1)^{n-1} \sum_{k=0}^{n-1} k! \binom{p-(n+2)}{k} \binom{n-1}{n-k-1} (-x)^k.$$

It is simply seen that, the finite set $\left\{\frac{d}{dx}\left(N_j^{(p>2n+1)}(x)\right)\right\}_{j=1}^n$ is orthogonal with respect to the weight function $w^{(p-2)}(x) = x^{2-p}e^{-\frac{1}{x}}$ on the half line $\mathbb{R}_+ := [0, \infty)$, i.e.,

$$\int_0^{\infty} x^{2-p} \mathrm{e}^{-\frac{1}{x}} \frac{d}{dx}\left(N_n^{(p)}(x)\right) \frac{d}{dx}\left(N_m^{(p)}(x)\right) dx = 0\,, \quad \Leftrightarrow \quad \begin{cases} m \neq n,\, p > 2\ell+1, \\ \ell = \max\{m,n\}, \end{cases} \tag{16}$$

and

$$\eta_n^{(p,1)} := \int_0^{\infty} x^{2-p} \mathrm{e}^{-\frac{1}{x}} \left(\frac{d}{dx}\left(N_n^{(p)}(x)\right)\right)^2 dx = \frac{(n-1)!(p-2-n)!}{p-1-2n}. \tag{17}$$

Also, equation (15) leads to the following three-term recurrence relation

$$\frac{d}{dx}\left(N_{n+1}^{(p)}(x)\right) = \left(a_n^{(1)} x - b_n^{(1)}\right) \frac{d}{dx}\left(N_n^{(p)}(x)\right) - c_n^{(1)} \frac{d}{dx}\left(N_{n-1}^{(p)}(x)\right), \tag{18}$$

where

$$\begin{aligned} a_n^{(1)} &= \frac{(p-(2n+1))(p-(2n+2))}{p-(n+2)}, \\ b_n^{(1)} &= \frac{(p+2)(p-(2n+1))}{(p-2n)(p-(n+2))}, \\ c_n^{(1)} &= \frac{(n-1)(p-(2n+2))}{(p-2n)(p-(n+2))}. \end{aligned}$$

Remark 2.5 Using the above procedure, we can also obtain

$$\int_0^{\infty} x^{2\mathrm{k}-p} \mathrm{e}^{-\frac{1}{x}} \frac{d^k}{dx^k}\left(N_n^{(p)}(x)\right) \frac{d^k}{dx^k}\left(N_m^{(p)}(x)\right) dx = 0, \quad \Leftrightarrow \quad \begin{cases} m \neq n,\ p > 2\ell+1, \\ \ell = \max\{m,n\}, \end{cases} \tag{19}$$

and

$$\eta_n^{(p,k)} = \int_0^{\infty} x^{2\mathrm{k}-p} \mathrm{e}^{-\frac{1}{x}} \left(\frac{d^k}{dx^k}\left(N_n^{(p)}(x)\right)\right)^2 dx = \frac{(n-k)!(p-(n+k+1))!}{p-1-2n}. \tag{20}$$

We note that

$$\gamma_n^{(p)} = \binom{n}{k} \prod_{i=0}^{k-1} (p-(n-i+1))\eta_n^{(p,k)}.$$

Now, by differentiating the three-term recurrence relation (5) and using (18), we derive

$$N_n^{(p)}(x) = \tilde{a}_n^{(p)} \frac{d}{dx}\left(N_{n-1}^{(p)}(x)\right) + \tilde{b}_n^{(p)} \frac{d}{dx}\left(N_n^{(p)}(x)\right) + \tilde{c}_n^{(p)} \frac{d}{dx}\left(N_{n+1}^{(p)}(x)\right), \tag{21}$$

where

$$\tilde{a}_n^{(p)} = \frac{c_n}{a_n} - \frac{c_n^{(1)}}{a_n^{(1)}}, \qquad \tilde{b}_n^{(p)} = \frac{b_n}{a_n} - \frac{b_n^{(1)}}{a_n^{(1)}}, \qquad \tilde{c}_n^{(p)} = \frac{1}{a_n} - \frac{1}{a_n^{(1)}}.$$

Hence, using (14) and (21), the coefficients $\{\tilde{u}_j^{(1)}\}$ can be computed in terms of

