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The Chromatic Number of the Square of the Cartesian Product of Cycles and Paths

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Abstract. Given any graph G, its square graph G^2 has the same vertex set $V(G)$, with two vertices adjacent in G^2 whenever they are at distance 1 or 2 in G. A set $S \subseteq V(G)$ is a 2-distance independent set of a graph G if the distance between every two vertices of S is greater than 2. The 2-distance independence number $\alpha_2(G)$ of G is the maximum cardinality over all 2-distance independent sets in G. In this paper, we establish the 2-distance independence number and 2-distance chromatic number for $C_3 \square C_6 \square C_m$, $C_n \square P_3 \square P_3$ and $C_4 \square C_7 \square C_n$ where $m \equiv 0\,(mod\,3)$ and $n, m \geqslant 3$.

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1. Introduction

Let $G = (V, E)$ be a finite and simple graph. For any graph G, we denote the vertex-set and the edge-set of G by $V(G)$ and $E(G)$, respectively. A proper vertex k-coloring of a graph G is a mapping $c : V(G) \to \{1, \ldots, k\}$, with the property that $c(u) \neq c(v)$ whenever $uv \in E(G)$. The smallest k for which there exists a k-coloring of G, called the chromatic number of G, is denoted by $\chi(G)$, see [1, 7] for more details. The square of a graph G, denoted by G^2, is a graph with $V(G) = V(G^2)$, in which two vertices are adjacent if their distance in G is at most two. A 2-distance coloring of G is a vertex coloring of G such that any two

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distinct vertices at distance less than or equal to 2 are assigned different colors. The 2-distance chromatic number of a graph G is the minimum number of colors necessary to have a 2-distance coloring of G, which is denoted by $\chi_2(G)$. Hence $\chi_2(G)$ is equal to $\chi(G^2)$. The 2-distance coloring of graphs was introduced by Wegner in [16]. The problem of determining the chromatic number of the square of particular graphs has attracted a lot of attention, with a particular focus on the square of planar graphs (see, e.g., [4, 5, 8, 10, 15]). The Cartesian product of graphs $G_1, G_2, \ldots, G_k$ is the graph $G_1 \Box G_2 \Box \cdots \Box G_k = \Box_{i=1}^{k} G_i$ with vertex set $\{(x_1, x_2, \ldots, x_k) | x_i \in V(G_i)\}$ and for which two vertices $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ are adjacent whenever $x_i y_i \in E(G_i)$ for exactly one index $1 \leqslant i \leqslant k$ and $x_j = y_j$ for each index1 $\leqslant j \leqslant k$ that $i \neq j$. The subgraph of $G \Box H$ induced by $\{u\} \times V(H)$ is isomorphic to H. It is called an H-fiber and is denoted by H^u. A set $S \subseteq V(G)$ is a k-distance independent set of a graph G if the distance between every two vertices of S is greater than k. The k-distance independence number $\alpha_k(G)$ of G is the maximum cardinality over all k-distance independent sets in G. For $k = 1$, we use $\alpha_k(G)$ as $\alpha(G)$. There are many results for the chromatic number of the square of the Cartesian product of tree, paths, and cycles (see, e.g., [2, 3, 6, 9, 11, 13]). Shao et al. [12] established that the 2-distance chromatic number of G equals $\lceil \frac{|V(G)|}{\alpha(G^2)} \rceil$ for $G = C_m \Box C_n \Box C_k$ where $k \geqslant 3$ and $(m, n) \in \{(3, 3), (3, 4), (3, 5), (4, 4)\}$ or k, m, and n are all multiples of seven. Moreover, it is shown that the 2-distance chromatic number of the three-dimensional square lattice is equal to seven and proved the following theorems.

