International Journal of
Mathematical Modelling & Computations
Vol. 11, No. 01, Winter 2021, 29- 47

The Lindley-Lindley Distribution: Characterizations, Copulas, Properties, Bayesian and Non-Bayesian Estimations

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Abstract. A new continuous distribution called Lindley-Lindley distribution is defined and studied. Relevant mathematical properties are derived. We present three characterizations of the new distribution based on the truncated moments of certain functions of the random variable; the hazard function and in terms of the conditional expectation of a function of the random variable. Some new bivariate type distributions using Farlie Gumbel Morgenstern copula, modified Farlie Gumbel Morgenstern copula and Clayton copula are introduced. The main justification of this paper is to show how different frequentist estimators of the new model perform for different sample sizes and different parameter values and to provide a guideline for choosing the best estimation method for the parameters of the proposed model. The unknown parameters of the new distribution are estimated using the maximum likelihood, ordinary least squares, Cramer-Von-Mises, weighted least squares and Bayesian methods. The obtained estimators are compared using Markov Chain Monte Carlo simulations and observed that Bayesian estimators are generally more efficient than the other estimators.

Received: 03 October 2020, Revised: 25 October 2020, Accepted: 16 November 2020.

Keywords: Different methods of estimations; Markov chain Monte Carlo simulations; Bayesian estimation; Cramer-Von-Mises; Lindley distribution; Copula, Characterizations.

AMS Subject Classification: 62N01, 62N02, 62E10.

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1. Introduction

The probability density function (PDF) and the cumulative distribution function (CDF) of the Lindley (Li) distribution are given by

$$g_{\xi_1}(x)|_{(x>0 \text{ and } \xi_1>0)} = \frac{\xi_1^2}{1+\xi_1}(1+x)\exp(-\xi_1 x), \tag{1}$$

and

$$G_{\xi_1}(x) = 1 - \frac{1+\xi_1+\xi_1 x}{1+\xi_1}\exp(-\xi_1 x), \tag{2}$$

respectively (see [28]). The shape parameter ξ_1 can result in either a unimodal or monotone decreasing (i.e., consistently decreasing) model. Because it decreases exponentially for large values of x, the corresponding Li model has a thin right tail. Hence, it is one way to describe the lifetime of a process or device. It can also be used in a wide variety of fields such as engineering, reliability, physics, biology and medicine. Among the most famous references, [10] discussed various properties of the Li model, [12] obtained size-biased and zero-truncated version of Poisson-Lindley distribution with various properties and applications and [11] has shown that the Li model is especially useful for modeling in mortality studies.

The goal of this article is to study the Lindley-Lindley (LiLi) distribution, with a focus on its applicability in a statistical scenario. First, the new distribution is derived from the odd Lindley-G (OLi-G) family of distributions introduced in [39]. Specifically, the PDF of the OL-G family of distributions is given by

$$f_{\xi_2,\underline{\mathbf{V}}}(x)|_{(x\in\mathbb{R} \text{ and } \xi_2>0)} = \frac{\xi_2^2}{(1+\xi_2)}\frac{g_{\underline{\mathbf{V}}}(x)\exp\left[-\xi_2\mathbf{O}_{\underline{\mathbf{V}}}(x)\right]}{\overline{G}_{\underline{\mathbf{V}}}(x)^3}, \tag{3}$$

where

$$\mathbf{O}_{\underline{\mathbf{V}}}(x) = \frac{G_{\underline{\mathbf{V}}}(x)}{\overline{G}_{\underline{\mathbf{V}}}(x)},$$

the function $G_{\underline{\mathbf{V}}}(x)$ refers to the CDF of the baseline model, $\overline{G}_{\underline{\mathbf{V}}}(x) = 1 - G_{\underline{\mathbf{V}}}(x)$ is the corresponding survival function (SF), $g_{\underline{\mathbf{V}}}(x) = dG_{\underline{\mathbf{V}}}(x)/dx$ is the PDF of the baseline distribution and $\underline{\mathbf{V}}$ refers to the parameter vector of the baseline model. Also, the CDF of the OL-G family is

$$F_{\xi_2,\underline{\mathbf{V}}}(x)|_{(x\in\mathbb{R} \text{ and } \xi_2>0)} = 1 - \frac{\xi_2+\overline{G}_{\underline{\mathbf{V}}}(x)}{(1+\xi_2)\,\overline{G}_{\underline{\mathbf{V}}}(x)}\exp\left[-\xi_2\mathbf{O}_{\underline{\mathbf{V}}}(x)\right]. \tag{4}$$

To this end, we use (1), (2), (3) and (4) to obtain the LiLi CDF and PDF as

$$F_{\xi_1,\xi_2}(x)|_{(x>0)} = 1 - \frac{\xi_2+\exp(-\xi_1 x)\frac{\mathbf{\Delta}_{\xi_1}(x)}{1+\xi_1}}{(1+\xi_2)\exp(-\xi_1 x)\frac{\mathbf{\Delta}_{\xi_1}(x)}{1+\xi_1}}\varrho_{\xi_1,\xi_2}(x), \tag{5}$$

where $\varrho_{\xi_1,\xi_2}(x) = \exp\left\{-\xi_2\left[\frac{(1+\xi_1)\exp(\xi_1 x)}{\mathbf{\Delta}_{\xi_1}(x)} - 1\right]\right\}$, $\mathbf{\Delta}_{\xi_1}(x) = 1 + \xi_1 + \xi_1 x$ and

$$f_{\xi_1,\xi_2}(x)|_{(x>0)} = \frac{\xi_1^2\xi_2^2(1+x)(1+\xi_1)^2}{(1+\xi_2)\,\mathbf{\Delta}_{\xi_1}^3(x)\exp(-2\xi_1 x)}\varrho_{\xi_1,\xi_2}(x), \tag{6}$$

respectively. The discrete analogue of (5) could be derived, studied and used for modeling the real count data sets in a separate article. For $\xi_2 = 1$, the LiLi distribution is reduced to the one parameter double Li distribution proposed in [21]. The hazard rate functions (HRF) can be derived from $h_{\xi_1,\xi_2}(x) = f_{\xi_1,\xi_2}(x)/\left[1 - F_{\xi_1,\xi_2}(x)\right]$. The PDF in (6) can be easily expressed as

$$f_{\xi_1,\xi_2}(x) = \sum_{\hbar,\kappa=0}^{\infty} c_{\hbar,\kappa} h_{\kappa^{\cdot},\xi_1}(x)\,|_{(\kappa^{\cdot}=(1+\hbar+\kappa))}, \tag{7}$$

where

$$c_{\hbar,\kappa} = \frac{(-1)^{\kappa}\,\xi_2^{2+\kappa}\Gamma(\kappa^{\cdot}+2)}{\kappa^{\cdot}\hbar!\kappa!\Gamma(\kappa+3)(1+\xi_2)},$$

$h_{\kappa^{\cdot},\xi_1}(x)$ is the PDF of exponentiated Li (Exp-Li) model with positive parameters $\kappa^{\cdot}$ and ξ_1, and $\Gamma(x)$denotes the standard gamma function (see [39]). We can use Equation (7) to derive some properties of the LiLi distribution such as ordinary moments, incomplete moments, means residual life and moments of residual life. The corresponding CDF can be given by integrating (7) as

$$F_{\xi_1,\xi_2}(x) = \sum_{\hbar,\kappa=0}^{\infty} c_{\hbar,\kappa} H_{\kappa^{\cdot},\xi_1}(x), \tag{8}$$

where $H_{\kappa^{\cdot},\xi_1}(x)$ is the CDF of Exp-Li model with positive parameters $\kappa^{\cdot}$ and ξ_1. Figure 1 gives some plots of the LiLi PDF for different values of ξ_1 and ξ_2. These plots show that the new PDF can be "unimodal with left skewness" and "symmetric" shapes. Figure 2 gives some plots of the LiLi HRF for different values of ξ_1 and ξ_2. These plots show that the HRF of the LiLi distribution can be "increasing" and "**J**-HRF".