$\{\tilde{u}_j\}_{j=0}^n$ as follows:

$$u'(x) = \sum_{j=0}^{n-1} \tilde{u}_j^{(1)} N_n^{(p)}(x) = \sum_{j=0}^{n-1} \tilde{u}_j^{(1)} \left(\tilde{a}_j^{(p)} \frac{d}{dx} \left(N_{j-1}^{(p)}(x) \right) + \tilde{b}_j^{(p)} \frac{d}{dx} \left(N_j^{(p)}(x) \right) + \tilde{c}_j^{(p)} \frac{d}{dx} \left(N_{j+1}^{(p)}(x) \right) \right)$$

$$= \sum_{j=1}^{n-1} \left(\tilde{a}_{j+1}^{(p)} \tilde{u}_{j+1}^{(1)} + \tilde{b}_j^{(p)} \tilde{u}_j^{(1)} + \tilde{c}_{j-1}^{(p)} \tilde{u}_{j-1}^{(1)} \right) \frac{d}{dx} \left(N_j^{(p)}(x) \right) + \tilde{c}_{n-1}^{(p)} \tilde{u}_{n-1}^{(1)} \frac{d}{dx} \left(N_n^{(p)}(x) \right).$$

So, we conclude

$$\tilde{u}_{j-1}^{(1)} = \frac{1}{\tilde{c}_{j-1}^{(p)}} \left(\tilde{u}_j - \tilde{a}_{j+1}^{(p)} \tilde{u}_{j+1}^{(1)} - \tilde{b}_j^{(p)} \tilde{u}_j^{(1)} \right), \qquad j = n-1, \ldots, 1,$$
$$\tilde{u}_n^{(1)} = 0, \qquad \tilde{u}_{n-1}^{(1)} = \frac{1}{\tilde{c}_{n-1}^{(p)}} \tilde{u}_n.$$

3. Approximation by $N_n^{(p)}(x)$ polynomials

This section is devoted to the analysis of approximations by $N_n^{(p)}(x)$ polynomials. These results will be useful for error analysis of spectral methods for unbounded domains.

3.1 *Inverse inequalities*

We first present two inverse inequalities associated with $N_n^{(p)}(x)$ polynomials.

Theorem 3.1 *For $p > 2n+1$ and any $\phi \in P_n$,*

$$\|\phi\|_{w^{(p)}} \leqslant (np)^{1/m} \left\| \frac{d^m}{dx^m} \phi \right\|_{w^{(p-2m)}}, \qquad m \geqslant 1.$$

Proof For any $\phi \in P_n$, we have

$$\phi(x) = \sum_{j=0}^{n} \tilde{\phi}_j^{(p)} N_j^{(p)}(x), \quad \text{with} \quad \tilde{\phi}_j^{(p)} = \frac{1}{\gamma_j^{(p)}} \int_0^\infty \phi(x) N_j^{(p)}(x) w^{(p)}(x) dx. \quad (22)$$

Hence, by the orthogonality of $\{N_j^{(p)}(x)\}$,

$$\|\phi\|_{w^{(p)}}^2 = \sum_{j=0}^{n} \gamma_j^{(p)} \left| \tilde{\phi}_j^{(p)} \right|^2.$$

Differentiating (22) and using the orthogonality (16)-(17), we obtain

$$\phi'(x) = \sum_{j=1}^{n} \tilde{\phi}_j^{(p)} \frac{d}{dx} \left(N_j^{(p)}(x) \right),$$

and

$$\left\|\phi'\right\|^2_{w^{(p-2)}} = \sum_{j=1}^{n} \eta_j^{(p,1)} \left|\tilde{\phi}_j^{(p)}\right|^2.$$

Since $\gamma_j^{(p)} = j(p-1-j)\eta_j^{(p,1)}$, we obtain

$$\|\phi\|^2_{w^{(p)}} = \sum_{j=0}^{n} \gamma_j^{(p)} \left|\tilde{\phi}_j^{(p)}\right|^2 = \sum_{j=0}^{n} j(p-1-j)\eta_j^{(p,1)} \left|\tilde{\phi}_j^{(p)}\right|^2 \leqslant np \left\|\phi'\right\|^2_{w^{(p-2)}},$$

and

$$\|\phi\|_{w^{(p)}} \leqslant \sqrt{np} \left\|\phi'\right\|_{w^{(p-2)}}.$$

Using the above inequality recursively leads to

$$\|\phi\|_{w^{(p)}} \leqslant (np)^{1/m} \left\|\frac{d^m}{dx^m}\phi\right\|_{w^{(p-2m)}}.$$

■

Next, we derive an inverse inequality involving the same weight function for derivatives of different order.