Theorem 1.1 [12] *If* $j, k, l \geqslant 1$, *then*

$$\alpha_2(C_{7j} \Box C_{7k} \Box C_{7l}) = 49jkl.$$

Theorem 1.2 [12] *If* $j, k, l \geqslant 1$, *then*

$$\chi_2(C_{7j} \Box C_{7k} \Box C_{7l}) = 7.$$

In this paper, as an extension of Theorems 1.1 and 1.2, we establish the 2-distance independence number and 2-distance chromatic number for $C_3 \Box C_6 \Box C_m$, $C_n \Box P_3 \Box P_3$ and $C_4 \Box C_7 \Box C_n$ where $m \equiv 0 \ (mod\, 3)$ and $n, m \geqslant 3$.

2. Main results

The aim of this section is to find lower and upper bounds and exact values for the spcial cases 2-distance chromatic number of the families $G = \{C_3 \Box C_6 \Box C_m,\ C_n \Box P_3 \Box P_3, C_4 \Box C_7 \Box C_n \ where\ m \equiv 0\ (mod\, 3)\ and\ n, m \geqslant 3.\}$ The following two lemmas are essential for proving the main theorems.

Let G be a graph and f be a proper 2-coloring of G. Since every color class under f is a 2-independent set, we have the following lemma,

Lemma 2.1 *If G is a graph, then* $\chi_2(G) \geqslant \lceil \frac{|V(G)|}{\alpha(G^2)} \rceil$.

Let H be a graph, $m \geqslant 3$ and f denote a proper t-coloring of $(C_m \Box H)^2$. We denote by $f_{i,p}, 0 \leqslant i \leqslant m - p$ and $1 \leqslant p \leqslant m$, the restriction of f to $V(H^i), \ldots, V(H^{i+p-1})$. The following lemma is a natural generalization of [11, Lemma 1].

Lemma 2.2 *Let $m, n, p \geqslant 3, s \geqslant 1$ and let f be a proper t-coloring of $(C_m \Box H)^2$. If $f_{0,p}$ is a proper t-coloring of $(C_p \Box H)^2$, then*

$$\chi((C_{m+(s-1)p} \Box H)^2) \leqslant t.$$

Proof Let $f' : V(C_{m+(s-1)p} \Box H) \longrightarrow \{1, 2, ..., k\}$ be a function and f'_i the restriction of f' to $V(H^i)$. We define the function f' by

$$f'_i = \begin{cases} f_i & i < m, \\ f_{(i-m) \, mod \, p} & i \geqslant m. \end{cases}$$

Consider first the vertex (j, m). In this case vertex (j, m) is adjacent to $\{(j - l, m - 1); l \in \{0, 1, -1\}\}$ and $(j, m - 2)$ in the subgraph induced by $V(H^0), ..., V(H^{m-1})$, as illustrated in Figure 1. By definition f'_i we have $f'(j, m) = f(j, 0)$. Since f is a proper t-coloring of $(C_m \Box H)^2$ and $(j, 0)$ is adjacent to $\{(j - l, m - 1); l \in 0, 1, -1\}$ and $(j, m - 2)$ in $(C_n \Box H)^2$, this case is settled. Similarly for any two adjacent vertices (x, y) and $(x', y') \in \{(j, m + 1), (j, m + sp), (j, m + sp + 1), \; s \geqslant 1\}$ of $V(C_{m+(s-1)p} \Box H)^2$, we have $f'(x, y) \neq f'(x', y')$ and can be proved analogously. Therefore the proof is completed.

■

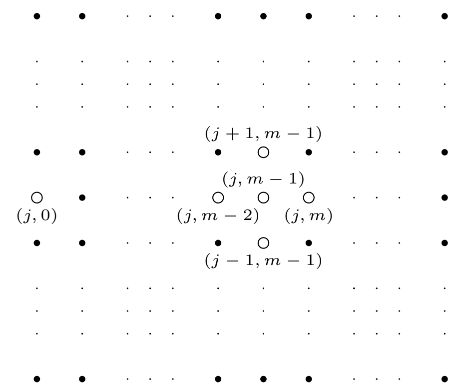

Figure 1. vertex-set of $(C_{m+(s-1)p} \Box H)^2$ for $s \geqslant 1$.