In the literature, certain generalizations of the Li distribution are proposed and studied, see [10], [7], [33], [11], [5], [31], [38], [3], [34], [30], [35], [6], [4], [1], [41], [22], [2], [24], [25] and [26], among others. The main motivation of the paper is to show how the different frequentist estimators of the LiLi distribution perform for different sample sizes and different parameter values and to provide a guideline in choosing the best estimation method for the LiLi model. The unknown parameters of the LiLi distribution are estimated using the maximum likelihood (ML) method, ordinary least squares (OLS) method, weighted least squares (WLS) method, Cramer-Von-Mises (CVM) method and Bayesian method. The obtained estimators are compared using Markov Chain Monte Carlo (MCMC) simulations and we will observe that the Bayesian estimators are more efficient compared to other estimators.

2. Characterizations of the LiLi distribution

To understand the behavior of certain data obtained through a given process, we need to be able to describe this behavior via its approximate distribution. This,

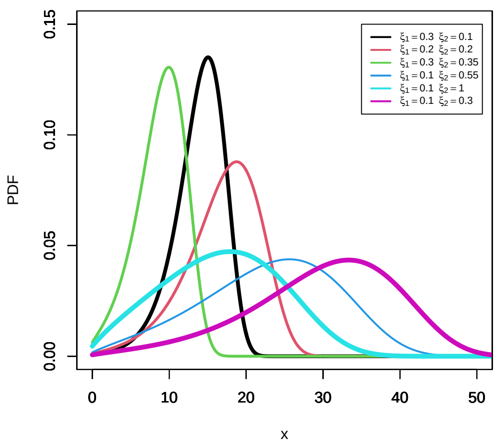

Figure 1. Plots of the PDF of the the LiLi model.

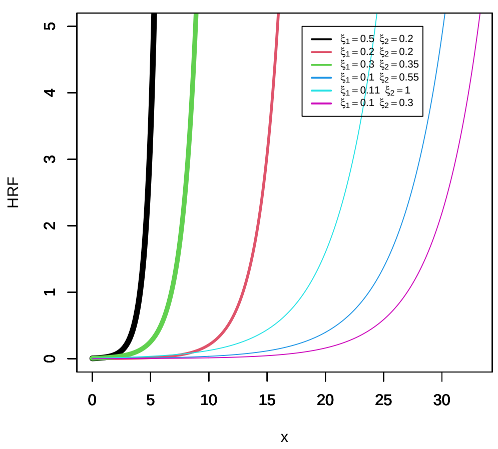

Figure 2. Plots of the HRF of the the LiLi model.

however, requires establishing conditions which govern the required distribution. In other words, we need to have certain conditions under which we may be able to recover the distribution of the data. So, characterization of a probability distribution is important in applied sciences, where an investigator is vitally interested in finding out if their model follows the selected model. Therefore, the investigator relies on conditions under which their model would follow a specified distribution. A probability model can be characterized in many directions, one of which is based on the truncated moments (see [9], [27], [15], [13], [16] and [23]). For example, [23] proposed a credibility theory based on the truncation of the loss data to estimate conditional mean loss for a given risk function. It should also be mentioned that characterization results are mathematically challenging and elegant. In this section, we present three characterizations of the LiLi distribution based on: (i) truncated moments of certain functions of the random variable; (ii) the HRF and (iii) in terms of the conditional expectation of a function of the random variable.

2.1 *Characterizations based on two truncated moments*

This subsection is devoted to the characterizations of the LiLi distribution in terms of a simple relationship between two truncated moments. We will employ Theorem 1 of [13] given in the Appendix A. As shown in [14], this characterization is stable in the sense of weak convergence.

Proposition 2.1 *Let X is be a continuous random variable and let*

$$q_1(x)\mid_{(x>0)}= \frac{e^{-2\xi_1 x}}{1+x}\exp\left[\frac{\xi_2(1+\xi_1)e^{\xi_1 x}}{\boldsymbol{\Delta}_{\xi_1}(x)}\right]$$

and $q_2(x)\mid_{(x>0)}= q_1(x)\boldsymbol{\Delta}_{\xi_1}^{-1}(x)$. Then X has the PDF specified by (6) if and only if the function $\eta(x)$ defined in [13, Theorem 1] is of the following form:

$$\eta(x) = \frac{2}{3}\boldsymbol{\Delta}_{\xi_1}^{-1}(x)\mid_{(x>0)}.$$

Proof If X has the PDF given as (6), then

$$[1-F_{\xi_1,\xi_2}(x)]\,\mathbb{E}\left[q_1(x)|X\geqslant x\right] = \frac{\xi_1\xi_2^2(1+\xi_1)^2\exp(\xi_2)}{2(1+\xi_2)}\boldsymbol{\Delta}_{\xi_1}^{-2}(x)|_{(x>0)},$$

and

$$[1-F_{\xi_1,\xi_2}(x)]\,\mathbb{E}\left[q_2(x)|X\geqslant x\right] = \frac{\xi_1\xi_2^2(1+\xi_1)^2\exp(\xi_2)}{3(1+\xi_2)}\boldsymbol{\Delta}_{\xi_1}^{-3}(x)|_{(x>0)},$$

and hence

$$\eta(x) = \frac{2}{3}\boldsymbol{\Delta}_{\xi_1}^{-1}(x)\mid_{(x>0)}.$$

We also have

$$\eta(x)q_1(x) - q_2(x) = -\frac{1}{3}q_1(x)\boldsymbol{\Delta}_{\xi_1}^{-1}(x) < 0|_{(x>0)}.$$

Conversely, if $\eta(x)$ is of the above form, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)} = \frac{2\xi_1}{\boldsymbol{\Delta}_{\xi_1}(x)}|_{(x>0)},$$

and $s(x) = \log\left[\boldsymbol{\Delta}_{\xi_1}^{2}(x)\right]$. Now, according to [13, Theorem 1] (see Appendix), X has the PDF defined in (6). ■

Corollary 2.1 Suppose that X is a continuous random variable. Let $q_1(x)$ be as in Proposition 2.1. Then X has the PDF given as (6) if and only if there exist functions $q_2(x)$ and $\eta(x)$ defined in Theorem 1 for which the following first order differential equation holds:

$$\frac{\xi'(x)q_1(x)}{\xi(x)q_1(x)-q_2(x)} = \frac{2\xi_1}{\boldsymbol{\Delta}_{\xi_1}(x)}|_{(x>0)}.$$

Corollary 2.2 The differential equation in Corollary 2.1 has the following general solution:

$$\eta(x) = \mathbf{\Delta}_{\xi_1}^2(x) \left[-2\xi_1 \int_0^\infty \mathbf{\Delta}_{\xi_1}^{-1}(x) \left(q_1(x)\right)^{-1} q_2(x) dx + D \right],$$

where D is a constant. A set of functions satisfying the above differential equation is given in Proposition 2.1.1 with $D = 0$. Clearly, there are other triplets $(q_1(x), q_2(x), \eta(x))$ satisfying the conditions of [13, Theorem 1].