Theorem 3.2 *For $p > 2n+1$ and any $\phi \in P_n$,*

$$\left\|\frac{d^m}{dx^m}\phi\right\|_{w^{(p)}} \leqslant Cn^m \left(\max_{0\leqslant k\leqslant n-1} \left\{\frac{\gamma_k^{(p)}}{\gamma_*^{(p)}}\right\}\right)^{1/m} \|\phi\|_{w^{(p)}},$$

where $\gamma_*^{(p)} = \min_k \{\gamma_{k+1}^{(p)}, \gamma_{k+2}^{(p)}, \dots, \gamma_n^{(p)}\}$.

Proof For any $\phi \in P_n$, we have

$$\phi(x) = \sum_{j=0}^{n} \tilde{\phi}_j^{(p)} N_j^{(p)}(x), \quad \text{with} \quad \tilde{\phi}_j^{(p)} = \frac{1}{\gamma_j^{(p)}} \int_0^\infty \phi(x) N_j^{(p)}(x) w^{(p)}(x) dx.$$

Hence, using differentiation in the frequency space, we get

$$\phi'(x) = \sum_{j=1}^{n} \tilde{\phi}_j^{(p)} \frac{d}{dx}\left(N_j^{(p)}(x)\right) = \sum_{j=1}^{n} \tilde{\phi}_j^{(p)} \left(\sum_{k=0}^{j-1} c_k N_j^{(p)}(x)\right) = \sum_{k=0}^{n-1} \left(\sum_{j=k+1}^{n} \tilde{\phi}_j^{(p)}\right) c_k N_k^{(p)}(x),$$

and

$$\left\|\phi'\right\|^2_{w^{(p)}} = \sum_{k=0}^{n-1} \left(\sum_{j=k+1}^{n} \tilde{\phi}_j^{(p)}\right)^2 c_k^2 \gamma_k^{(p)}.$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{j=k+1}^{n}\tilde{\phi}_j^{(p)}\right)^2 \leqslant \left(\sum_{j=k+1}^{n}\gamma_j^{(p)}\left|\tilde{\phi}_j^{(p)}\right|^2\right)\left(\sum_{j=k+1}^{n}\left(\gamma_j^{(p)}\right)^{-1}\right),$$

we conclude that

$$\begin{aligned}\left\|\phi'\right\|^2_{w^{(p)}} &= \sum_{k=0}^{n-1}\left(\sum_{j=k+1}^{n}\tilde{\phi}_j^{(p)}\right)^2 c_k^2\gamma_k^{(p)}\\ &\leqslant \|\phi\|^2_{w^{(p)}}\sum_{k=0}^{n-1}c_k^2\gamma_k^{(p)}\left(\sum_{j=k+1}^{n}\left(\gamma_j^{(p)}\right)^{-1}\right) \leqslant Cn\,\|\phi\|^2_{w^{(p)}}\sum_{k=0}^{n-1}\frac{\gamma_k^{(p)}}{\min\limits_k\{\gamma_{k+1}^{(p)},\gamma_{k+2}^{(p)},\dots,\gamma_n^{(p)}\}}\\ &\leqslant Cn\,\|\phi\|^2_{w^{(p)}}\sum_{k=0}^{n-1}\frac{\gamma_k^{(p)}}{\gamma_*^{(p)}} = Cn^2\,\|\phi\|^2_{w^{(p)}}\max_{0\leqslant k\leqslant n-1}\{\frac{\gamma_k^{(p)}}{\gamma_*^{(p)}}\}\end{aligned}$$

and

$$\left\|\phi'\right\|_{w^{(p)}} \leqslant Cn\sqrt{\max_{0\leqslant k\leqslant n-1}\{\frac{\gamma_k^{(p)}}{\gamma_*^{(p)}}\}}\,\|\phi\|_{w^{(p)}}.$$

Using the above inequality recursively leads to

$$\left\|\frac{d^m}{dx^m}\phi\right\|_{w^{(p)}} \leqslant Cn^m\left(\max_{0\leqslant k\leqslant n-1}\{\frac{\gamma_k^{(p)}}{\gamma_*^{(p)}}\}\right)^{1/m}\|\phi\|_{w^{(p)}}.$$

■

3.2 *Orthogonal projections*

A common procedure in error analysis is to compare the numerical solution u_n with a suitable orthogonal projection $\Pi_n u$ (or interpolation $I_n u$) of the exact solution u in some appropriate Sobolev space with the norm $\|\,.\,\|_S$, and use the triangle inequality,

$$\|u-u_n\|_S \leqslant \|u-\Pi_n u\|_S + \|\Pi_n u - u_n\|_S.$$

Hence, one needs to estimate the errors $\|u-\Pi_n u\|_S$ and $\|\Pi_n u - u_n\|_S$. Such estimates involving $N_n^{(p)}(x)$ polynomials will be the main concern of this subsection.