Before presenting our main results we need to obtain the 2-distance independent number of families G. We first mention two lemmas that need for proof of next lemmas. Let H be a graph. If I is a d-distance independent set of $C_k \Box H$, then, for $i = 0, \ldots, k - 1$, we set $I^i := I \cap V(H^i)$, that is, I^i is the subset of I induced by the vertices of H^i.

Lemma 2.3 [12] *Let H be a graph, $k, p \geqslant 3$ and $s \geqslant 1$. If I is a d-distance independent set of $C_k \Box H$ and $I^0 \cup I^1 \cup \cdots \cup I^{p-1}$ is a d-distance independent set of $C_p \Box H$ such that $|I^0| + |I^1| + \cdots + |I^{p-1}| = l$, then $\alpha_d(C_{k+(s-1)p} \Box H) \geqslant |I| + (s - 1)l$.*

Lemma 2.4 [12] *Let H be a graph, $k \geqslant 3$, $k \geqslant q \geqslant 1$ and $d \geqslant 1$. Then $\alpha_d(C_k \Box H) \leqslant \dfrac{k\alpha_d(P_q \Box H)}{q}$.*

Lemma 2.5 *If k is an integer, then $\alpha_2(C_3 \Box C_6 \Box C_{3k}) = 6k$.*

Proof By computer calculations, we have $\alpha_2(C_3\square C_6\square C_3) = 6$ and $\alpha_2(C_3\square C_6\square P_3) = 6$. Thus by Lemma 2.4 and since $\alpha_2(C_3\square C_6\square P_3) = 6$, we have,

$$\alpha_2(C_3\square C_6\square C_{3k}) \leqslant \frac{3k\alpha_2(C_3\square C_6\square P_3)}{3} = 6k.$$

To prove the lower bound, it is enough to find a 2-independent set with cardinality $6k$ for graph $(C_3\square C_6\square C_{3k})$.

We define S as follows,

$$S = \{(i\ (mod\, 6), i\ (mod\, 3), i), (i+3\ (mod\, 6), i\ (mod\, 3), i), \quad i = 0, 1, \ldots, 3k-1\}.$$

It is obvious that the cardinality of set S is $6k$. We show that S is a 2-distance independent set of $(C_3\square C_6\square C_{3k})$. Let $A = (x_1, x_2, x_3)$ and $B = (x'_1, x'_2, x'_3)$ be two arbitrary vertices in S. We show that the distance between them is greater than 2. Then we consider the following cases.

Case 1: If $x_3 = x'_3$, then by the definition of Cartesian product, $|A - B| = |x_1 - x'_1| + |x_2 - x'_2| = 3$. Thus the distance between A and B is greater than 2.

Case 2: If $x_3 - x'_3 = 1$, then there are 2 cases for A and B vertices.

(1) If both A and B are $(i\ (mod\, 6), i\ (mod\, 3), i)$, or $(i+3\ (mod\, 6), i\ (mod\, 3), i)$ then $|A - B| = |x_1 - x'_1| + |x_2 - x'_2| + 1 \geqslant 3$.

(2) If A is $(i\ (mod\, 6), i\ (mod\, 3), i)$, and B is $(i+3\ (mod\, 6), i\ (mod\, 3), i)$ then $|A - B| = |x_1 - x'_1| + |x_2 - x'_2| + 1 \geqslant 6$.