2.2 *Characterization based on HRF*

The HRF $h_F(x)$ associated to a twice differentiable CDF $F(x)$ with PDF denoted by $f(x)$ satisfies the following trivial differential equation:

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x).$$

The following proposition establishes a non-trivial characterization of the LiLi distribution based on the HRF.

Proposition 2.2 *Suppose that X is a continuous random variable. Then, X has the PDF in (6) if and only if its HRF $h_F(x)$ satisfies the following first order differential equation:*

$$h_F'(x) - \xi_1 h_F(x) = \xi_1^2 \xi_2^2 (1+\xi_1)^2 e^{\xi_1 x} \frac{d}{dx} \left\{ \frac{(1+x)\,\mathbf{\Delta}_{\xi_1}^{-2}(x)}{\xi_2 (1+\xi_1)\,\mathbf{\Delta}_{\xi_1}(x) \exp(-\xi_1 x)} \right\} |_{(x>0)} \, .$$

The proof is omitted.

2.3 *Characterizations based on the conditional expectation of a function of the random variable*

Reference [19] established the following proposition which can be used to characterize the LiLi distribution.

Proposition 2.3 *Suppose that $X : \Omega \to (a, b)$ is a continuous random variable with CDF given as $F(x)$. If $\psi(x)$ is a differentiable function on (a, b) with* $\lim_{x\to a^+} \psi(x) = 1$. *Then, for $\delta \neq 1$, we have*

$$\mathbb{E}\left[\psi(x) | X \geqslant x\right] = \delta \psi(x)|_{(x \in (a,b))},$$

if and only if

$$\psi(x) = \left[1 - F(x)\right]^{1/\delta - 1} |_{(x \in (a,b))}.$$

Remark 2.1 Now, let $(a, b) = (0, \infty)$,

$$\psi(x) = \left[\frac{\xi_2 (1+\xi_1)\,\mathbf{\Delta}_{\xi_1}(x) \exp(-\xi_1 x)}{(1+\xi_2)\,\mathbf{\Delta}_{\xi_1}(x) \exp(-\xi_1 x)} \right]^{1/\xi_2} \exp\left\{ - \left[\frac{(1+\xi_1)\, e^{\xi_1 x}}{\mathbf{\Delta}_{\xi_1}(x)} - 1 \right] \right\}$$

and $\delta = \xi_2/(\xi_2+1)$, then Proposition 2.3 presents a characterization of the LiLi distribution. Clearly, there are other suitable functions than the one we employed for simplicity.

3. Copulas

In this section, we derive some new bivariate type LiLi (Biv-LiLi) distributions using Farlie Gumbel Morgenstern (FGM) copula (see [32], [8], [18] and [17]), modified FGM copula (see [37]) and Clayton copula. The multivariate LiLi (MvLiLi) type distribution is also presented. However, future work may be allocated to the study of these new distributions.

3.1 *Biv-LiLi type via FGM copula*

First, the FGM copula is defined by

$$\mathcal{H}_{\boldsymbol{\Delta}}(\mathbf{u},\mathbf{v}) = \mathbf{uv}\left(1+\boldsymbol{\Delta}\bar{\mathbf{u}}\bar{\mathbf{v}}\right)|_{\bar{\mathbf{u}}=1-\mathbf{u}},$$

with the marginal functions $\mathbf{u} = F_{\xi_1,\xi_2}(x_1)$, $\mathbf{v} = F_{\xi_3,\xi_4}(x_2)$, and $\boldsymbol{\Delta} \in (-1,1)$ is a dependence parameter. For every $\mathbf{u},\mathbf{v} \in (0,1)$, note that $\mathcal{H}_\Delta(\mathbf{u},0) = \mathcal{H}_\Delta(0,\mathbf{v}) = 0$ which is "grounded minimum" and $\mathcal{H}_\Delta(\mathbf{u},1) = \mathbf{u}$ and $\mathcal{H}_\Delta(1,\mathbf{v}) = \mathbf{v}$ which is "grounded maximum", $\mathcal{H}_\Delta(\mathbf{u}_1,\mathbf{v}_1)+\mathcal{H}_\Delta(\mathbf{u}_2,\mathbf{v}_2)-\mathcal{H}_\Delta(\mathbf{u}_1,\mathbf{v}_2)-\mathcal{H}_\Delta(\mathbf{u}_2,\mathbf{v}_1) \geqslant 0$.

3.2 *Biv-LiLi type via modified FGM copula*

The modified FGM copula is defined as

$$\mathcal{H}_{\boldsymbol{\Delta}}(\mathbf{u},\mathbf{v}) = \mathbf{uv}\left[1+\boldsymbol{\Delta}\boldsymbol{\Upsilon}(\mathbf{u})\boldsymbol{\Theta}(\mathbf{v})\right]|_{\boldsymbol{\Delta}\in(-1,1)}$$

or equivalently: $\mathcal{H}_{\boldsymbol{\Delta}}(\mathbf{u},\mathbf{v}) = \mathbf{uv} + \boldsymbol{\Delta}\widetilde{\boldsymbol{\Upsilon}}_{\mathbf{u}}\widetilde{\boldsymbol{\Theta}}_{\mathbf{v}}|_{\boldsymbol{\Delta}\in(-1,1)}$, where $\widetilde{\boldsymbol{\Upsilon}}_{\mathbf{u}} = \mathbf{u}\boldsymbol{\Upsilon}(\mathbf{u})$, and $\widetilde{\boldsymbol{\Theta}}_{\mathbf{v}} = \mathbf{v}\boldsymbol{\Theta}(\mathbf{v})$ and $\boldsymbol{\Upsilon}(\mathbf{u})$ and $\boldsymbol{\Theta}(\mathbf{v})$ are two continuous functions on $(0,1)$ with $\boldsymbol{\Upsilon}(0) = \boldsymbol{\Upsilon}(1) = \boldsymbol{\Theta}(0) = \boldsymbol{\Theta}(1) = 0$. Let

$$a_1 = \inf\left\{\widetilde{\boldsymbol{\Upsilon}}_{\mathbf{u}} : \frac{\partial}{\partial\mathbf{u}}\widetilde{\boldsymbol{\Upsilon}}_{\mathbf{u}}|_{\sigma_1}\right\} < 0, \quad a_2 = \sup\left\{\widetilde{\boldsymbol{\Upsilon}}_{\mathbf{u}} : \frac{\partial}{\partial\mathbf{u}}\widetilde{\boldsymbol{\Upsilon}}_{\mathbf{u}}|_{\sigma_1}\right\} < 0,$$

