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Numerical Solution Two-Dimensional Volterra-Fredholm Integral Equations of the Second Kind with Block-Pulse Functions Based on Legendre Polynomials

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Abstract. In this paper, we present a new numerical technique based on Block-pulse functions to solve two-dimensional Volterra-Fredholm integral equations of the second kind. To produce Block-pulse functions, the orthogonal Legendre polynomials is used. Furthermore, operational matrix is applied to convert two-dimensional Volterra-Fredholm integral equations to a linear algebraic system. The convergence analysis of the new method is discussed. Finally, some numerical examples are given to confirm the applicability and efficiency of the new method for solving two-dimensional Volterra-Fredholm integral equations of the second kind.

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Index to information contained in this paper

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1. Introduction

In this paper, we consider two-dimensional Volterra-Fredholm integral equations (`TDVFIE`) of the second kind in the form

$$f(x,t) = g(x,t) + \int_0^x \int_0^1 k(x,t,y,s) f(y,s) ds dy, \quad x \in [0,1), \tag{1}$$

where $f(x,t) \in L^2(\Omega)$ is an unknown function, the function $g(x,t) \in L^2(\Omega)$, kernel $k(x,t,y,s) \in L^2(\Omega \times \Omega)$ are given and $\Omega = [0,1) \times [0,1)$. Two-dimensional Volterra-Fredholm integral equations arise in many phenomena in physics and engineering fields [8, 21]. The existence and uniqueness of the solution for the two-dimensional integral equation (1) are discussed in [13, 18]. Many integral equations are usually difficult to solve analytically. So, we obtain an approximate solution for them. In recent years, significant progress has been made in numerical analysis for linear two-dimensional mixed Volterra-Fredholm integral equations [12]. Furthermore, efficient numerical methods are given for the nonlinear integral equations and especially for two-dimensional models in [9, 17]. Also, the collocation and discretization method [6], the particular trapezoidal Nyström method [11] and the Adomian decomposition method [7] are applied for solving two-dimensional linear and nonlinear integral equations.

In this study, we use two-dimensional Block-pulse functions based on Legendre polynomials to solve (1). Also, we compute the operational matrix of integration and then approximate known and unknown functions in integral equations (1) by Block-pulse functions. Therefore, we get a linear system.

The rest of this paper is organized as follows. In Section 2, two-dimensional Block-pulse functions based on Legendre polynomials are introduced. The operational matrix of integration have been constructed in Section 3. The new method is presented in Section 4 which will convert the integral equation into a linear algebraic system. Convergence analysis for the proposed method is provided in Section 5. Some numerical results are reported to show the applicability and reliability of our method in Section 6. Finally, concluding remarks are drawn in Section 7.

2. Two-dimensional block-pulse functions

We define two-dimensional Block-pulse functions based on Legendre polynomials on Ω as

$$\psi_{ij}(x,t) = \begin{cases} L_i(2Nx - 2i + 1) L_j(2Nt - 2j + 1), & \frac{i-1}{N} \leqslant x < \frac{i}{N}, \quad \frac{j}{M} \leqslant t < \frac{j+1}{M}, \\ 0, & otherwise \end{cases} \tag{2}$$

where $i = 1, 2, ..., N$, $j = 0, 1, ..., M-1$, N and M are positive integers. Here L_i and L_j are the well-known Legendre polynomials of order i and j, respectively, which are defined on the interval $[-1, 1]$ and can be determined by the following formulae:

$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= x, \\ &\vdots \\ L_{m+1}(x) &= \frac{2m+1}{m+1} x L_m(x) - \frac{m}{m+1} L_{m-1}(x), \quad m = 1, 2, ... \end{aligned}$$

Some properties of two-dimensional Block-pulse functions (2) are as follows:

- Disjointness:

$$\psi_{ij}(x,t)\psi_{i'j'}(x,t) = \begin{cases} \psi_{ij}^2(x,t), & i = i',\, j = j', \\ 0. & otherwise \end{cases} \tag{3}$$