Case 3: If $x_3 - x'_3 = 2$, it is clear that the distance between A and B is equal or greater than 3. This implies that $\alpha_2(C_3\square C_6\square C_{3k}) \geqslant 6k$ and the proof is completed. ∎

Lemma 2.6 *$\alpha_2(C_3\square C_{6k}) = 3k$ and $\alpha_2(C_3\square P_{6k}) = 3k$.*

Proof Since $\alpha_2(C_3\square P_6) = 3$, then

$$\alpha_2(C_3\square P_{6k}) \leqslant \alpha_2(C_3\square P_6) \times k = 3k.$$

To reach the lower bounds, we define the set S as follows,

$$S = \{(6i, 0), (1, 2+6i), (2, 4+6i) | i = 0, 1, \cdots, k-1\}.$$

By definition of the Cartesian product, if (g, h) and (g', h') are vertices of S, $d_{G\square H}((g, h), (g', h')) = d_G(g, g') + d_H(h, h') \geqslant 3$, hence S is a 2-distance independent set of $(C_3\square P_{6k})$ with cardinality $3k$. Therefore $\alpha_2(C_3\square P_{6k}) \geqslant 3k$ and this completes the proof of the first statement. To prove the second part, by Lemma 2.4 we have,

$$\alpha_2(C_3\square C_{6k}) \leqslant \alpha_2(C_3\square P_{6k}) \leqslant k\alpha_2(C_3\square P_6) = 3k.$$

Also, since $\alpha_2(C_3\square C_6) = 3$, Lemma 2.3 implies

$$\alpha_2(C_3\square C_{6k}) \geqslant 3 + (s-1) \times 3 = 3s.$$

∎

Lemma 2.7 *Let $G = C_k\square P_3\square P_3$ and $k = 3t$, then $\alpha_2(G) = 5t$.*

Proof We obtain,

$\alpha_2(P_3\Box P_3\Box P_3)=5, \alpha_2(C_3\Box P_3\Box P_3)=5, \alpha_2(C_6\Box P_3\Box P_3)=10, \alpha_2(C_9\Box P_3\Box P_3)=15$

by a computer search. Since $k=3t$, then Lemma 2.4 implies

$$\alpha_2(C_{3t}\Box P_3\Box P_3)\leqslant \alpha_2(P_{3t}\Box P_3\Box P_3)\leqslant t\alpha_2(P_3\Box P_3\Box P_3)=5t.$$

In order to prove the lower bound when $k=3t$, from Lemma 2.3 and since $\alpha_2(C_3\Box P_3\Box P_3)=5$, depicted in Fig. 2 we have,

$$\alpha_2(C_{3t}\Box P_3\Box P_3)\geqslant 5+(t-1)5=5t.$$

This assertion completes the proof. ■

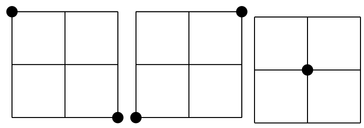

Figure 2. 5 vertices of 2-distance independent set of $(C_3\Box P_3\Box P_3)$

Lemma 2.8 *If* $G=C_4\Box C_7\Box C_n$, $n=4k$ *then* $\alpha_2(G)=14k$

Proof Using a computer program, we have $\alpha_2(C_4\Box C_7)=4, \alpha_2(C_4\Box C_7\Box P_4)=14$, $\alpha_2(C_4\Box C_7\Box C_4)=14$. Since $n=4k$, then by Lemma 2.4 we have

$$\alpha_2(C_4\Box C_7\Box C_{4k})\leqslant \alpha_2(C_4\Box C_7\Box P_{4k})\leqslant k\alpha_2(C_4\Box C_7\Box P_4)=14k.$$

To reach the lower bounds, we found fourteen vertices of 2-distance independent set of $(C_4\Box C_7\Box C_4)$ depicted in Fig. 3. Therefore, by Lemma 2.3 we have,

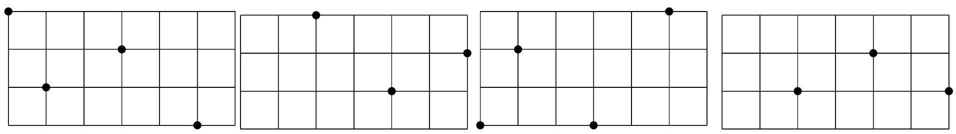

Figure 3. $14\, vertices\, of\, a\, 2-distance\, independent\, set\, of (C_4\Box C_7\Box C_4)$

