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# Hermite-Hadamard Type Inequalities for $M_\varphi A$-Convex Functions

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**Abstract.** This article deals with the different classes of convexity and generalizations. Firstly, we reveal the new generalization of the definition of convexity that can reduce many order of convexity. We have showed features of algebra for this new convex function. Then after we have constituted Hermite-Hadamard type inequalities for this class of functions. Finally the identity has been revealed for its by us and by using this identity, then theorems and corollaries have been obtained.

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## 1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the

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mapping $f$. Both inequalities hold in the reversed direction if $f$ is concave. For some results which generalize, improve and extend the inequalities (1) we refer the reader to the recent papers (see [4, 8, 11, 12, 14, 15, 17, 18]).

For $r \in \mathbb{R}$ the power mean $M_r(a,b)$ of order $r$ of two positive numbers $a$ and $b$ is defined by

$$M_r = M_r(a,b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{1/r}, & r \neq 0 \\ \sqrt{ab}, & r = 0 \end{cases}.$$

It is well-known that $M_r(a,b)$ is continuous and strictly increasing with respect to $r \in \mathbb{R}$ for fixed $a,b>0$ with $a \neq b$.

Let $L = L(a,b) = (b-a)/(\ln b - \ln a)$, $I = I(a,b) = \frac{1}{e}\left(a^a/b^b\right)^{1/a-b}$, $A = A(a,b) = (a+b)/2$, $G = G(a,b) = \sqrt{ab}$ and $H = H(a,b) = 2ab/(a+b)$ be the logarithmic, identric, arithmetic, geometric, and harmonic means of two positive real numbers $a$ and $b$ with $a \neq b$, respectively. Then

$$\begin{aligned} \min\{a,b\} &< H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < L(a,b) \\ &< I(a,b) < A(a,b) = M_1(a,b) < \max\{a,b\}. \end{aligned}$$

Let $\mathfrak{M}$ be the family of all mean values of two numbers in $\mathbb{R}_+ = (0,\infty)$. Given $M,N \in \mathfrak{M}$, we say that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is $(M,N)$-convex if $f(M(x,y)) \leqslant N(f(x),f(y))$ for all $x,y \in \mathbb{R}_+$. The concept of $(M,N)$-convexity has been studied extensively in the literature from various points of view (see e.g. [2, 3, 5, 19]),

Let $A(a,b;t) = ta+(1-t)b$, $G(a,b;t) = a^t b^{1-t}$, $H(a,b;t) = ab/(ta+(1-t)b)$ and $M_p(a,b;t) = (ta^p+(1-t)b^p)^{1/p}$ be the weighted arithmetic, geometric, harmonic , power of order $p$ means of two positive real numbers $a$ and $b$ with $a \neq b$ for $t \in [0,1]$, respectively.

The most used class of means is quasi-arithmetic mean, which are associated to a continuous and strictly monotonic function $\varphi : I \to \mathbb{R}$ by the formula

$$M_\varphi(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \text{ for } x,y \in I.$$

Weighted quasi-arithmetic mean is given by the formula

$$M_\varphi(x,y;t) = \varphi^{-1}\left(t\varphi(x)+(1-t)\varphi(y)\right), \text{ for } x,y \in I,\ t \in [0,1].$$

Here $t \in (0,1)$ and $x<y$ always implies $x < M_\varphi(x,y;t) < y$. The function $\varphi$ is called *Kolmogoroff-Naguma function of* $M$. Of special interest are the power means $M_p$ on $\mathbb{R}_+$, defined by

$$\varphi_p(x) := \begin{cases} x^p, & p \neq 0 \\ \ln x, & p = 0 \end{cases}.$$

For $p=1$, we get the arithmetic mean $A = M_1$, for $p=0$, we get the geometric mean $G = M_0$ and for $p=-1$, we get the harmonic mean $H = M_{-1}$.

For any two quasi-arithmetic means $M,N$ ( with *Kolmogoroff-Naguma function* $\varphi,\psi$ defined on intervals $I,J$, respectively ), a function $f : I \to J$ can be called $(M_\varphi, M_\psi)$-convex if it satisfies

$$f(M_\varphi(x,y;t)) \leqslant M_\psi(f(x),f(y);t) \tag{2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (2), then $f$ is said to be $(M_\varphi, M_\psi)$-concave. If $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(x) = x$, (i.e., $M_\psi(f(x), f(y); t) = A(a, b; t)$ ), then we just say that $f$ is $M_\varphi A$-convex.