- Orthogonality:

$$\int_0^1 \int_0^1 \psi_{ij}(x,t)\psi_{i'j'}(x,t) dx dt = \begin{cases} \int_0^1 \int_0^1 \psi_{ij}^2(x,t) dx dt, & i = i',\, j = j', \\ 0. & otherwise \end{cases} \tag{4}$$

Let

$$\Psi(x,t) = \left[\psi_{10}(x,t) \ldots \psi_{1(M-1)}(x,t) \ldots \psi_{N0}(x,t) \ldots \psi_{N(M-1)}(x,t)\right]^T. \tag{5}$$

Hence, it is clear that

$$\Psi(x,t)\Psi^T(x,t) = \begin{bmatrix} \psi_{10}^2(x,t) & 0 & \ldots & 0 \\ 0 & \psi_{11}^2(x,t) & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & \ldots & 0 & \psi_{N(M-1)}^2(x,t) \end{bmatrix}_{NM \times NM} \tag{6}$$

3. Operational matrix

The integration of Block-pulse functions based on Legendre polynomials should be expandable into Block-pulse functions with the coefficient matrix P. Take

$$\Psi(x) = \left[\psi_{10}(x) \ldots \psi_{1(M-1)}(x) \ldots \psi_{N0}(x) \ldots \psi_{N(M-1)}(x)\right]^T,$$

where $\psi_{ij}(x)$ are one-dimensional of Block-pulse functions based on Legendre polynomials [15]. Then, we get

$$\int_0^x \Psi(y) dy = P\Psi(x),$$

where P is an operational matrix as follows [15]:

$$P = \begin{bmatrix} S & U & U & \dots & U \\ 0 & S & U & \dots & U \\ 0 & 0 & S & \dots & U \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & S \end{bmatrix}_{NM \times NM}$$

and

$$U = \frac{1}{N} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{M \times M}$$

$$S = \frac{1}{2N} \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{-1}{2M-3} & 0 & \frac{1}{2M-3} \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{-1}{2M-1} & 0 \end{bmatrix}_{M \times M}$$

4. New method

In this section, we use two-dimensional Block-pulse functions based on Legendre polynomials (2) to solve following `TDVFIF`

$$f(x,t) = g(x,t) + \int_0^x \int_0^1 k(x,t,y,s) f(y,s) ds dy, \quad x \in [0,1). \tag{7}$$

We approximate known and unknown functions in (7) by two-dimensional Block-pulse functions. Therefore, we get

$$g(x,t) \approx \sum_{i=1}^{N} \sum_{j=0}^{M-1} g_{ij} \psi_{ij}(x,t) = G^T \Psi(x,t), \tag{8}$$

in which

$$G = \left[g_{10} \cdots g_{1(M-1)} \; g_{20} \cdots g_{2(M-1)} \cdots g_{N0} \cdots g_{N(M-1)} \right]^T,$$

and

$$g_{ij} = \frac{\int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j}{M}}^{\frac{j+1}{M}} g(x,t)\psi_{ij}(x,t)dtdx}{\int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j}{M}}^{\frac{j+1}{M}} |\psi_{ij}(x,t)|^2 dtdx}. \tag{9}$$

Hence, we can approximate the function $k(x,t,y,s)$, $g(x,t)$ and $f(x,t)$ as follows

$$\begin{aligned} k(x,t,y,s) &\approx \Psi^T(x,t)K\Psi(y,s), \\ g(x,t) &\approx \Psi^T(x,t)G, \\ f(x,t) &\approx \Psi^T(x,t)F. \end{aligned}$$

Note that K is an $NM \times NM$ matrix and its element are obtained by

$$K_{ij} = \frac{\int_0^1\int_0^1\int_0^1\int_0^1 k(x,t,y,s)\Psi_{(i)}(x,t)\Psi_{(j)}(y,s)dsdydtdx}{\Big(\int_0^1\int_0^1 |\Psi_{(i)}(x,t)|^2 dxdt\Big)\Big(\int_0^1\int_0^1 |\Psi_{(j)}(y,s)|^2 dyds\Big)}, \tag{10}$$

where $\Psi_{(i)}(x,t)$ denotes the i-th element of the $\Psi(x,t)$. By substituting these functions in (7), we obtain

$$\begin{aligned} \Psi^T(x,t)F &= \Psi^T(x,t)G + \int_0^x\int_0^1 \Psi^T(x,t)K\Psi(y,s)\Psi^T(y,s)Fdsdy \\ &= \Psi^T(x,t)G + \Psi^T(x,t)K\Big(\int_0^x\int_0^1 \Psi(y,s)\Psi^T(y,s)dsdy\Big)F. \end{aligned} \tag{11}$$