$\alpha_2(C_4\Box C_7\Box C_{4k})=\alpha_2(C_4\Box C_7\Box C_{4+(k-1)4})\geqslant |I|+(k-1)|i|=14+(k-1)14=14k,$

which completes our proof. ■

Given two integers x and y, let $S(x,y)$ denote the set of all nonnegative integer combinations of x and y defined as follows,

$$S(x,y)=\{\alpha x+\beta y:\alpha,\beta \textit{ are nonnegative integers}\}.$$

Lemma 2.9 [14] *Let* x *and* y *be relatively prime integers greater than 1, then* $n\in S(x,y)$ *for all* $n\geqslant (x-1)(y-1)$.

Theorem 2.10 *If* $k\geqslant 1$, *then*

$$\chi(C_3\Box C_6\Box C_{3k})^2=9.$$

Proof Fig. 4 presents a proper 9-coloring of $(C_3\square C_6\square C_3)^2$. By Lemmas 2.2 and

1 4 2 3 6 5	7 3 6 8 9 2	8 9 5 1 4 3
2 5 1 4 8 7	6 8 9 7 5 1	4 7 3 6 2 9
3 6 7 5 1 4	9 1 4 2 3 8	5 2 8 9 7 6

Figure 4. *A proper* $9-$*coloring of* $\chi_2(C_3\square C_6\square C_3)$.

2.9 we have,

$$\chi((C_3\square C_6)\square C_{3+(k-1)3})^2 = \chi((C_3\square C_6)\square C_{3k})^2 \leqslant 9,$$

where $k \geqslant 1$. It is sufficient to show that $\chi(C_3\square C_6\square C_{3k})^2 \geqslant 9$. By Lemmas 2.1 and 2.5, we have,

$$\chi_2(C_3\square C_6\square C_{3k}) \geqslant \lceil V(C_3\square C_6\square C_{3k})/\alpha_2(C_3\square C_6\square C_{3k})\rceil = \lceil (3\times 6\times 3k)/6k\rceil = 9.$$

■

Theorem 2.11 *If* $k \geqslant 1$*, then* $\chi_2(C_n\square P_3\square P_3) \geqslant 6$ *if* $n = 3k$ *and*

$$\chi_2(C_n\square P_3\square P_3) \leqslant \begin{cases} 7 & n = 3k, \\ 8 & n = 3k+1, \\ 9 & n = 3k+2. \end{cases}$$

Proof Let $n = 3k$, by Lemmas 2.1 and 2.7 we have,

$$\chi_2(C_{3k}\square P_3\square P_3) \geqslant \lceil V(C_{3k}\square P_3\square P_3)/\alpha_2(C_{3k}\square P_3\square P_3)\rceil = \lceil 3k\times 3\times 3/5k\rceil = 6.$$

For an upper bound, Fig. 5 presents a proper 7-coloring of $(C_3\square P_3\square P_3)^2$. By

1 2 3	3 4 5	5 6 1
4 5 6	2 1 7	7 3 2
3 7 1	5 6 3	1 4 5

Figure 5. *A proper* $7-$*coloring of* $\chi_2(C_3\square P_3\square P_3)$.

Lemmas 2.2 and 2.9, we have $\chi_2(C_{3k}\square P_3\square P_3) \leqslant 7$. Therefore,

$$6 \leqslant \chi_2(C_{3k}\square P_3\square P_3) \leqslant 7.$$

A proper 8-coloring of $(C_4\square P_3\square P_3)^2$ is illustrated in Fig. 6 such that the leftmost three blocks induce a proper 8-coloring of $(C_3\square P_3\square P_3)^2$. Thus, for $n = 3k+1$, by Lemmas 2.2 and 2.9, we get $\chi_2(C_{3k+1}\square P_3\square P_3) \leqslant 8$.