Let $f$ be a $M_\varphi A$-convex.

i) If we take $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$, then $M_\varphi A$-convexity deduce usual convexity.

ii) If we take $\varphi : I \subset (0, \infty) \to \mathbb{R}$, $\varphi(x) = \ln x$, then $M_\varphi A$-convexity deduce GA-convexity. (see [20, 21])

iii) If we take $\varphi : I \subset (0, \infty) \to \mathbb{R}$, $\varphi(x) = x^{-1}$, then $M_\varphi A$-convexity deduce Harmonically convexity. (see [13])

iv) If we take $\varphi : I \subset (0, \infty) \to \mathbb{R}$, $\varphi(x) = x^p$, $p \in \mathbb{R} \setminus \{0\}$, then $M_\varphi A$-convexity deduce $p$-convexity. (see [16]).

The theory of $(M_\varphi, M_\psi)$-convex functions can be deduced from the theory of usual convex functions.

**Lemma 1.1** *[1] If $\varphi$ and $\psi$ are two continuous and strictly monotonic functions (on intervals $I$ and $J$ respectively) and $\psi$ is increasing then a function $f : I \to J$ is $(M_\varphi, M_\psi)$-convex if and only if $\psi \circ f \circ \varphi^{-1}$ is convex on $\varphi(I)$ in the usual sense.*

There is a lot of works in this area. Lots of authors found out theorems and corollary about convex, $GA$-convex and $p$-convex functions as follows:

**Theorem 1.2** *[? ] Let $f : I^o \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^o$, $a, b \in I^o$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leqslant \frac{b-a}{8} \left[ |f'(a)| + |f'(b)| \right] \tag{3}$$

**Theorem 1.3** *[? ] Let $f : I^o \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^o$, $a, b \in I^o$ with $a < b$, and let $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then we have*

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \qquad (4) \\ &\leqslant \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{1/p} \left[ \left( |f'(a)|^{p/(p-1)} + 3 |f'(b)|^{p/(p-1)} \right)^{(p-1)/p} \right. \\ &\left. + \left( 3 |f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)} \right)^{(p-1)/p} \right] \end{aligned}$$

**Theorem 1.4** *[? ] Let $f : I^o \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^o$, $a, b \in I^o$ with $a < b$ and let $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leqslant \left(\frac{b-a}{4}\right) \left(\frac{4}{p+1}\right)^{1/p} \left[ |f'(a)| + |f'(b)| \right] \tag{5}$$

**Lemma 1.5** *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on $I^o$ and*

*$a,b\in I^o$ with $0<a<b$. If $f'\in L([a,b])$, then*

$$\frac{1}{lnb-lna}\int\limits_a^b \frac{f(x)}{x}dx - f\left(\sqrt{a}b\right) \tag{6}$$

$$= \frac{lnb-lna}{4}\int\limits_0^1 t\left[a^{t/2}b^{1-t/2}f'\left(a^{t/2}b^{1-t/2}\right) - a^{1-t/2}b^{t/2}f'\left(a^{1-t/2}b^{t/2}\right)\right]dt.$$

**Corollary 1.6** *[10] Let $f: I\subseteq\mathbb{R}_+\to\mathbb{R}$ be a differentiable mapping on $I^o$ and $f'\in L[a,b]$ with $a<b$. If $|f'|$ is GA-convex on $[a,b]$, then*

$$\left|f\left(\sqrt{a}b\right) - \frac{1}{lnb-lna}\int\limits_a^b \frac{f(x)}{x}dx\right| \leqslant ln\left(\frac{b}{a}\right)\left[C_1(1)\left|f'(a)\right| + C_2(1)\left|f'(b)\right|\right], \tag{7}$$

*where*

$$C_1(1) = \int\limits_0^{\frac{1}{2}} u\left[(1-u)(a^{1-u}b^u) + u(a^u b^{1-u})\right]du,$$

$$C_2(1) = \int\limits_0^{\frac{1}{2}} u\left[u(a^{1-u}b^u) + (1-u)(a^u b^{1-u})\right]du.$$