From (6), we have

$$\int_0^x\int_0^1 \Psi(y,s)\Psi^T(y,s)dsdy = \begin{bmatrix} \int_0^x\int_0^1 \psi_{10}^2(y,s)dsdy & & 0 \\ & \ddots & \\ 0 & & \int_0^x\int_0^1 \psi_{N(M-1)}^2(y,s)dsdy \end{bmatrix} \tag{12}$$

in which

$$\begin{aligned}\int_0^x\int_0^1 \psi_{ij}^2(y,s)dsdy &= \int_0^x\int_0^1 L_i^2(2Ny-2i+1)L_j^2(2Ns-2j+1)dsdy\\ &= \int_0^x L_i^2(2Ny-2i+1)\Big(\int_0^1 L_j^2(2Ns-2j+1)ds\Big)dy\\ &= \int_0^x L_i^2(2Ny-2i+1)\underbrace{\Big(\int_{\frac{j}{M}}^{\frac{j+1}{M}} L_j^2(2Ns-2j+1)ds\Big)}_{\Upsilon_j}dy\\ &= \Upsilon_j\int_0^x L_i^2(2Ny-2i+1)dy\\ &= \Upsilon_j R_i\Psi(x),\end{aligned}$$

where R_i is the i-th row of operational matrix. Hence

$$\int_0^x\int_0^1 \Psi(y,s)\Psi^T(y,s)dsdy = \begin{bmatrix}\Upsilon_0 R_1\Psi(x) & & 0\\ & \ddots & \\ 0 & & \Upsilon_{M-1}R_N\Psi(x)\end{bmatrix} := R. \tag{13}$$

Finally, by subsuiting (13) in (11), we have

$$\Psi^T(x,t)F = \Psi^T(x,t)G + \Psi^T(x,t)KRF,$$

or

$$(I-KR)F = G.$$

5. Convergence analysis

In this section, we show that our method to solve two-dimensional Volterra-Fredholm integral equations of the second kind (1) is convergent. By two-dimensional Block-pulse functions based on Legendre polynomials, the unknown function $f(x,t) \in L^2(\Omega)$ can be approximated by

$$f(x,t) \approx \sum_{i=1}^{N}\sum_{j=0}^{M-1} f_{ij}\psi_{ij}(x,t) = \Psi^T(x,t)F, \tag{14}$$

where the vector $F \in \mathbb{R}^{N\times M}$ is given by

$$F = \big[f_{10}\dots f_{1(M-1)}\ f_{20}\dots f_{2(M-1)}\dots f_{N0}\dots f_{N(M-1)}\big]^T, \tag{15}$$

and the Block-pulse coefficients are computed by

$$f_{ij} = \frac{\displaystyle\int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j}{M}}^{\frac{j+1}{M}} f(x,t)\psi_{ij}(x,t)dtdx}{\displaystyle\int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j}{M}}^{\frac{j+1}{M}} |\psi_{ij}(x,t)|^2 dtdx}. \tag{16}$$

Theorem 5.1 *Let*

$$f_{NM}(x,t) = \sum_{i=1}^{N} \sum_{j=0}^{M-1} f_{ij}\psi_{ij}(x,t),$$

be the approximated solution of (1) *on*

$$\Omega_{NM} = \Big\{(x,t) \in \Omega \mid \frac{i-1}{N} \leqslant x < \frac{i}{N},\ \frac{j}{M} \leqslant t < \frac{j+1}{M}\Big\}.$$

Then,

$$\int_0^1 \int_0^1 \Big(f(x,t) - f_{NM}(x,t)\Big)^2 dtdx$$

achieves its minimum value. Moreover, we have

$$\int_0^1 \int_0^1 f^2(x,t)dtdx = \sum_{i=1}^{N} \sum_{j=0}^{M-1} f_{ij}^2 \|\psi_{ij}(x,t)\|_2^2.$$

Proof See [16]. ■

Now, we get the main result of convergence analysis in the following theorem. We will show that the error of approximate solution is bounded for two-dimensional Volterra-Fredholm integral equations based on two-dimensional Block-pulse functions of Legendre polynomials.