Consider the $L^2_{w^{(p)}}$-orthogonal projection $\Pi_{n,p} : L^2_{w^{(p)}}(\mathbb{R}_+) \to P_n$, defined by

$$(\Pi_{n,p}u - u, v_n)_{w^{(p)}} = 0, \qquad \forall v_n \in P_n,$$

so we have

$$\Pi_{n,p}u(x) = \sum_{j=0}^{n}\tilde{u}_j^{(p)}N_j^{(p)}(x) \quad \text{with} \quad \tilde{u}_j^{(p)} = \frac{1}{\gamma_j^{(p)}}\int_{\mathbb{R}_+}u(x)N_j^{(p)}(x)w^{(p)}(x)dx.$$

Introduce the space

$$B_p^m(\mathbb{R}_+) = \left\{ u : \frac{d^k}{dx^k}(u) \in L^2_{w^{(p-2k)}}(\mathbb{R}_+),\ 0 \leqslant k \leqslant m \right\},$$

equipped with the norm and semi-norm

$$\|u\|_{B_p^m} = \left(\sum_{k=0}^{m} \left\| \frac{d^k}{dx^k}(u) \right\|^2_{L^2_{w^{(p-2k)}}} \right)^{1/2}, \quad |u|_{B_p^m} = \left\| \frac{d^k}{dx^k}(u) \right\|_{L^2_{w^{(p-2k)}}}.$$

The space $B_p^m(\mathbb{R}_+)$ distinguishes itself from the usual weighted Sobolev space $H^m_{w^{(p)}}(\mathbb{R}_+)$ by involving different weight functions for derivatives of different orders [5, 10, 11]. It is obvious that $H^m_{w^{(p)}}(\mathbb{R}_+)$ is a subspace of $B_p^m(\mathbb{R}_+)$, that is, for any $m \geqslant 0$,

$$\|u\|_{B_p^m(\mathbb{R}+)} \leqslant c\|u\|_{H^m_{w^{(p)}}(\mathbb{R}+)}.$$

Now, we are ready to state the first fundamental result.

Theorem 3.3 *Let* $0 \leqslant \ell \leqslant m \leqslant n+1 < \frac{p+1}{2}$. *If* $u \in B_p^m(\mathbb{R}_+)$, *we have*

$$\left\| \frac{d^\ell}{dx^\ell} \left(\Pi_{n,p} u - u \right) \right\|_{w^{(p-2\ell)}} \leqslant \sqrt{\frac{((p+1)/2-\ell)!}{((p+1)/2-m)!}} \left\| \frac{d^m}{dx^m}(u) \right\|_{w^{(p-2m)}}.$$

Proof Thanks to the orthogonality (19),

$$\left\| \frac{d^k}{dx^k}(u) \right\|^2_{w^{(p-2k)}} = \sum_{j=k}^{\infty} \eta_j^{(p,k)} \left| \tilde{u}_j^{(p)} \right|^2, \qquad k \geqslant 0,$$

so we have

$$\begin{aligned}
\left\| \frac{d^\ell}{dx^\ell} \left(\Pi_{n,p} u - u \right) \right\|^2_{w^{(p-2\ell)}} &= \sum_{j=n+1}^{\infty} \eta_j^{(p,\ell)} \left| \tilde{u}_j^{(p)} \right|^2 = \sum_{j=n+1}^{\infty} \eta_j^{(p,m)} \frac{\eta_j^{(p,\ell)}}{\eta_j^{(p,m)}} \left| \tilde{u}_j^{(p)} \right|^2 \\
&\leqslant \max_{j \geqslant n+1} \left\{ \frac{\eta_j^{(p,\ell)}}{\eta_j^{(p,m)}} \right\} \sum_{j=n+1}^{\infty} \eta_j^{(p,m)} \left| \tilde{u}_j^{(p)} \right|^2 \\
&\leqslant \max_{j \geqslant n+1} \left\{ \frac{(j-\ell)!(p-(j+\ell+1))!}{(j-m)!(p-(j+m+1))!} \right\} \left\| \frac{d^m}{dx^m}(u) \right\|^2_{w^{(p-2m)}} \\
&\leqslant \max_{j \geqslant n+1} \left\{ \frac{(j-\ell)!}{(j-m)!} \right\} \left\| \frac{d^m}{dx^m}(u) \right\|^2_{w^{(p-2m)}} \leqslant \frac{((p+1)/2-\ell)!}{((p+1)/2-m)!} \left\| \frac{d^m}{dx^m}(u) \right\|^2_{w^{(p-2m)}},
\end{aligned}$$