**Corollary 1.7** *[10] Let $f: I\subset\mathbb{R}_+\to\mathbb{R}$ be a differentiable mapping on $I^o$ and $f'\in L[a,b]$ with $a<b$. If $|f'|^q$, $q\geqslant 1$, is GA-convex on $[a,b]$, then the following Hermite-Hadamard type inequality for GA-convex function is obtained*

$$\left|f\left(\sqrt{a}b\right) - \frac{1}{lnb-lna}\int\limits_a^b \frac{f(x)}{x}dx\right|$$

$$\leqslant \frac{lnb-lna}{2^{3\left(1-\frac{1}{q}\right)}}\left\{\left[C_3(1)\left|f'(a)\right|^q + C_4(1)\left|f'(b)\right|^q\right]^{\frac{1}{q}} + \left[C_5(1)\left|f'(a)\right|^q + C_6(1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}. \tag{8}$$

In this study, we have given the new generalization of the definition of convexity that can reduce many order of convexity. First of all, we have revealed the identity. By taking advantage of this identity, we have given theorems and corollaries for $M_\varphi A$-convex functions. The studies on this article have scrutinised the relationships with previous studies.

## 2. Main results

### 2.1 *MφA convex functions*

**Definition 2.1** Let $I$ be a interval, $\varphi: I\to\mathbb{R}$ be a continuous and strictly monotonic function. $f: I\to\mathbb{R}$ is said to be $M_\varphi A$ convex, if

$$f\left(\varphi^{-1}\left(t\varphi(x) + (1-t)\varphi(y)\right)\right) \leqslant tf(x) + (1-t)f(y) \tag{9}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (9) is reversed, then $f$ is said to be $M_\varphi A$-concave function.

**Proposition 2.2**

*i i. For $\varphi : I \to \mathbb{R}$, $\varphi(x) = mx + n$, $m \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{R}$, $M_\varphi A$-convexity reduces to ordinary convexity on $I$.*
*ii ii. For $\varphi : I \to (0,\infty)$, $\varphi(x) = lnx$, $M_\varphi A$-convexity reduces to ordinary GA-convexity on $I$.*
*iii iii. For $\varphi : I \to (0,\infty)$, $\varphi(x) = x^{-1}$, $M_\varphi A$-convexity reduces to ordinary harmonically convexity on $I$.*
*iv iv. For $\varphi : I \to (0,\infty)$, $\varphi(x) = x^p$, $p \in \mathbb{R} \setminus \{0\}$, $M_\varphi A$-convexity reduces to ordinary $p$- convexity on $I$.*

**Theorem 2.3** *$I \subset \mathbb{R}$ is an interval, $a, b \in I$ with $a < b$. If $f : [a,b] \to \mathbb{R}$ be a $M_\varphi A$-convex function on $[a,b]$ and finite on $[a,b]$, then it is bounded.*

***Proof*** Firs of all, a function $M_\varphi A$ convex and finite on closed $[a,b]$ is bounded from above by $M = max\{f(a), f(b)\}$, since for any $w = \varphi^{-1}(\lambda\varphi(a) + (1-\lambda)\varphi(b))$ in the interval,

$$f(w) \leqslant \lambda f(a) + (1-\lambda) f(b) \leqslant \lambda M + (1-\lambda) M = M.$$

It must be showed that this function is bounded from below. So we see by writing an arbitrary in the form $\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2} + t\right)$ as follow:

$$\begin{aligned} & f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \\ &= f\left(\varphi^{-1}\left(\frac{1}{2}\varphi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2} - t\right)\right) + \frac{1}{2}\varphi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2} + t\right)\right)\right)\right) \\ &\leqslant \frac{1}{2} f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2} - t\right)\right) + \frac{1}{2} f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2} + t\right)\right) \end{aligned}$$

$$\begin{aligned} 2f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) &\leqslant f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2} - t\right)\right) + f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2} + t\right)\right) \\ f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2} + t\right)\right) &\geqslant 2f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) - f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2} - t\right)\right). \end{aligned}$$

On the other hand it is known that

$$-f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2} - t\right)\right) \geqslant -M. \tag{10}$$

■

**Theorem 2.4** *If $f : I \to \mathbb{R}$ be a $M_\varphi A$-convex, $\varphi : I \to \varphi(I)$ be continuous, strictly monotonic function and $\varphi$ satifies L-Lipschitz condition then $f$ is a Lipschitzian function on closed interval $[a,b]$ with $a < b$ contained in the interior $I^o$ of $I$.*