Theorem 5.2 *Let $f_{NM}(x,t)$ be the approximate solution for the two-dimensional Volterra-Fredholm integral equation* (7)*. Then, there exists a positive constant L such that*

$$\|f(x,t) - f_{NM}(x,t)\|_2 \leqslant L.$$

Proof We define the representation error between $f(x,t)$ and its expansion, $f_{NM}(x,t)$, as follows:

$$E_{ij}(x,t) = f_{ij}\psi_{ij}(x,t) - f(x,t), \qquad i = 1, \cdots, N, \quad j = 0, 1, \cdots, M-1.$$

By the mean-value theorem for integrals, we obtain

$$\begin{aligned}
\|E_{ij}\|_2^2 &= \int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j}{M}}^{\frac{j+1}{M}} E_{ij}^2(x,t)dtdx \\
&= \int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j}{M}}^{\frac{j+1}{M}} \Big(f_{ij}\psi_{ij}(x,t) - f(x,t)\Big)^2 dtdx \\
&= \Big(f_{ij}\psi_{ij}(\alpha,\beta) - f(\alpha,\beta)\Big)^2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j}{M}}^{\frac{j+1}{M}} dtdx \qquad (\alpha,\beta) \in \Omega_{NM} \\
&= \frac{1}{NM}\Big(f_{ij}\psi_{ij}(\alpha,\beta) - f(\alpha,\beta)\Big)^2.
\end{aligned} \tag{17}$$

Now, (16) and mean-value theorem for integrals, give us

$$\begin{aligned}
f_{ij} &= \frac{\displaystyle\int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j}{M}}^{\frac{j+1}{M}} f(x,t)\psi_{ij}(x,t)dtdx}{\displaystyle\int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j}{M}}^{\frac{j+1}{M}} |\psi_{ij}(x,t)|^2 dtdx} \\
&= \frac{f(\eta,\zeta)\psi_{ij}(\eta,\zeta)}{\psi_{ij}^2(\eta,\zeta)} \cdot \frac{\displaystyle\int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j}{M}}^{\frac{j+1}{M}} dtdx}{\displaystyle\int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j}{M}}^{\frac{j+1}{M}} dtdx} \qquad (\eta,\zeta) \in \Omega_{NM} \\
&= \frac{f(\eta,\zeta)}{\psi_{ij}(\eta,\zeta)}.
\end{aligned} \tag{18}$$

Substituting (18) into (17), we have

$$\|E_{ij}\|_2^2 = \frac{1}{NM}\Big(\frac{f(\eta,\zeta)}{\psi_{ij}(\eta,\zeta)}\psi_{ij}(\alpha,\beta) - f(\alpha,\beta)\Big)^2 \leqslant \frac{M_{ij}^2}{NM}, \tag{19}$$

where M_{ij} is the upper bound for

$$\frac{f(\eta,\zeta)}{\psi_{ij}(\eta,\zeta)}\psi_{ij}(\alpha,\beta) - f(\alpha,\beta).$$

Now, for any $(x,t) \in \Omega_{NM}$, the error between $f(x,t)$ and $f_{NM}(x,t)$ is

$$E(x,t) = f_{NM}(x,t) - f(x,t).$$

Hence

$$\|E(x,t)\|_2^2 = \int_0^1\int_0^1 E^2(x,t)dtdx$$
$$= \int_0^1\int_0^1 \Big(\sum_{i=1}^{N}\sum_{j=0}^{M-1} E_{ij}(x,t)\Big)^2 dtdx$$
$$= \int_0^1\int_0^1 \sum_{i=1}^{N}\sum_{j=0}^{M-1} E_{ij}^2(x,t)dtdx$$
$$+ 2\sum_{i<j}\sum_{i'<j'}\int_0^1\int_0^1 E_{ij}(x,t)E_{i'j'}(x,t)dtdx.$$