$$b_1 = \inf\left\{\widetilde{\boldsymbol{\Theta}}_{\mathbf{v}} : \frac{\partial}{\partial\mathbf{v}}\widetilde{\boldsymbol{\Theta}}_{\mathbf{v}}|_{\sigma_2}\right\} > 0, \quad b_2 = \sup\left\{\widetilde{\boldsymbol{\Theta}}_{\mathbf{v}} : \frac{\partial}{\partial\mathbf{v}}\widetilde{\boldsymbol{\Theta}}_{\mathbf{v}}|_{\sigma_2}\right\} > 0,$$

where $\mathbf{u}[\partial\boldsymbol{\Upsilon}(\mathbf{u})/\partial\mathbf{u}] = \partial\widetilde{\boldsymbol{\Upsilon}}_{\mathbf{u}}/\partial\mathbf{u} - \boldsymbol{\Upsilon}(\mathbf{u})$, $\sigma_1 = \left\{\mathbf{u} : \mathbf{u} \in (0,1) \,|\, \partial\widetilde{\boldsymbol{\Upsilon}}_{\mathbf{u}}/\partial\mathbf{u} \quad \text{exists}\right\}$ and $\sigma_2 = \left\{\mathbf{v} : \mathbf{v} \in (0,1) \,|\, \frac{\partial}{\partial\mathbf{v}}\widetilde{\boldsymbol{\Theta}}_{\mathbf{v}} \quad \text{exists}\right\}$. Then, we have $1 \leqslant \min(a_1a_2, b_1b_2) < \infty$. Based on this scheme, several types of Biv-LiLi distributions are listed below.

Biv-LiLi (Type-I): Let $\widetilde{\boldsymbol{\Upsilon}}_{\mathbf{u}} = \mathbf{u}(1-\mathbf{u})$ and $\widetilde{\boldsymbol{\Theta}}_{\mathbf{v}} = \mathbf{v}(1-\mathbf{v})$. Then, the Biv-LiLi-FGM (Type-I) can be derived from

$$\mathcal{H}_{\boldsymbol{\Delta}}(\mathbf{u},\mathbf{v}) = \mathbf{uv} + \boldsymbol{\Delta}\widetilde{\boldsymbol{\Upsilon}}_{\mathbf{u}}\widetilde{\boldsymbol{\Theta}}_{\mathbf{v}}|_{\boldsymbol{\Delta}\in(-1,1)}.$$

Biv-LiLi (Type-II): Let $\boldsymbol{\Upsilon}(\mathbf{u})^{\cdot}|_{(\boldsymbol{\Delta}_1\in(0,1))} = \mathbf{u}^{\boldsymbol{\Delta}_1}(1-\mathbf{u})^{1-\boldsymbol{\Delta}_1}$ and $\boldsymbol{\Theta}(\mathbf{v})^{\cdot}|_{(\boldsymbol{\Delta}_2\in(0,1))} = \mathbf{v}^{\boldsymbol{\Delta}_2}(1-\mathbf{v})^{1-\boldsymbol{\Delta}_2}$. Then, the corresponding Biv-LiLi-FGM (Type-II) can be derived from

$$\mathcal{H}_{\boldsymbol{\Delta},\boldsymbol{\Delta}_1,\boldsymbol{\Delta}_2}(\mathbf{u},\mathbf{v}) = \mathbf{uv}\left[1+\boldsymbol{\Delta}\boldsymbol{\Upsilon}(\mathbf{u})^{\cdot}\,\boldsymbol{\Theta}(\mathbf{v})^{\cdot}\right].$$

Biv-LiLi (Type-III): Let $\widetilde{\boldsymbol{\Upsilon}^{\cdot}(\mathbf{u})} = \mathbf{u}\log(1+\bar{\mathbf{u}})|_{\bar{\mathbf{u}}=1-\mathbf{u}}$ and $\widetilde{\boldsymbol{\Theta}^{\cdot}(\mathbf{v})} = \mathbf{v}\log(1+\bar{\mathbf{v}})|_{\bar{\mathbf{v}}=1-\mathbf{v}}$. In this case, one can also derive a closed form expression for the associated CDF of the Biv-LiLi-FGM (Type-III) from

$$\mathcal{H}_{\boldsymbol{\Delta}}(\mathbf{u},\mathbf{v}) = \mathbf{uv}\left(1+\boldsymbol{\Delta}\widetilde{\boldsymbol{\Upsilon}^{\cdot}(\mathbf{u})}\widetilde{\boldsymbol{\Theta}^{\cdot}(\mathbf{v})}\right).$$

Biv-LiLi (Type-IV): The CDF of the Biv-LiLi-FGM (Type-IV) distribution can be derived from

$$\mathcal{H}(\mathbf{u},\mathbf{v}) = \mathbf{u}F^{-1}_{\xi_3,\xi_4}(\mathbf{v}) + \mathbf{v}F^{-1}_{\xi_1,\xi_2}(\mathbf{u}) - F^{-1}_{\xi_1,\xi_2}(\mathbf{u})F^{-1}_{\xi_3,\xi_4}(\mathbf{v}),$$

where $F^{-1}_{\xi_1,\xi_2}(\mathbf{u})$ and $F^{-1}_{\xi_3,\xi_4}(\mathbf{v})$ denotes the quantile function of the LiLi distribution with parameters ξ_1 and ξ_2, and ξ_3 and ξ_4, respectively. We can cite [36] for more information on this type.

3.3 *Biv-LiLi and Mv-LiLi type via Clayton copula*

The Clayton copula can be considered as

$$\mathcal{H}(\mathbf{v}_1,\mathbf{v}_2) = \left[\mathbf{v}_1^{-\boldsymbol{\Delta}} + \mathbf{v}_2^{-\boldsymbol{\Delta}} - 1\right]^{-\boldsymbol{\Delta}^{-1}}|_{\boldsymbol{\Delta}\in(0,\infty)}.$$

Setting $\mathbf{v}_1 = F_{\xi_1,\xi_2}(x_1)$ and $\mathbf{v}_2 = F_{\xi_3,\xi_4}(x_2)$, the Biv-LiLi type can be derived from $\mathcal{H}(\mathbf{v}_1,\mathbf{v}_2) = \mathcal{H}(F_{\xi_1,\xi_2}(x_1), F_{\xi_3,\xi_4}(x_2))$. Similarly, the Mv-LiLi (m -dimensional extension) from the above can be derived from

$$\mathcal{H}(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_m) = \left(\sum_{\hbar=1}^{m}\mathbf{v}_{\hbar}^{-\boldsymbol{\Delta}} + 1 - m\right)^{-\boldsymbol{\Delta}^{-1}}.$$

4. Mathematical properties

Let X be a random variable with the LiLi distribution. The r^{th} ordinary moment of X is given by $\mu'_r = \mathbb{E}(X^r) = \int_0^\infty x^r f_{\xi_1,\xi_2}(x)dx$. Using (7), we obtain

$$\mu'_r = \sum_{\hbar,\kappa=0}^{\infty} c_{\hbar,\kappa}\frac{\kappa^{\cdot}\xi_1^2}{1+\xi_1}\mathbb{K}(\kappa^{\cdot},\xi_1,r,\xi_1)$$

where

$$\mathbb{K}(a,b,r,\delta) = \int_0^\infty x^r(1+x)\left[1-\frac{\boldsymbol{\Delta}_b(x)}{1+b}\exp(-bx)\right]^{a-1}\exp(-\delta x)\,dx,$$

which can be expressed as

$$\mathbb{K}(a,b,r,\delta)=\sum_{w=0}^{\infty}\sum_{j=0}^{w}\sum_{m=0}^{j+1}\varsigma_{w,j,m}^{(r,a,b,\delta)}\Gamma(1+r+m),$$

where

$$\varsigma_{w,j,m}^{(r,a,b,\delta)}=(-1)^{w}\,b^{j}\,(1+b)^{-w}\,(bw+\delta)^{-(1+r+m)}\binom{a-1}{w}\binom{w}{j}\binom{j+1}{m}.$$