***Proof*** Firstly, we take $\varphi$ function with strictly increasing function. Choose $\epsilon > 0$ so that $\varphi(a-\epsilon)$ and $\varphi(b+\epsilon)$ belong to $\varphi(I)$, and let $m$ and $M$ be the lower and upper bounds for $f$ on $[a-\epsilon, b+\epsilon]$. If $x$ and $y$ distinct points of $[a,b]$, set

$$\varphi(z) = \frac{|\varphi(y)-\varphi(x)|+\epsilon}{|\varphi(y)-\varphi(x)|}\varphi(y) - \frac{\epsilon}{|\varphi(y)-\varphi(x)|}\varphi(x), \qquad \lambda = \frac{|\varphi(y)-\varphi(x)|}{\epsilon+|\varphi(y)-\varphi(x)|}$$

then it is gotten

$$z \in [a-\epsilon, b+\epsilon], \qquad \varphi(y) = \lambda\varphi(z) + (1-\lambda)\varphi(x),$$

and we have

$$\begin{aligned} f(y) = f\left(\varphi^{-1}\left(\lambda\varphi(z)+(1-\lambda)\varphi(x)\right)\right) \leqslant \lambda f(z) + (1-\lambda)f(x) &= \lambda\left(f(z)-f(x)\right)+f(x) \\ f(y)-f(x) &\leqslant \lambda\left(f(z)-f(x)\right) \leqslant \lambda\left(M-m\right) \\ &= \frac{|\varphi(y)-\varphi(x)|}{\epsilon+|\varphi(y)-\varphi(x)|}\left(M-m\right) \\ &\leqslant \frac{L\,|y-x|}{\epsilon}(M-m) \\ &\leqslant K\,|y-x|, \quad K := \frac{L(M-m)}{\epsilon} \end{aligned}$$

■

**Theorem 2.5** *Let $I$ is an interval of $\mathbb{R}$, $\varphi : I \to \mathbb{R}$ be a continuous and strictly increasing function. If $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are $M_\varphi A$-convex functions and $\alpha \geqslant 0$, then $f+g$ and $\alpha f$ are $M_\varphi A$-convex on $I$.*

***Proof*** Let function of $f$ and $g$ is a $M_\varphi A$-convex on $[a,b] \subset I$ and $a \leqslant b$ with $a, b \in \mathbb{R}$. We get, for $\lambda \in (0,1)$,

$$\begin{aligned} &(f+g)\left(\varphi^{-1}\left(\lambda\varphi(a)+(1-\lambda)\varphi(b)\right)\right) \\ &= f\left(\varphi^{-1}\left(\lambda\varphi(a)+(1-\lambda)\varphi(b)\right)\right) + g\left(\varphi^{-1}\left(\lambda\varphi(a)+(1-\lambda)\varphi(b)\right)\right) \\ &\leqslant \lambda f(a) + (1-\lambda)f(b) + \lambda g(a) + (1-\lambda)g(b) = \lambda(f+g)(a) + (1-\lambda)(f+g)(b), \end{aligned}$$

and $\lambda \geqslant 0$,

$$\begin{aligned} (\lambda f)\left(\varphi^{-1}\left(\lambda\varphi(a)+(1-\lambda)\varphi(b)\right)\right) &= \alpha f\left(\varphi^{-1}\left(\lambda\varphi(a)+(1-\lambda)\varphi(b)\right)\right) \\ &\leqslant \alpha\left(\lambda f(a)+(1-\lambda)f(b)\right) \\ &= \lambda\left(\alpha f\right)(a) + (1-\lambda)\left(\alpha f\right)(b) \end{aligned}$$

■

**Theorem 2.6** *Let $I$ be an interval of $\mathbb{R}$ and $\varphi : I \to \mathbb{R}$ be a continuous, strictly increasing function. If $f, g : I \to \mathbb{R}$ are both nonnegative, decreasing (increasing) and $M_\varphi A$-convex, then $h(x) = f(x)g(x)$ also expose these properties.*

***Proof*** We begin by noting that for $x < y$,

$$\left[f(x)-f(y)\right]\left[g(y)-g(x)\right] \leqslant 0$$

which implies that

$$f(x)f(y) + f(y)g(x) \leqslant f(x)g(x) + f(y)g(y)$$

an inequality we use blow. Now if $\alpha > 0$,$\beta > 0$ and $\alpha + \beta = 1$,

$$\begin{aligned}
&f\left(\varphi^{-1}\left(\alpha\varphi(x) + \beta\varphi(y)\right)\right) g\left(\varphi^{-1}\left(\alpha\varphi(x) + \beta\varphi(y)\right)\right) \\
&\leqslant \left(\alpha f(x) + \beta f(y)\right)\left(\alpha g(x) + \beta g(y)\right) \\
&= \alpha^2 f(x)g(x) + \alpha\beta\left[f(x)g(y) + f(y)g(x)\right] + \beta^2 f(y)g(y) \\
&\leqslant \alpha^2 f(x)g(x) + \alpha\beta\left[f(x)g(x) + f(y)g(y)\right] + \beta^2 f(y)g(y) \\
&= \alpha f(x)g(x) + \beta f(y)g(y).
\end{aligned}$$