Since the Block-pulse functions are distinct, we get

$$\|E(x,t)\|_2^2 = \sum_{i=1}^{N}\sum_{j=0}^{M-1}\int_0^1\int_0^1 E_{ij}^2(x,t)dtdx$$
$$= \sum_{i=1}^{N}\sum_{j=0}^{M-1}\|E_{ij}\|_2^2$$
$$\leqslant \sum_{i=1}^{N}\sum_{j=0}^{M-1}\frac{M_{ij}^2}{NM}$$
$$\leqslant L^2\sum_{i=1}^{N}\sum_{j=0}^{M-1}\frac{1}{NM} = L^2,$$

in which $L = \max\{M_{ij} \mid i = 1, \cdots, N,\ j = 0, \cdots, M-1\}$. Hence

$$\|E(x,t)\|_2 = \|f(x,t) - f_{NM}(x,t)\|_2 \leqslant L.$$

■

6. Numerical examples

To clarify the accuracy and effectiveness of the new method, some numerical examples are provided. We compare the numerical results of our method with Bernoulli collocation method (BCM) [14] for $N = M = 3$, successive approximations method (SAM) with $m_1 = m_2 = 18$ [19] and rationalized Haar functions (RHF) for $n = 4$ [10]. We introduce the error function as

$$E_{N,M}(x,t) = |f(x,t) - f_{NM}(x,t)|, \qquad \forall (x,t) \in \Omega,$$

where $f(x,t)$ denotes the exact solution and $f_{NM}(x,t)$ is the approximate solution of TDVFIE obtained by our method. All codes are implemented in Matlab 2017 programming environment on a 2.3Hz Intel core i3 processor laptop and 4GB of RAM.

Example 6.1 Consider the following two-dimensional Volterra-Fredholm integral equations of the second kind

$$f(x,t) = g(x,t) + \int_0^x \int_0^1 (x+y) f(y,s) ds dy, \quad (x,t) \in \Omega$$

where

$$g(x,t) = xt - \frac{5}{12}x^3.$$

The exact solution is $f(x,t) = xt$. For this example, the numerical results of the new method are reported in Table 1.

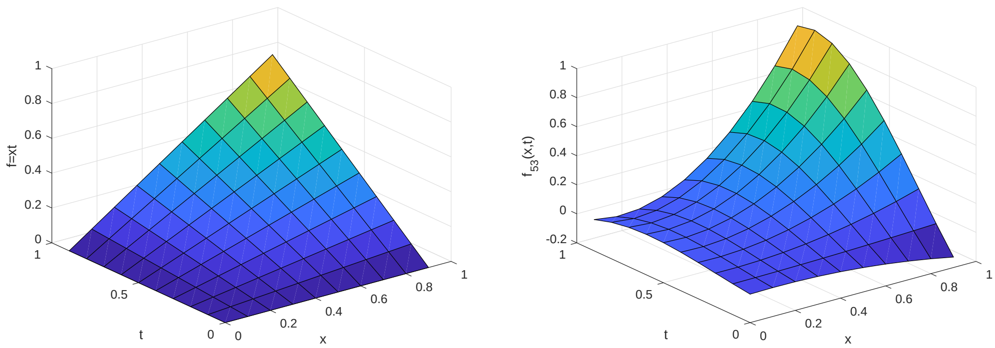

Figure 1. Plot of the exact solution and approximate solution $f_{53}(x,t)$ for Example 6.1.

Example 6.2 Consider the following two-dimensional Volterra-Fredholm integral equations of the second kind

$$f(x,t) = g(x,t) + \int_0^x \int_0^1 (xy^2 + \cos s) f(y,s) ds dy, \quad (x,t) \in \Omega$$

where

$$g(x,t) = x \sin t - \frac{1}{4}x^2 \sin^2 1 + \frac{1}{4}x^5(\cos 1 - 1).$$

The exact solution is $f(x,t) = x \sin t$. The numerical results of the new method for Example 6.2 are reported in Table 2.