and

$$\left\| \frac{d^\ell}{dx^\ell} \left(\Pi_{n,p} u - u \right) \right\|_{w^{(p-2\ell)}} \leqslant \sqrt{\frac{((p+1)/2-\ell)!}{((p+1)/2-m)!}} \left\| \frac{d^m}{dx^m}(u) \right\|_{w^{(p-2m)}}.$$

■

Since $H^m_{w^{(p)}}(\mathbb{R}_+)$ is a Hilbert space, the best approximation polynomial for u is the orthogonal projection of u upon P_n under the inner product

$$(u,v)_{m,w^{(p)}} = \sum_{k=0}^{m} \left(\frac{d^k}{dx^k}u, \frac{d^k}{dx^k}v \right)_{w^{(p)}},$$

which induces the norm $\| \,.\, \|_{m,w^{(p)}}$ of $H^m_{w^{(p)}}(\mathbb{R}_+)$. In fact, this type of approximation results are often needed in analysis of spectral methods for second-order elliptic PDEs [5, 11–13]. Therefore, we consider below the $H^1_{w^{(p)}}$-orthogonal projection. Define the orthogonal projection $\Pi^1_{n,p} : H^1_{w^{(p)}}(\mathbb{R}_+) \to P_n$ by

$$\left(\Pi^1_{n,p}u - u, v_n\right)_{1,w^{(p)}} = 0, \qquad \forall v_n \in P_n. \tag{23}$$

By definition, $\Pi^1_{n,p}u$ is the best approximation of u in the sense that

$$\left\|\Pi^1_{n,p}u - u\right\|_{1,w^{(p)}} = \inf_{\phi \in P_n} \|\phi - u\|_{1,w^{(p)}}. \tag{24}$$

Using Theorem 3.3, we can derive the following estimate.

Theorem 3.4 *Let $1 \leqslant m \leqslant n+1 < \frac{p+1}{2}$. If $\frac{d}{dx}(u) \in B^{m-1}_p(\mathbb{R}_+)$, we have*

$$\left\|\Pi^1_{n,p}u - u\right\|_{1,w^{(p)}} \leqslant c\sqrt{\frac{((p-1)/2)!}{((p+1)/2-m)!}} \left\| \frac{d^m}{dx^m}(u) \right\|_{w^{(p-2m-2)}},$$

where c is a positive constant independent of m, n, p and u.

Proof Let $\Pi_{n-1,p}$ be the $L^2_{w^{(p)}}$ -orthogonal projection upon P_{n-1}. Set

$$\phi(x) = \phi(0) + \int_0^x \Pi_{n-1,p}u'(y)dy,$$

where the constant $\phi(0)$ is chosen such that $\phi(0) = u(0)$. In view of (24), we derive from the Poincare inequality and Theorem 3.3 that

$$\begin{aligned} \left\|\pi^1_{n,p}u - u\right\|_{1,w^{(p)}} &\leqslant \|\phi - u\|_{1,w^{(p)}} \leqslant c\left\|(\phi - u)'\right\|_{w^{(p)}} \\ &\leqslant c\left\|\pi_{N-1,p}u' - u'\right\|_{w^{(p)}} \\ &\leqslant c\sqrt{\frac{((p-1)/2)!}{((p+1)/2-m)!}} \left\| \frac{d^m}{dx^m}(u) \right\|_{w^{(p-2m-2)}}. \end{aligned}$$

■

The approximation results in the Sobolev norms are of great importance for spectral approximation of boundary value problems. Oftentimes, it is necessary to take the boundary conditions into account and consider the projection operators onto the space of polynomials built in homogeneous boundary data [11]. To this end, denote

$$\begin{aligned} &H^1_{0,w^{(p)}}(\mathbb{R}_+) = \left\{ u \in H^1_{w^{(p)}}(\mathbb{R}_+) \,:\, u(0) = 0 \right\}, \\ &P^0_n = \left\{ \phi \in P_n \,:\, \phi(0) = 0 \right\}. \end{aligned}$$