■

**Theorem 2.7** *Let $I$ is an interval. If $f, g : I \to \mathbb{R}$ be a $M_\varphi A$-convex, a convex function and $\varphi : I \to \mathbb{R}$ is continuous and strictly increasing function, then $g \circ f$ be a $M_\varphi A$-convex function.*

***Proof*** If we go out of hypothesis, then we get

$$\begin{aligned}
(g \circ f)\left(\varphi^{-1}\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right)\right) &= g\left(f\left(\varphi^{-1}\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right)\right)\right) \\
&\leqslant g\left(\lambda f(x) + (1-\lambda)f(y)\right) \leqslant \lambda g\left(f(x)\right) + (1-\lambda)g\left(f(y)\right) \\
&= \lambda\left(g \circ f\right)(x) + (1-\lambda)\left(g \circ f\right)(y).
\end{aligned}$$

■

**Theorem 2.8** *Let $f_\alpha : I \to \mathbb{R}$ be an arbitrary family of $M_\varphi A$-convex functions and $\varphi : J \to \varphi\{J\}$ be a continuous, strictly increasing function and let $f(z) =$ $\sup_\alpha f_\alpha(z)$. If $J = \{z \in I : f(z) < \infty\}$ is nonempty, then $J$ is an interval and $f$ is $M_\varphi A$-convex on $J$.*

***Proof*** Firstly It is showed that $J$ is an interval. So, we must proof that $[x, y] \subset J$ for $\forall x, y \in J$. Let $x, y \in J$ and $z \in [x, y]$ are arbitrary then there exist $\lambda \in [0, 1]$ such that

$$z = \varphi^{-1}\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right) \in [x, y] \subset J.$$

and since $f(x), f(y) < \infty$ we have $z \in J$

$$\begin{aligned}
f\left(\varphi^{-1}\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right)\right) &= \sup_\alpha f_\alpha\left(\varphi^{-1}\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right)\right) \\
&\leqslant \sup_\alpha\left[\lambda f_\alpha(x) + (1-\lambda)f_\alpha(y)\right] \\
&\leqslant \lambda\sup_\alpha f_\alpha(x) + (1-\lambda)\sup_\alpha f_\alpha(y) \\
&= \lambda f(x) + (1-\lambda)f(y) < \infty.
\end{aligned}$$

This simultaneously that $J$ is an interval (since it contains every point between any two of its points) and that $f$ is $M_\varphi A$-convex on it. ■

**Theorem 2.9** *Let $I$ is an interval and $x_1, x_2, x3 \in I$. $\varphi : I \to \mathbb{R}$ be a continuous, strictly monotonic function and $f : I \to \mathbb{R}$ be a $M_\varphi A$-convex function.*

*(1) If $\varphi$ is an increasing function, then we get for $x_1 < x_2 < x_3$*

$$\begin{vmatrix} 1 & \varphi(x_1) & f(x_1) \\ 1 & \varphi(x_2) & f(x_2) \\ 1 & \varphi(x_3) & f(x_3) \end{vmatrix} \geqslant 0.$$

*(2) If $\varphi$ is an decreasing function, then we get for $x_1 < x_2 < x_3$*

$$\begin{vmatrix} 1 & \varphi(x_3) & f(x_3) \\ 1 & \varphi(x_2) & f(x_2) \\ 1 & \varphi(x_1) & f(x_1) \end{vmatrix} \geqslant 0.$$

***Proof***

(1) If it is determination that is amplified in case of first line and thirdly column, we get

$$\begin{aligned} &\begin{vmatrix} 1 & \varphi(x_3) & f(x_3) \\ 1 & \varphi(x_2) & f(x_2) \\ 1 & \varphi(x_1) & f(x_1) \end{vmatrix} && (11) \\ &= f(x_1)\left(\varphi(x_3) - \varphi(x_2)\right) - f(x_2)\left(\varphi(x_3) - \varphi(x_1)\right) + f(x_3)\left(\varphi(x_2) - \varphi(x_1)\right) \end{aligned}$$