Example 6.3 Consider the following two-dimensional Volterra-Fredholm integral equations of the second kind

$$f(x,t) = g(x,t) + \int_0^x \int_0^1 xty^2s^2 f(y,s) ds dy, \quad (x,t) \in \Omega$$

where

$$g(x,t) = x^2 + xt - \frac{1}{15}tx^6 - \frac{1}{16}tx^5.$$

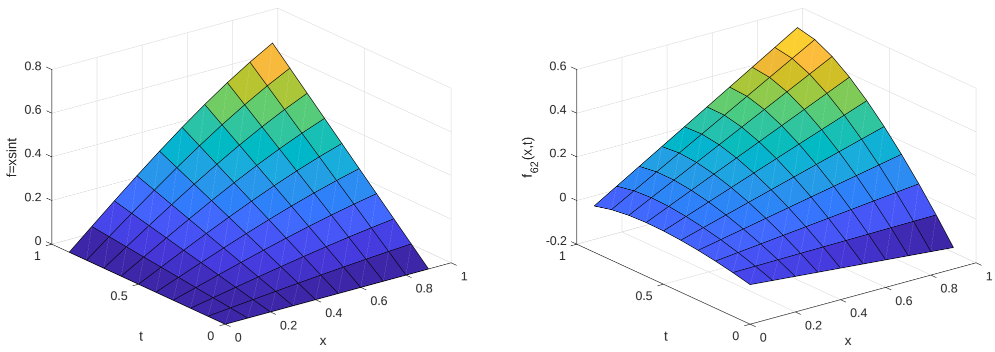

Figure 2. Plot of the exact solution and approximate solution $f_{62}(x,t)$ for Example 6.2.

The exact solution is $f(x,t) = x^2 + xt$. For this example, the numerical results of the new method are reported in Table 3.

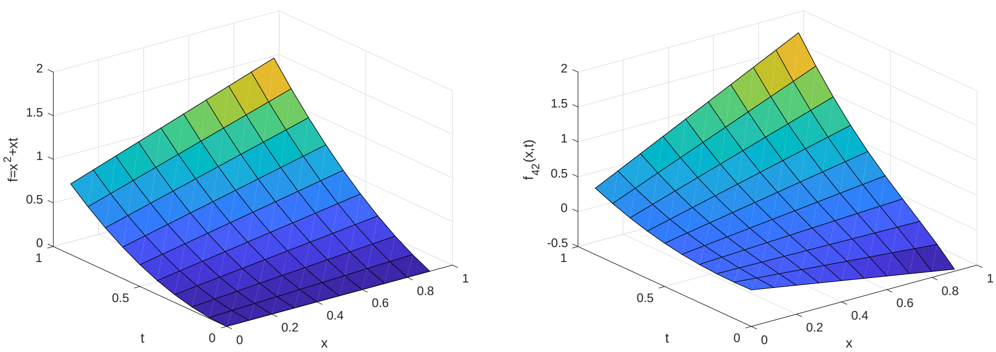

Figure 3. Plot of the exact solution and approximate solution $f_{42}(x,t)$ for Example 6.3.

7. Conclusion

This paper has introduced a new method for solving two-dimensional Volterra-Fredholm integral equations of the second kind. Our approach is based on Block-pulse functions introduced with orthogonal Legendre polynomials. Operational matrix are used to convert two-dimensional integral equations into linear algebraic equations. We compare our method with BCM for solving two-dimensional Volterra-Fredholm integral equations where absolute errors are reported in Tables 1-3. These numerical results confirm excellent performance of new the method to solve TDVFIE.

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Table 1. Absolute errors for Example 6.1.