The s^{th} incomplete moment of X, say $\varphi_{s,X}(t)$, is given by $\varphi_{s,X}(t)=\int_0^t x^s f_{\xi_1,\xi_2}(x)dx$. Using Equation (7), we obtain

$$\varphi_{s,X}(t)=\sum_{\hbar,\kappa=0}^{\infty}c_{\hbar,\kappa}\frac{\kappa^{\cdot}\xi_1^2}{1+\xi_1}\left[\mathbb{K}(\kappa^{\cdot},\xi_1,s,\xi_1)-\mathbb{L}(\kappa^{\cdot},\xi_1,s,\xi_1,t)\right],$$

where

$$\mathbb{L}(a,b,s,\delta,t)=\int_t^{\infty}x^s(1+x)\left[1-\frac{\mathbf{\Delta}_b(x)}{1+b}\exp(-bx)\right]^{a-1}\exp(-\delta x)\,dx,$$

then

$$\mathbb{L}(a,b,s,\delta,t)=\sum_{w=0}^{\infty}\sum_{j=0}^{w}\sum_{m=0}^{j+1}\varsigma_{w,j,m}^{(r,a,b,\delta)}\Gamma(1+s+m,(bw+\delta)\,t)$$

and $\Gamma(\mathbf{\Delta},z)\,|_{(\mathbf{\Delta}>0\text{ and }z>0)}=\int_z^{\infty}t^{\mathbf{\Delta}-1}\exp(-t)\,dt$ denotes the complementary incomplete gamma function. The n^{th} moment of the residual life is given by $z_{n,X}(t)=\mathbb{E}\left[(X-t)^n|_{(X>t)}\right]$, that is $z_{n,X}(t)=[1-F_{\xi_1,\xi_2}(t)]^{-1}\int_t^{\infty}(x-t)^n f_{\xi_1,\xi_2}(x)dx$. We can write

$$z_{n,X}(t)=[1-F_{\xi_1,\xi_2}(t)]^{-1}\sum_{\hbar,\kappa=0}^{\infty}\sum_{r=0}^{n}c_{\hbar,\kappa}(-t)^{n-r}\binom{n}{r}\int_t^{\infty}x^r h_{\kappa^{\cdot},\xi_1}(x)dx.$$

Then

$$z_{n,X}(t)=[1-F_{\xi_1,\xi_2}(t)]^{-1}\sum_{\hbar,\kappa=0}^{\infty}\sum_{r=0}^{n}c_{\hbar,\kappa}(-t)^{n-r}\binom{n}{r}\frac{\kappa^{\cdot}\xi_1^2}{1+\xi_1}\mathbb{L}(\kappa^{\cdot},\xi_1,r,\xi_1,t).$$

The mean residual life (MRL), or the life expectation at age t, of X can be obtained by setting $n=1$ in the last equation and it represents the expected additional life length for a unit which is alive at age t. The n^{th} moment of the reversed residual life is defined by $Z_{n,X}(t)=\mathbb{E}\left[(t-X)^n|_{(X\leqslant t)}\right]$, that is $Z_{n,X}(t)=F_{\xi_1,\xi_2}^{-1}(t)\int_0^t(t-x)^n f_{\xi_1,\xi_2}(x)dx$. So

$$Z_{n,X}(t)=F_{\xi_1,\xi_2}^{-1}(t)\sum_{\hbar,\kappa=0}^{\infty}\sum_{r=0}^{n}c_{\hbar,\kappa}(-1)^r\binom{n}{r}t^{n-r}\int_0^t x^r h_{\kappa^{\cdot},\xi_1}(x)dx.$$

Then,

$$Z_{n,X}(t) = F^{-1}_{\xi_1,\xi_2}(t) \sum_{\hbar,\kappa=0}^{\infty} \sum_{r=0}^{n} c_{\hbar,\kappa} (-1)^r \binom{n}{r} t^{n-r} \frac{\kappa^{\cdot} \xi_1^2}{1+\xi_1} \begin{bmatrix} \mathbb{K}(\kappa^{\cdot}, \xi_1, r, \xi_1) \\ -\mathbb{L}(\kappa^{\cdot}, \xi_1, r, \xi_1, t) \end{bmatrix}.$$

The mean inactivity time (MIT) also called the mean reversed residual life function represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$ and it is defined by $Z_*(t) = \mathbb{E}\left[(t-X)|_{(X\leqslant t)}\right]$. Hence, it can be obtained easily by setting $n=1$ in $Z_{n,X}(t)$.

5. Classical estimation

The estimation of the parameters of the LiLi model is the subject of this section.

5.1 *Maximum likelihood method*

Let $x_1, x_2, \cdots, x_n$ be an observed sample from the LiLi distribution with parameters ξ_1 and ξ_2 Then, the log-likelihood function, say $\ell = \ell(\xi_1, \xi_1)$, is given by

$$\begin{aligned} \ell = n \log\left[\frac{\xi_1^2 \xi_2^2 (1+\xi_1)^2}{(1+\xi_2)}\right] + \sum_{\hbar=1}^{n} \log(1+x_\hbar) \\ -3\sum_{\hbar=1}^{n} \log\left[\mathbf{\Delta}_{\xi_1}(x_\hbar)\right] + 2\xi_1 \sum_{\hbar=1}^{n} x_\hbar + \sum_{\hbar=1}^{n} \log\left[\varrho_{\xi_1,\xi_2}(x_\hbar)\right]. \end{aligned} \quad (9)$$

Equation (9) can be maximized either directly by using the `R` (`optim` function), `SAS` (`PROC NLMIXED`) or `Ox` program (sub-routine `MaxBFGS`) or by solving the nonlinear likelihood equations obtained by differentiating ℓ with respect to the parameters.