Then the orthogonal projection $\Pi^{1,0}_{n,p}: H^1_{0,w^{(p)}}(\mathbb{R}_+) \to P^0_n$ is defined by

$$\left((\Pi^{1,0}_{n,p}u-u)', v'_n\right)_{w^{(p)}} = 0, \qquad \forall v_n \in P^0_n.$$

Theorem 3.5 *If $u \in H^1_{0,w^{(p)}}(\mathbb{R}_+)$ and $\frac{d}{dx}(u) \in B^{m-1}_p(\mathbb{R}_+)$, then for $1 \leqslant m \leqslant n+1 < \frac{p+1}{2}$, we have*

$$\left\|\Pi^{1,0}_{n,p}u-u\right\|_{1,w^{(p)}} \leqslant c\sqrt{\frac{((p-1)/2)!}{((p+1)/2-m)!}}\left\|\frac{d^m}{dx^m}(u)\right\|_{w^{(p-2m-2)}},$$

where c is a positive constant independent of m, n, p and u.

Proof The desired result can be proved as in Theorem 3.4 by taking $\phi(x) = \int_0^x \Pi_{n-1,p}u'(y)dy$. ■

4. Spectral methods using $N^{(p)}_n(x)$ polynomials

In this section, we consider spectral-Galerkin methods using $N^{(p)}_n(x)$ polynomials. An advantage of using $N^{(p)}_n(x)$ polynomials is that they are mutually orthogonal, so we can work with the usual variational formulation.

4.1 $N^{(p)}_n(x)$-*Galerkin method*

Consider the model equation:

$$\begin{cases} -u_{xx}+\gamma u = f, & x \in \mathbb{R}_+,\ \gamma > 0 \\ u(0)=0, & \lim\limits_{x\to+\infty} u(x) = 0. \end{cases} \tag{25}$$

Let $H^1_{0,w^{(p)}}(\mathbb{R}_+)$ and P^0_n be the spaces as defined before. Then, a weak formulation of (25) is

$$\left(u', v'\right) + \gamma\,(u, v) = (f, v)\,, \quad \forall v \in H^1_{0,w^{(p)}}(\mathbb{R}+),$$

where $(\cdot,\cdot)$ is the usual (non-weighted) inner product in L_2-space. The $N^{(p)}_n(x)$-Galerkin approximation to (25) is

$$\left(u'_n, v'\right) + \gamma\,(u_n, v) = (f, v)\,, \quad \forall v \in P^0_n.$$

As in the Laguerre and Hermite cases, the $N^{(p)}_n(x)$ polynomials are not very useful in practice due to its wild behavior at infinity. Therefore, we consider the polynomials functions defined by

$$\varphi^{(p)}_k(x) = \left(N^{(p)}_k(x) + N^{(p)}_{k+1}(x)\right)x^{\frac{-p}{2}}e^{\frac{-1}{2x}}, \qquad p > 2k+3.$$

So one verifies that

$$P^0_n = \operatorname{span}\left\{\varphi^{(p)}_0, \varphi^{(p)}_1, \ldots, \varphi^{(p)}_{n-1}\right\}.$$

Hence, by setting $u_n(x) = \sum_{k=0}^{n-1} u_k \varphi_k^{(p)}(x)$ and $v = \varphi_j^{(p)}(x)$ for $j = 0, 1, \cdots, n-1$, we get

$$\sum_{k=0}^{n-1} u_k \left(\left(\left(\varphi_k^{(p)}\right)', \left(\varphi_j^{(p)}\right)' \right) + \gamma \left(\varphi_k^{(p)}, \varphi_j^{(p)}\right) \right) = \left(f, \varphi_j^{(p)}\right).$$

and in matrix form

$$AU = F,$$

where

$$A = [a_{k,j}]_{n\times n}, \quad U = [u_j]_{n\times 1}, \quad F = [f_j]_{n\times 1},$$

in which

$$\begin{aligned} a_{k,j} &= \left(\left(\varphi_k^{(p)}\right)', \left(\varphi_j^{(p)}\right)' \right) + \gamma \left(\varphi_k^{(p)}, \varphi_j^{(p)}\right), \\ f_j &= \left(f, \varphi_j^{(p)}\right). \end{aligned} \tag{26}$$

Note that all integrals in (26), are solved by Gauss-Radau quadrature (see Theorem 2.2).

4.2 *Numerical results*

Now, we present some numerical results to illustrate the convergence behavior of the proposed schemes. We consider (25) with two sets of exact solutions having different decay properties.