Nodes	BCM	SAM	RHF	Present method		
				$M=2$, $N=6$	$M=2$, $N=4$	$M=3$, $N=5$
(x,t)	$E_{3,3}$	$E_{18,18}$	$E_{4,4}$	$E_{6,2}$	$E_{4,2}$	$E_{5,3}$
$(0,0)$	3.2969690×10^{-1}	5.196012×10^{-1}	2.251543×10^{-1}	2.34284×10^{-3}	1.956527×10^{-2}	6.58611×10^{-3}
$(0.1,0.1)$	3.2760781×10^{-1}	5.213247×10^{-1}	2.159587×10^{-1}	6.9594×10^{-4}	9.41065×10^{-3}	3.20930×10^{-3}
$(0.2,0.2)$	3.1576020×10^{-1}	5.185562×10^{-1}	2.085641×10^{-1}	6.54634×10^{-3}	3.25421×10^{-3}	4.31792×10^{-3}
$(0.3,0.3)$	2.9536746×10^{-1}	5.164139×10^{-1}	1.948571×10^{-1}	1.264851×10^{-2}	3.6006×10^{-4}	1.691187×10^{-2}
$(0.4,0.4)$	2.6797218×10^{-1}	5.141450×10^{-1}	1.665021×10^{-1}	1.500976×10^{-2}	4.40517×10^{-3}	2.407185×10^{-2}
$(0.5,0.5)$	2.3534264×10^{-1}	5.126711×10^{-1}	1.498517×10^{-1}	8.20457×10^{-3}	1.338096×10^{-2}	1.858205×10^{-2}
$(0.6,0.6)$	1.9937474×10^{-1}	5.102234×10^{-1}	1.422590×10^{-1}	1.462540×10^{-2}	3.329435×10^{-2}	1.95878×10^{-3}
$(0.7,0.7)$	1.6199934×10^{-1}	5.083669×10^{-1}	1.393651×10^{-1}	6.177137×10^{-2}	7.167914×10^{-2}	3.360778×10^{-2}
$(0.8,0.8)$	1.2509497×10^{-1}	5.063321×10^{-1}	1.321355×10^{-1}	1.4295737×10^{-1}	1.3758610×10^{-1}	6.454826×10^{-2}
$(0.9,0.9)$	9.040599×10^{-2}	5.044418×10^{-1}	1.272933×10^{-1}	2.6934026×10^{-1}	2.4158494×10^{-1}	7.356012×10^{-2}
$\|E(x,t)\|_2$	7.8739105×10^{-1}	1.622998	5.498208×10^{-1}	3.1226913×10^{-1}	2.9020713×10^{-1}	1.0951696×10^{-1}
$\|E(x,t)\|_\infty$	3.2969690×10^{-1}	5.213247×10^{-1}	2.251543×10^{-1}	2.69340260×10^{-1}	2.4158494×10^{-1}	7.356012×10^{-2}

Table 2. Absolute errors for Example 6.2.

Nodes	BCM	SAM	RHF	Present method		
				$M=2$, $N=6$	$M=2$, $N=4$	$M=3$, $N=5$
(x,t)	$E_{3,3}$	$E_{18,18}$	$E_{4,4}$	$E_{6,2}$	$E_{4,2}$	$E_{5,3}$
$(0,0)$	2.6755733×10^{-1}	3.104216×10^{-1}	3.002515×10^{-1}	3.222986×10^{-2}	1.8301704×10^{-1}	1.868999×10^{-2}
$(0.1,0.1)$	2.6657086×10^{-1}	3.064510×10^{-1}	2.902015×10^{-1}	1.423012×10^{-2}	1.4060045×10^{-1}	8.87016×10^{-3}
$(0.2,0.2)$	2.5730993×10^{-1}	3.014985×10^{-1}	2.859613×10^{-1}	1.602133×10^{-2}	1.0261597×10^{-1}	1.59991×10^{-3}
$(0.3,0.3)$	2.4081558×10^{-1}	2.945562×10^{-1}	2.796123×10^{-1}	2.469156×10^{-2}	7.303647×10^{-2}	2.07303×10^{-3}
$(0.4,0.4)$	2.1841221×10^{-1}	2.892110×10^{-1}	2.711755×10^{-1}	2.832266×10^{-2}	5.348672×10^{-2}	1.30046×10^{-3}
$(0.5,0.5)$	1.9161429×10^{-1}	2.926213×10^{-1}	2.595910×10^{-1}	1.647868×10^{-2}	4.417464×10^{-2}	4.83428×10^{-3}
$(0.6,0.6)$	1.6203911×10^{-1}	2.845532×10^{-1}	2.554674×10^{-1}	1.897809×10^{-2}	4.482817×10^{-2}	1.770842×10^{-2}
$(0.7,0.7)$	1.3132546×10^{-1}	2.715506×10^{-1}	2.523966×10^{-1}	8.256636×10^{-2}	5.563746×10^{-2}	3.967572×10^{-2}
$(0.8,0.8)$	1.0105840×10^{-1}	2.665001×10^{-1}	2.498872×10^{-1}	1.7318349×10^{-1}	7.820212×10^{-2}	7.470613×10^{-2}
$(0.9,0.9)$	7.269991×10^{-2}	2.592318×10^{-1}	2.375121×10^{-1}	2.8124502×10^{-1}	1.1648304×10^{-1}	1.2914853×10^{-1}
$\|E(x,t)\|_2$	6.4052209×10^{-1}	9.111215×10^{-1}	8.894447×10^{-1}	3.4561537×10^{-1}	3.1421215×10^{-1}	1.5686941×10^{-1}
$\|E(x,t)\|_\infty$	2.6755733×10^{-1}	3.104216×10^{-1}	3.002515×10^{-1}	2.8124502×10^{-1}	1.8301704×10^{-1}	1.2914853×10^{-1}