Figure 7 presents a proper 9-coloring of $(C_5\square P_3\square P_3)^2$ and the leftmost three blocks of Fig. 7 induce a proper 9-coloring of $(C_3\square P_3\square P_3)^2$. Hence, by Lemmas 2.2 and 2.9, we have

$$\chi_2(C_{3k+2}\square P_3\square P_3) \leqslant 9,$$

1 2 3	3 4 1	5 6 8	8 7 5
4 5 6	2 8 7	7 3 2	6 1 4
8 7 1	5 6 3	1 4 5	3 2 8

Figure 6. *A proper* $8-$ *coloring of* $\chi_2(C_4\Box P_3\Box P_3)$.

whenever $n = 3k + 2$. ■

1 2 9	4 5 6	9 3 8	6 4 1	8 7 5
3 4 7	6 7 2	8 1 5	7 2 3	5 6 8
2 9 6	1 3 8	5 7 4	3 8 9	4 1 2

Figure 7. *A proper* $9-$ *coloring of* $\chi_2(C_5\Box P_3\Box P_3)$.

Theorem 2.12 *If* $k \geqslant 1$ *then,* $\chi_2(C_4\Box C_7\Box C_n) = 8$ *if* $n = 4k$ *and,*

$$\chi_2(C_4\Box C_7\Box C_n) \leqslant \begin{cases} 11 & n = 4k+1, k \neq 1, \\ 11 & n = 4k+2, \\ 12 & n = 4k+3. \end{cases}$$

Proof Let $n = 4k$, by Lemma 2.1 and Lemma 2.8 we have,

$$\chi_2(C_4\Box C_7\Box C_n) \geqslant \lceil V(C_4\Box C_7\Box C_n)/\alpha_2(C_4\Box C_7\Box C_n)\rceil = \lceil 4\times 7\times 4k/14k\rceil = 8$$

Fig. 8 presents a proper 8-coloring of $(C_4\Box C_7\Box C_4)^2$, by Lemmas 2.2 and 2.9 we have $\chi_2(C_4\Box C_7\Box C_n) \leqslant 8$, therefore,

$$\chi_2(C_4\Box C_7\Box C_n) = 8.$$

Figure 9 presents a proper 11-coloring of $(C_4\Box C_7\Box C_9)^2$ such that the leftmost

1 2 3 4	6 5 8 7	3 4 1 2	8 7 6 5
3 4 1 2	8 7 6 5	1 2 3 4	6 5 8 7
5 6 7 8	2 1 4 3	7 8 5 6	4 3 2 1
1 2 3 4	6 5 8 7	3 4 1 2	8 7 6 5
3 4 1 2	8 7 6 5	1 2 3 4	6 5 8 7
5 6 7 8	2 1 4 3	7 8 5 6	4 3 2 1
7 8 5 6	4 3 2 1	5 6 7 8	2 1 4 3

Figure 8. *A proper* $8-$ *coloring of* $\chi_2(C_4\Box C_7\Box C_4)$.

four blocks induce a proper 11-coloring of $(C_4\Box C_7\Box C_4)^2$. Thus, by Lemmas 2.2 and 2.9, we get for $t > 1$,

$$\chi_2(C_4\Box C_7\Box C_{5+4s}) = \chi_2(C_4\Box C_7\Box C_{1+4t}) \leqslant 11.$$

A proper 11-coloring of $(C_4\Box C_7\Box C_6)^2$ is illustrated in Fig. 10 such that the leftmost three blocks of Fig. 10 induce a proper 11-coloring of $(C_4\Box C_7\Box C_4)^2$. Thus, for $n = 4k + 2$, by Lemmas 2.2 and 2.9, we get