5.2 *Method of ordinary least square and weighted least square estimation*

The theory of OLS and WLS is based on the minimization of the sum of the square of differences of theoretical cumulative distribution function and empirical distribution function. Let $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$ be the n ordered values of $x_1, x_2, \ldots, x_n$. Then, the OLS estimates (OLSEs) are obtained upon minimizing

$$\mathbf{OLS}(\xi_1,\xi_2) = \sum_{\hbar=1}^{n}\left[F_{\xi_1,\xi_2}(x_{\hbar:n}) - \frac{\hbar}{1+n}\right]^2.$$

Using (5), we have

$$\mathbf{OLS}(\xi_1,\xi_2) = \sum_{\hbar=1}^{n}\left(1 - \frac{\xi_2 + \exp(-\xi_1 x_{\hbar:n})^{\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}}{(1+\xi_2)\exp(-\xi_1 x_{\hbar:n})^{\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}}\varrho_{\xi_1,\xi_2}(x_{\hbar:n}) - \frac{\hbar}{1+n}\right)^2.$$

The OLSEs are obtained via solving the following non linear equations:

$$\sum_{\hbar=1}^{n}\left(1 - \frac{\xi_2 + \exp(-\xi_1 x_{\hbar:n})^{\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}}{(1+\xi_2)\exp(-\xi_1 x_{\hbar:n})^{\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}}\varrho_{\xi_1,\xi_2}(x_{\hbar:n}) - \frac{\hbar}{1+n}\right) d_{\xi_1}(x_{\hbar:n}) = 0$$

and

$$\sum_{\hbar=1}^{n}\left(1-\frac{\xi_2+\exp\left(-\xi_1 x_{\hbar:n}\right)\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}{(1+\xi_2)\exp\left(-\xi_1 x_{\hbar:n}\right)\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}\varrho_{\xi_1,\xi_2}(x_{\hbar:n})-\frac{\hbar}{1+n}\right)d_{\xi_2}(x_{\hbar:n})=0,$$

where $d_{\xi_1}(x)$ and $d_{\xi_2}(x)$ the values of first derivatives of the CDF of the LiLi distribution with respect to the parameters ξ_1 and ξ_2, respectively. The OLSEs of the parameters ξ_1 and ξ_2 are obtained by solving the above simultanenous equations by using any numerical approximation technique. The WLSEs are obtained by minimizing the given form of equation with respect to the parameters.

$$\mathbf{WLS}\left(\xi_1,\xi_2\right)=\sum_{\hbar=1}^{n} w_{\hbar}\left[F_{\xi_1,\xi_2}(x_{\hbar:n})-\frac{\hbar}{1+n}\right]^2.$$

So, the WLSEs of the parameters are obtained by solving the following non-linear equations:

$$\sum_{\hbar=1}^{n} w_{\hbar}\left(1-\frac{\xi_2+\exp\left(-\xi_1 x_{\hbar:n}\right)\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}{(1+\xi_2)\exp\left(-\xi_1 x_{\hbar:n}\right)\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}\varrho_{\xi_1,\xi_2}(x_{\hbar:n})-\frac{\hbar}{1+n}\right)d_{\xi_1}(x_{\hbar:n})=0$$

and

$$\sum_{\hbar=1}^{n} w_{\hbar}\left(1-\frac{\xi_2+\exp\left(-\xi_1 x_{\hbar:n}\right)\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}{(1+\xi_2)\exp\left(-\xi_1 x_{\hbar:n}\right)\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}\varrho_{\xi_1,\xi_2}(x_{\hbar:n})-\frac{\hbar}{1+n}\right)d_{\xi_2}(x_{\hbar:n})=0,$$

where $w_{\hbar}=(n+1)^2(n+2)/\left[\hbar(n-\hbar+1)\right]$.

5.3 *Method of Cramer-Von-Mises estimation*

The Cramer-Von-Mises estimates (CVMEs) of the parameters are based on the theory of minimum distance estimation. It was first proposed by [29] and justified that the bias of the corresponding estimators is smaller than the one of other minimum distance estimators. So, the CVMEs of the parameters ξ_1 and ξ_2 are obtained by minimizing the following expression with respect to the parameters ξ_1 and ξ_2:

$$\mathbf{CVM}(\xi_1,\xi_2)=\frac{1}{12n}+\sum_{\hbar=1}^{n}\left[F_{\xi_1,\xi_2}(x_{\hbar:n})-\frac{2\hbar-1}{2n}\right]^2.$$

From (5), we obtain

$$\mathbf{CVM}(\xi_1,\xi_2)=\sum_{\hbar=1}^{n}\left(1-\frac{\xi_2+\exp\left(-\xi_1 x_{\hbar:n}\right)\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}{(1+\xi_2)\exp\left(-\xi_1 x_{\hbar:n}\right)\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}\varrho_{\xi_1,\xi_2}(x_{\hbar:n})-\frac{2\hbar-1}{2n}\right)^2.$$

The CVMEs of the parameters are obtained by solving the following non-linear equations:

$$\sum_{\hbar=1}^{n}\left(1-\frac{\xi_2+\exp\left(-\xi_1 x_{\hbar:n}\right)\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}{\left(1+\xi_2\right)\exp\left(-\xi_1 x_{\hbar:n}\right)\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}\varrho_{\xi_1,\xi_2}(x_{\hbar:n})-\frac{2\hbar-1}{2n}\right)d_{\xi_1}(x_{\hbar:n})=0$$

and

$$\sum_{\hbar=1}^{n}\left(1-\frac{\xi_2+\exp\left(-\xi_1 x_{\hbar:n}\right)\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}{\left(1+\xi_2\right)\exp\left(-\xi_1 x_{\hbar:n}\right)\frac{\mathbf{\Delta}_{\xi_1}(x_{\hbar:n})}{1+\xi_1}}\varrho_{\xi_1,\xi_2}(x_{\hbar:n})-\frac{2\hbar-1}{2n}\right)d_{\xi_2}(x_{\hbar:n})=0.$$

6. Bayesian estimation

In this section, we use Bayesian procedures to construct the estimators of the unknown parameters of the LiLi model. There are many situations where maximum likelihood estimator does not converge, especially with higher dimension models. In such cases, the use of Bayesian methods is sought. At first sight, Bayesian methods seem to be very complex as the estimators involve intractable integrals. However, the advanced MCMC techniques make it possible to apply Bayesian methods even in higher dimension models. Under Bayesian estimation, we update the likelihoods with prior knowledge to explore the posterior probabilities of the parameters. Here we assume that the gamma priors (Gam-Ps) for the parameters ξ_1 and ξ_2 are of the following forms:

$$\pi_1\left(\xi_1\right)\sim \mathrm{Gam}(\varsigma_1,\varsigma_2) \text{ and } \pi_2\left(\xi_2\right)\sim \mathrm{Gam}(\varsigma_3,\varsigma_4),$$

where $\mathrm{Gam}(\varsigma_1,\varsigma_2)$ stands for gamma distribution with shape parameter ς_1 and scale parameter ς_2. It is further assumed that the parameters are independently distributed. The joint prior distribution is given by

$$\pi(\xi_1,\xi_2)=\frac{\varsigma_2^{\varsigma_1}\varsigma_4^{\varsigma_3}}{\Gamma(\varsigma_1)\Gamma(\varsigma_3)}\xi_1^{\varsigma_1-1}\xi_2^{\varsigma_3-1}\exp\left[-\left(\xi_1\varsigma_2+\xi_2\varsigma_4\right)\right].$$

Also, the posterior distribution of the parameters is defined by

$$\pi\left(\xi_1,\xi_2|\underline{x}\right)\propto \text{likelihood}(\xi_1,\xi_2|\underline{x})\times\pi(\xi_1,\xi_2).$$

Under squared error loss, Bayesian estimators of parameters ξ_1 and ξ_2 are the means of their marginal posteriors and defined by

$$\widehat{\xi}_{i\text{Bayesian}}=\int_{\xi_i}\xi_i\pi\left(\xi_1,\xi_2|\underline{x}\right)d\xi_i \tag{10}$$