Example 4.1 Exponential decay with oscillation at infinity:

$$u(x) = e^{-x}\sin(x), \qquad x \in (0, +\infty).$$

The exact solution $u(x) = e^{-x}\sin(x)$ and the approximate solution $u_n(x)$ for $n = 10, p = 22$ and $n = 26, p = 54$ are demonstrated in Figure 1. At the end, the plot of logarithm error function $|u(x) - u_n(x)|$ at different points in intervals $[0, 1]$ and $[0, 1000]$ for $n = 26, p = 54$ is shown in Figure 2. In these figures, we clearly observe that the desired solution is provided by the approximate solution as well.

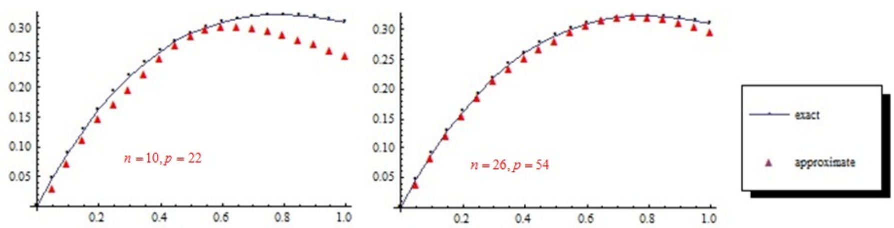

Figure 1. The plots of exact solution $u(x) = e^{-x}\sin(x)$ and the approximate solution $u_n(x)$ for $n = 10, p = 22$ and $n = 26, p = 54$.

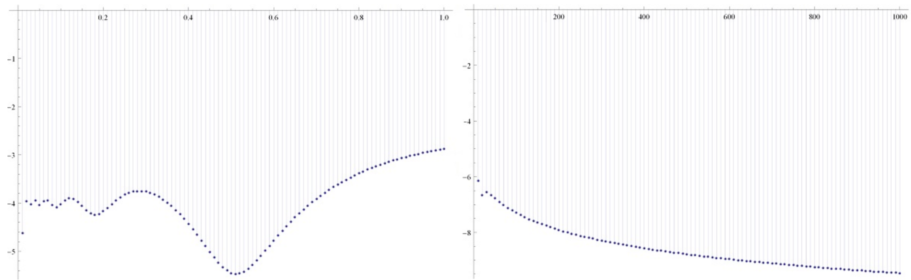

Figure 2. The plot of logarithm error function $|e^{-x}\sin(x) - u_n(x)|$ at different points in intervals $[0, 1]$ and $[0, 1000]$ for $n = 26, p = 54$.

Remark 4.2 Note that, according to [6, 14], the following relationship between Laguerre polynomials $L_n^{(p)}(x)$ and $N_n^{(p)}(x)$ can be derived easily

$$N_n^{(p)}(x) = n!x^n L_n^{(p-(2n+1))}\left(\frac{1}{x}\right) \quad \Leftrightarrow \quad L_n^{(p)}(x) = \frac{x^n}{n!} N_n^{(p+2n+1)}\left(\frac{1}{x}\right).$$

Hence, the numerical results based on $N_n^{(p)}(x)$-Galerkin method are approximately similar to Laguerre-Galerkin Method.

Example 4.3 Algebraic decay without oscillation at infinity:

$$u(x) = \frac{x}{1+x^2}, \qquad x \in (0, +\infty).$$

The exact solution $u(x) = \frac{x}{1+x^2}$ and the approximate solution $u_n(x)$ for $n = 10, p = 22$ and $n = 26, p = 54$ are demonstrated in Figure 3. At the end, the plot of logarithm error function $|u(x) - u_n(x)|$ at different points in intervals $[0, 1]$ and $[0, 1000]$ for $n = 26, p = 54$ is shown in Figure 4. In these figures, we clearly observe that the desired solution is provided by the approximate solution as well.

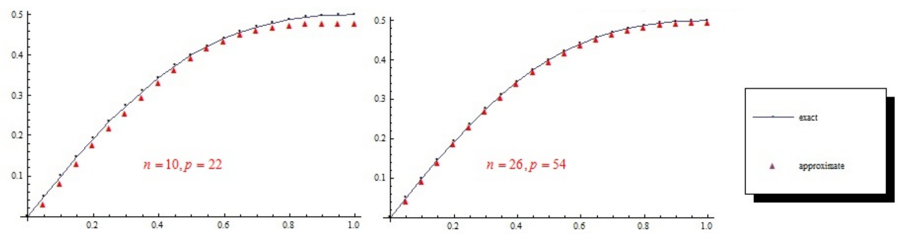

Figure 3. The plots of exact solution $u(x) = \frac{x}{1+x^2}$ and the approximate solution $u_n(x)$ for $n = 10, p = 22$ and $n = 26, p = 54$.