$$\chi_2(C_4\Box C_7\Box C_n) \leqslant 11.$$

1 10 11 4	6 9 8 10	3 4 1 2	8 7 6 5	1 2 3 4	6 5 8 7	9 4 1 2	8 11 6 5	3 5 2 7
9 4 1 2	8 11 6 5	1 2 3 4	6 5 8 7	3 4 1 2	8 7 6 5	1 2 10 4	10 9 8 7	11 7 3 6
5 6 7 11	10 1 9 3	7 8 5 6	4 3 2 1	5 6 7 8	2 1 4 3	7 8 5 6	4 3 2 1	2 10 4 8
1 2 10 4	6 5 8 7	3 4 1 2	8 7 6 5	1 2 3 4	6 5 8 7	3 4 9 2	11 7 6 5	9 8 1 3
3 4 11 9	8 7 6 5	1 2 3 4	6 5 8 7	3 4 1 2	8 7 6 5	1 2 3 4	6 9 8 10	11 5 2 7
5 9 7 8	11 1 4 3	7 8 5 6	4 3 2 1	5 6 7 8	2 1 4 3	7 8 5 6	4 3 11 9	2 6 10 1
7 8 5 6	4 3 2 1	5 6 7 8	2 1 4 3	7 8 5 6	4 3 2 1	5 6 7 8	9 2 4 3	10 1 9 11

Figure 9. $A\,proper\ 11 - coloring\ of\ \chi_2(C_4\Box C_7\Box C_9)$.

A proper 12-coloring of $(C_4\Box C_7\Box C_7)^2$ is illustrated in Fig. 11 such that the left-

1 2 3 4	6 5 8 7	3 4 1 2	8 7 6 5	4 3 2 1	5 6 7 8
10 4 1 2	8 7 6 5	1 2 3 4	6 5 8 7	2 1 9 10	9 8 5 6
5 9 10 8	11 1 4 3	7 8 5 6	4 3 2 1	9 6 4 11	3 10 2 7
1 2 3 4	10 5 8 7	3 4 10 2	8 7 6 9	10 5 1 3	6 4 9 5
3 11 1 2	8 7 6 5	1 2 3 11	6 5 8 7	2 9 11 4	9 8 10 6
5 6 7 8	2 1 4 3	7 8 5 6	9 3 2 1	8 7 6 5	1 2 3 4
7 8 5 6	4 3 2 1	5 6 7 8	2 1 4 3	6 5 8 7	3 4 1 2

Figure 10. $A\,proper\ 11 - coloring\ of\ \chi_2(C_4\Box C_7\Box C_6)$.

most four blocks of Fig. 11 induce a proper 12-coloring of $(C_4\Box C_7\Box C_4)^2$. Thus, for $n = 4k + 3$, by Lemmas 2.2 and 2.9 we have,

$$\chi_2(C_4\Box C_7\Box C_n) \leqslant 12.$$

This assertion completes the proof. ∎

1 10 3 4	6 5 8 7	3 4 1 2	8 7 6 5	1 2 3 4	11 5 8 7	3 7 9 12
9 4 1 2	8 7 9 5	1 2 3 4	6 5 8 7	3 4 1 2	8 9 10 5	10 11 6 8
5 6 7 11	2 1 4 3	7 8 6 5	4 3 2 1	5 6 7 10	2 1 4 3	4 3 5 9
1 10 3 4	9 5 10 7	3 4 1 2	8 7 6 5	1 2 3 4	9 5 11 7	6 10 2 8
3 11 1 10	8 7 9 5	1 2 3 4	6 5 8 7	3 10 1 2	8 7 10 5	2 9 4 6
9 6 7 8	2 1 4 3	7 8 5 6	4 3 2 1	5 6 7 8	10 1 4 3	7 5 11 2
11 8 9 6	4 3 2 1	5 6 7 8	2 1 4 3	7 8 5 6	4 3 11 9	6 2 10 5

Figure 11. $A\,proper\ 12 - coloring\ of\ \chi_2(C_4\Box C_7\Box C_7)$.

The following theorem summarizes the above discussion.

Theorem 2.13 *Let* $G = C_m\Box C_n\Box C_k$ *if* $k \geqslant 1$ *and* $(m, n, k) \in S = \{(3, 6, 3t), (4, 7, 4t), t \geqslant 1\}$ *then,*

$$\chi_2(G) = \lceil \frac{|V(G)|}{\alpha_2(G)} \rceil.$$

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