It is not easy to calculate Bayesian estimates through equation (10) so the numerical approximation techniques are needed. Therefore, we propose the use of MCMC techniques namely Gibbs sampler and Metropolis Hastings (MH) algorithm (see [20]). Since the conditional posteriors of the parameters can not be obtained in any standard forms, we therefore used a hybrid MCMC strategy for drawing samples from the joint posterior of the parameters. To implement the Gibbs algorithm, the full conditional posteriors of ξ_1 and ξ_2 are given by

$$\pi_1(\xi_1, \xi_2|\underline{x}) \propto \xi_1^{n+\varsigma_1-1}\exp\left[-\left(\xi_1\varsigma_2 + \xi_2\varsigma_4\right)\right]\prod_{\hbar=1}^{n}\Upsilon_\hbar,$$

and

$$\pi_2(\xi_1, \xi_2|\underline{x}) \propto \xi_2^{n+\varsigma_3-1}\exp\left[-\left(\xi_1\varsigma_2 + \xi_2\varsigma_4\right)\right]\prod_{\hbar=1}^{n}\Upsilon_\hbar,$$

where

$$\Upsilon_\hbar = \frac{\frac{\xi_1^2(1+x_\hbar)}{1+\xi_1}\exp\left(-\xi_1 x_\hbar\right)}{2\left[\frac{\mathbf{\Delta}_{\xi_1}(x_\hbar)}{1+\xi_1}\exp\left(-\xi_1 x_\hbar\right)\right]^3}\varrho_{\xi_1,\xi_2}(x_\hbar)$$

The simulation algorithm is as follows:

(1) Provide initial values, say $\xi_{1(0)}$ and $\xi_{2(0)}$ then at the $\hbar^{\text{th}}$ stage,
(2) Using MH algorithm, generate $\xi_{1(\hbar)} \sim \pi_1\left(\xi_{1(\hbar-1)}|, \underline{x}\right)$ and $\xi_{2(\hbar)} \sim \pi_3\left(\xi_{2(\hbar-1)}|, \underline{x}\right)$,
(3) Repeat steps 2, $M(= 10000)$ times to get the samples of size M from the corresponding posteriors of interest.
(4) Obtain the Bayesian estimates of ξ_1 using the following formula:

$$\widehat{\xi}_{1\text{Bayesian}} = \frac{1}{M - M_0}\sum_{j=M_0+1}^{M}\xi_{1(j)}$$

and

$$\widehat{\xi}_{2\text{Bayesian}} = \frac{1}{M - M_0}\sum_{j=M_0+1}^{M}\xi_{2(j)},$$

(5) where $M_0 \approx 2000$ is the burn-in period of the generated Markov Chains.

7. Simulation study for comparing methods

A MCMC simulation study is conducted in this section to compare the performance of different classical estimators of the unknown parameters of the LiLi model with the Bayesian estimators. The performance of all estimation methods is evaluated based on their mean squared errors (MSEs). All the computations in this section are done by **Mathcad program Version 15.0**.

The following algorithm is used for all classical methods in this paper:

(1) We generate $N = 1000$ samples of sizes n =50, 100, 200, 300, 500 from the LiLi distribution using the initials $\mathbf{I} : \xi_{10} = 0.6, \xi_{20} = 5$.
(2) Compute the MLEs for the 1000 samples, say

$$\left[\widehat{\xi_{1i}}, \widehat{\xi_{2i}}\right]|_{(i=1,2,\ldots,1000)},$$

(3) Compute the average values (AVs) of the 1000 estimations for $\widehat{\xi_{1i}}$ and $\widehat{\xi_{2i}}$ for each sizes n = 50, 100, 200, 300, 500.
(4) Compute the MSEs by the following equations:

$$MSE_{\underline{\varepsilon}}(n) = \frac{1}{1000}\sum_{i=1}^{1000}\left(\widehat{\underline{\varepsilon}}_i - \underline{\varepsilon}\right)^2|_{\underline{\varepsilon}=\xi_1,\xi_2},$$

(5) Repeat 1-4 for the initials **II** : $\xi_{10} = 1.5, \xi_{20} = 0.8$.
(6) Repeat 1-4 for the initials **III** : $\xi_{10} = 1.5, \xi_{20} = 1.5$.

Whereas the algorithm related to the Bayesian method is mentioned at the end of Section 6. The AVS of the estimates and the MSEs of MLEs, LSEs, WLSEs, CVMEs and Bayesian estimators are obtained and reported in Tables 1, 2, 3 and 4. The Bayesian estimators of the parameters are evaluated with flexible Gam-Ps under the SELF by using the MCMC technique. The values of the hyperparameter are assumed to be known and chosen in such a way that the prior mean is equal to the true value, and prior variance is unity.

Table 1. AVs and the corresponding MSEs (in parentheses) for $n = 50$.

Parameters	MLE	Bayesian	LS	WLS	CVM
$\xi_1 = 0.6$	0.60419 (0.00270)	0.59531 (0.00190)	0.60212 (0.00352)	0.59938 (0.00281)	0.60158 (0.00363)
$\xi_2 = 5$	5.05688 (0.45377)	5.07663 (0.18495)	5.04743 (0.54350)	5.01224 (0.45995)	5.03813 (0.56242)
$\xi_1 = 1.5$	1.50618 (0.00773)	1.44863 (0.00801)	1.50215 (0.01184)	1.49900 (0.00880)	1.50398 (0.013386)
$\xi_2 = 0.8$	0.80585 (0.00786)	0.77175 (0.00748)	0.80532 (0.00998)	0.80201 (0.00869)	0.80522 (0.00993)
$\xi_1 = 1.5$	1.50959 (0.01154)	1.55207 (0.01081)	1.50343 (0.01625)	1.50353 (0.01407)	1.50776 (0.01884)
$\xi_2 = 1.5$	1.51383 (0.03152)	1.61419 (0.04832)	1.50955 (0.03561)	1.51084 (0.03718)	1.51321 (0.04151)

From Tables 1, 2, 3 and 4, we observe that all the estimates show the property of consistency, i.e., the MSEs decrease and approach 0 as sample size (n) increases. Also, the MSEs of the Bayesian estimators are generally smaller as compared to the rest of the estimators for all n = 50, 100, 200, 300 and 500.

8. Conclusions

A new continuous distribution called Lindley-Lindley distribution is defined and studied. The new model is derived based on the odd Lindley family of distributions

Table 2. AVs and the corresponding MSEs (in parentheses) for $n = 100$.

Parameters	MLE	Bayesian	LS	WLS	CVM
$\xi_1 = 0.6$	0.60206	0.58010	0.59955	0.60151	0.60124
	(0.00143)	(0.00135)	(0.00186)	(0.00161)	(0.00195)
$\xi_2 = 5$	5.03003	4.72237	5.00561	5.03035	5.02526
	(0.22097)	(0.15857)	(0.27048)	(0.24763)	(0.28280)
$\xi_1 = 1.5$	1.50390	1.44435	1.50099	1.50193	1.50374
	(0.00487)	(0.00579)	(0.00678)	(0.00523)	(0.00690)
$\xi_2 = 0.8$	0.80472	0.75979	0.80263	0.80327	0.80458
	(0.00422)	(0.00452)	(0.00500)	(0.00447)	(0.00515)
$\xi_1 = 1.5$	1.50294	1.44264	1.50347	1.50258	1.50229
	(0.00659)	(0.00705)	(0.00961)	(0.00757)	(0.00950)
$\xi_2 = 1.5$	1.50479	1.38945	1.50758	1.50626	1.50459
	(0.01651)	(0.02277)	(0.01987)	(0.01887)	(0.01958)

Table 3. AVs and the corresponding MSEs (in parentheses) for $n = 200$.