Table 3. Absolute errors for Example 6.3.

Nodes	BCM	SAM	RHF	Present method		
				$M=2$ $N=6$	$M=2$ $N=4$	$M=3$ $N=5$
(x,t)	$E_{3,3}$	$E_{18,18}$	$E_{4,4}$	$E_{6,2}$	$E_{4,2}$	$E_{5,3}$
$(0,0)$	9.7520627×10^{-1}	8.211116×10^{-1}	5.412113×10^{-1}	1.245482×10^{-2}	2.777764×10^{-2}	2.425156×10^{-2}
$(0.1,0.1)$	9.1831788×10^{-1}	7.982315×10^{-1}	5.125010×10^{-1}	2.49766×10^{-3}	7.96351×10^{-3}	2.776649×10^{-2}
$(0.2,0.2)$	8.3312768×10^{-1}	7.900254×10^{-1}	4.321851×10^{-1}	6.06823×10^{-3}	1.280602×10^{-2}	6.286305×10^{-2}
$(0.3,0.3)$	7.2818378×10^{-1}	7.861215×10^{-1}	6.112349×10^{-1}	4.90589×10^{-3}	2.595216×10^{-2}	8.588845×10^{-2}
$(0.4,0.4)$	6.1214343×10^{-1}	7.115298×10^{-1}	4.231546×10^{-1}	1.422899×10^{-2}	2.297013×10^{-2}	9.615989×10^{-2}
$(0.5,0.5)$	4.9316703×10^{-1}	6.443223×10^{-1}	3.936684×10^{-1}	5.932749×10^{-2}	4.43112×10^{-3}	8.894710×10^{-2}
$(0.6,0.6)$	3.7843036×10^{-1}	5.865597×10^{-1}	3.495821×10^{-1}	1.3783869×10^{-1}	6.407226×10^{-2}	5.695801×10^{-2}
$(0.7,0.7)$	2.7375490×10^{-1}	4.221917×10^{-1}	3.315560×10^{-1}	2.5621643×10^{-1}	1.6289229×10^{-1}	8.17563×10^{-3}
$(0.8,0.8)$	1.8335642×10^{-1}	3.895517×10^{-1}	3.175123×10^{-1}	4.1926478×10^{-1}	3.0634507×10^{-1}	1.1441260×10^{-1}
$(0.9,0.9)$	1.0971175×10^{-1}	2.908871×10^{-1}	3.098841×10^{-1}	6.2924593×10^{-1}	4.9756594×10^{-1}	2.6781650×10^{-1}
$\Vert E(x,t)\Vert_2$	1.9749504	2.057574	1.370756	8.1260336×10^{-1}	6.1178272×10^{-1}	3.4347196×10^{-1}
$\Vert E(x,t)\Vert_\infty$	9.7520627×10^{-1}	8.211116×10^{-1}	6.112349×10^{-1}	6.2924593×10^{-1}	4.9756594×10^{-1}	2.6781650×10^{-1}