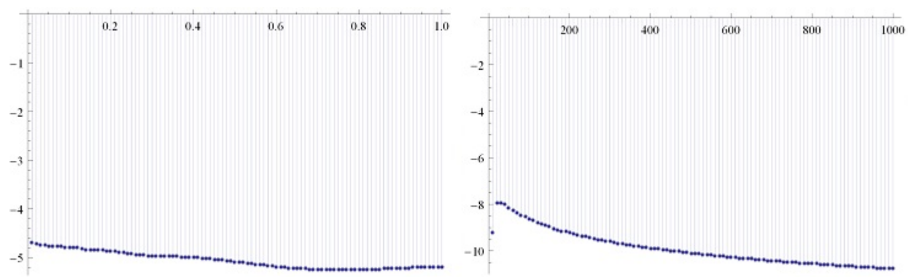

Figure 4. The plot of logarithm error function $|\frac{x}{1+x^2} - u_n(x)|$ at different points in intervals $[0, 1]$ and $[0, 1000]$ for $n = 26, p = 54$.

5. Conclusion

In this paper, we proposed a spectral method using the finite class of orthogonal polynomial $N_n^{(p)}(x)$ related to inverse Gamma distribution. Almost all properties of this class of orthogonal polynomials were studied. In addition, error analysis and convergence of the proposed spectral method were given. It is worth to point out, in particular, this class is directly related to generalized Bessel polynomials and has also relation with the Laguerre polynomials. Hence, the numerical results based on $N_n^{(p)}(x)$-Galerkin method are approximately similar to Laguerre-Galerkin method and we see a resemblance between the two numerical approaches.

Declarations

Ethical Approval and Consent to participate

Author approves the Ethical Approval and Consent to participate.

Consent for publication

The Author hereby consents to publication of the work.

Availability of supporting data

There is no availability of supporting data.

Competing interests

Not Applicable.

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Authors' contributions

Not Applicable.

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References

[1] A. F. Nikiforov, V. B. Uvarov, Special functions of mathematical physics, Birkhauser, Basel-Boston, 1988.
[2] S. A. Chihara, An introduction to orthogonal polynomials, Gordon and Breach, New York, 1978.
[3] G. Szego, Orthogonal polynomials, 4th ed. Amer. Math. Soc., Providence, RI, 1975.
[4] W. Koepf, M. Masjed-Jamei, Two classes of special functions using Fourier transforms of some finite classes of classical orthogonal polynomials, Proc. Amer. Math. Soc., 135 (2007), no. 11, 3599–3606.
[5] J. Shen, T. Tang, L. Wang, Spectral methods: algorithms, analysis and applications, Heidelberg: Springer, 2011.

[6] M. Masjedjamei, Three finite classes of hypergeometric orthogonal polynomials and their application in functions approximation, Integral Transforms Spec. Funct, 13 (2002), no. 2, 169–191.

[7] P. Malik, A. Swaminathan, Derivatives of a finite class of orthogonal polynomials related to inverse gamma distribution, Appl. Math. Comput., 218 (2012), no. 11, 6251–6262.

[8] H. L. Krall, On derivatives of orthogonal polynomials, Bull. Amer. Math. Soc., 42 (1936), no. 6, 423–428.

[9] H. L. Krall, On higher derivatives of orthogonal polynomials, Bull. Amer. Math. Soc., 42 (1936), no. 12, 867–870.

[10] B. Guo, J. Shen, Laguerre-Galerkin method for nonlinear partial differential equations on a semi-infinite interval, Numer. Math., 86 (2000), no. 4, 635–654.

[11] J. Shen, Stable and efficient spectral methods in unbounded domains using Laguerre functions, SIAM J. Numer. Anal., 38 (2000), no. 4, 1113–1133.

[12] F. Liu, H. Li, Z. Wang, Spectral methods using generalized Laguerre functions for second and fourth order problems, Numer. Algorithms, 75 (2017), no. 4, 1005–1040.

[13] S. Moazzezi, A. H. Salehi Shayegan, A. Zakeri, A spectral element method for solving backward parabolic problems, Int. J. Comput. Methods Eng. Sci. Mech., 19 (2018), no. 5, 324–339.

[14] M. Masjedjamei, Special functions and generalized Sturm-Liouville problems, Springer Nature, 2020.