Parameters	MLE	Bayesian	LS	WLS	CVM
$\xi_1 = 0.6$	0.60121	0.60819	0.59857	0.60119	0.60138
	(0.00088)	(0.00059)	(0.00113)	(0.00094)	(0.00114)
$\xi_2 = 5$	5.02020	5.16237	4.98970	5.02299	5.02373
	(0.12106)	(0.13384)	(0.144851)	(0.13078)	(0.15160)
$\xi_1 = 1.5$	1.50023	1.49215	1.49702	1.50024	1.49970
	(0.00370)	(0.00140)	(0.00446)	(0.00385)	(0.00475)
$\xi_2 = 0.8$	0.80031	0.79210	0.79847	0.80065	0.80024
	(0.00255)	(0.00183)	(0.00278)	(0.00272)	(0.00305)
$\xi_1 = 1.5$	1.50161	1.51006	1.50015	1.49876	1.50066
	(0.00451)	(0.00218)	(0.00598)	(0.00454)	(0.00570)
$\xi_2 = 1.5$	1.50339	1.52161	1.50177	1.49916	1.50193
	(0.00957)	(0.00814)	(0.01118)	(0.00958)	(0.01052)

first introduced in [39]. Relevant mathematical properties are derived and analyzed. We present three characterizations of the new distribution based on: (i) truncated moments of certain functions of the random variable; (ii) the hazard function and

Table 4. AVs and the corresponding MSEs (in parentheses) for $n = 300$.

Parameters	MLE	Bayesian	LS	WLS	CVM
$\xi_1 = 0.6$	0.59952 (0.00068)	0.59341 (0.00039)	0.59998 (0.00087)	0.59964 (0.00071)	0.59944 (0.00087)
$\xi_2 = 5$	4.99710 (0.08492)	4.88079 (0.09592)	5.00512 (0.10694)	4.99974 (0.08989)	4.99799 (0.10815)
$\xi_1 = 1.5$	1.50029 (0.00311)	1.51469 (0.00108)	1.50013 (0.00369)	1.50085 (0.00322)	1.50035 (0.00377)
$\xi_2 = 0.8$	0.80095 (0.00181)	0.81472 (0.00139)	0.80152 (0.00205)	0.80167 (0.00192)	0.80108 (0.00212)
$\xi_1 = 1.5$	1.50223 (0.00371)	1.51121 (0.00145)	1.49887 (0.00467)	1.50133 (0.00376)	1.49873 (0.00464)
$\xi_2 = 1.5$	1.50369 (0.00698)	1.52968 (0.00566)	1.49978 (0.00805)	1.50354 (0.00711)	1.49910 (0.00798)

Table 5. AVs and the corresponding MSEs (in parentheses) for $n = 500$.

Parameters	MLE	Bayesian	LS	WLS	CVM
$\xi_1 = 0.6$	0.60001 (0.00057)	0.58772 (0.00035)	0.59965 (0.00066)	0.60022 (0.00058)	0.59932 (0.00066)
$\xi_2 = 5$	5.00208 (0.06285)	4.81844 (0.06092)	4.99991 (0.07414)	5.00563 (0.06534)	4.99506 (0.07337)
$\xi_1 = 1.5$	1.498203 (0.00275)	1.50713 (0.00058)	1.49854 (0.00309)	1.49905 (0.00279)	1.49890 (0.00308)
$\xi_2 = 0.8$	0.79910 (0.00130)	0.80397 (0.00078)	0.79953 (0.00146)	0.79989 (0.00136)	0.79991 (0.00145)
$\xi_1 = 1.5$	1.49945 (0.00308)	1.48623 (0.00101)	1.49875 (30.0073)	1.50045 (0.00316)	1.4982 (0.00372)
$\xi_2 = 1.5$	1.49979 (0.00493)	1.48023 (0.00293)	1.49918 (0.00578)	1.50138 (0.00523)	1.49777 (0.00577)

(iii) in terms of the conditional expectation of a function of the random variable. Some new bivariate type distributions obtained using Farlie Gumbel Morgenstern copula, modified Farlie Gumbel Morgenstern copula and Clayton copula. The main justification of this paper is to show how different frequentist estimators of the new model perform for different sample sizes and different parameter values and to provide a guideline for choosing the best estimation method for the proposed model. The unknown parameters of the new distribution are estimated using the maximum likelihood, ordinary least squares, Cramer-Von-Mises, weighted least squares and Bayesian methods. The obtained estimators are compared using Markov Chain Monte Carlo simulations and observed that Bayesian estimators are generally more efficient compared to the other estimators.

Acknowledgments

The authors are grateful to the anonymous reviewers for their suggestions and comments.

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Appendix

The following result was established by [14].

Theorem 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $a < b$, the infinite bounds being allowed. Let $X : \Omega \to H$ be a continuous random variable with the CDF $F(x)$ and let $q_1(x)$ and $q_2(x)$ be two real functions defined on H such that

$$\mathbb{E}\left[q_2(x)|X \geqslant x\right] = \mathbb{E}\left[q_1(x)|X \geqslant x\right]\eta(x), \quad x \in H,$$

is defined with some real function $\eta(x)$. Assume that $q_1(x), q_2(x) \in C^1(H)$, $\eta(x) \in C^2(H)$ and $F(x)$ is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta(x)q_1(x) = q_2(x)$ has no real solution in the interior of H. Then $F(x)$ is uniquely determined by the functions

$q_1(x), q_2(x)$ and $\eta(x)$, particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)q_1(u) - q_2(u)} \right| \exp\left(-s(u)\right) du,$$

where the function $s(u)$ is a solution of the differential equation $s'(u) = [\eta'(u)q_1(u)]/[\eta(u)q_1(u) - q_2(u)]$ and C is the distributional normalization constant.
Note: The goal is to have the function $\eta(x)$ as simple as possible.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see [14]), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with CDF $F(x)$ such that the functions $q_{1n}(x)$, $q_{2n}(x)$ and $\eta_n(x)$ $(n \in \mathbb{N})$ satisfy the conditions of Theorem 1 and let $q_{1n}(x) \to q_1(x)$, $q_{2n}(x) \to q_2(x)$ for some continuously differentiable real functions $q_1(x)$ and $q_2(x)$. Let, finally, X be a random variable with CDF $F(x)$. Under the condition that $q_{1n}(x)$ and $q_{2n}(x)$ are uniformly integrable and the family $F(x)$ is relatively compact, the sequence X_n converges to X in distribution if and only if $\eta_n(x)$ converges to $\eta(x)$, where

$$\eta(x) = \frac{\mathbb{E}\left[q_2(x)|X \geqslant x\right]}{\mathbb{E}\left[q_1(x)|X \geqslant x\right]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions $q_1(x)$, $q_2(x)$ and $\eta(x)$, respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lvy-Smirnov distribution if $\alpha \to \infty$.