On the other hand, from $x_1 < x_2 < x_3$, $\varphi$ is a strictly monotonic function and $f$ be a $M_\varphi A$-convex function, we get for $t \in (0, 1)$

$$x_2 = \varphi^{-1}\left(t\varphi(x_1) + (1-t)\varphi(x_3)\right) \tag{12}$$

$$f(x_2) = f\left(\varphi^{-1}\left(t\varphi(x_1) + (1-t)\varphi(x_3)\right)\right) \leqslant tf(x_1) + (1-t)f(x_3). \tag{13}$$

By using (12)-(13) in (11), it is gotten

$$\begin{vmatrix} 1 & \varphi(x_3) & f(x_3) \\ 1 & \varphi(x_2) & f(x_2) \\ 1 & \varphi(x_1) & f(x_1) \end{vmatrix} \tag{14}$$

$$\begin{aligned} &= f(x_1)\left(\varphi(x_3) - t\varphi(x_1) - (1-t)\varphi(x_3)\right) - f(x_2)\left(\varphi(x_3) - \varphi(x_1)\right) && (15) \\ &+f(x_3)\left(t\varphi(x_1) + (1-t)\varphi(x_3) - \varphi(x_1)\right) \\ &= \left(\varphi(x_3) - \varphi(x_1)\right)\left(tf(x_1) - f(x_2) + (1-t)f(x_3)\right) \geqslant 0. \end{aligned}$$

The proof is completed.

(2) It is be occured that this is proved the same as ((1)).

■

According to We have given definition of $M_\varphi A$-convexity as above. Presently, we will establish a new lemma for $M_\varphi A$-convexity. Using this identity, we will give new theorems and corollaries.

By benefiting the definition of $M_\varphi A$-convex functions, we constitute the Hermite-Hadamard inequality for this convexity as follow:

### 2.2 *Hermite-Hadamard inequalities for $M_\varphi A$-convex*

**Theorem 2.10** *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a $M_\varphi A$-convex function, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function and $a, b \in I$ with $a < b$. If $f, \varphi' \in L[a, b]$*

*the the following inequalities holds*

$$f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \leqslant \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx \leqslant \frac{f(a)+f(b)}{2}. \quad (16)$$

*The above inequlaties are sharp.*

***Proof*** Since $f : I \subset (0,\infty) \to \mathbb{R}$ be a $M_\varphi A$-convex function, we have, for all $x, y \in I$, (with $t = \frac{1}{2}$ in the inequality (9))

$$f\left(\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)\right) \leqslant \frac{f(x)+f(y)}{2}.$$

Choosing $x = \varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))$, $y = \varphi^{-1}(t\varphi(b)+(1-t)\varphi(a))$, we get

$$f\left(\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)\right) \leqslant \frac{f\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)+f\left(\varphi^{-1}((1-t)\varphi(a)+t\varphi(b))\right)}{2}$$

By integrating for $t \in [0,1]$, we have

$$\begin{aligned} & f\left(\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)\right) \quad (17) \\ & \leqslant \frac{1}{2}\left[\int_0^1 f\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)dt + \int_0^1 f\left(\varphi^{-1}((1-t)\varphi(a)+t\varphi(b))\right)dt\right] \\ & \leqslant \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx. \end{aligned}$$

We get the left hand side of the inequality (16). Furhermore, we observe that for all $t \in [0,1]$

$$f\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right) \leqslant tf(a)+(1-t)f(b).$$

By integrating this inequality with respect to $t$ over $[0,1]$, we have the right-hand side of the inequality (16). Let consider the function $f : (0,\infty) \to \mathbb{R}$, $f(x) = 1$. Thus

$$\begin{aligned} 1 &= f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \\ &= tf(a)+(1-t)f(b) = 1 \end{aligned}$$

for all $x, y \in I$ and $t \in [0,1]$. Therefore $f$ is $M_\varphi A$-convex on $I$. We also have

$$f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) = 1, \quad \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx = 1$$

$$\frac{f(a)+f(b)}{2} = 1$$

which reveals us the inequalities (16) are sharp. ■

**Lemma 2.11** *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable function on $I^o$, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function and $a, b \in I^o$ with $0 < a < b$. If $f' \in L([a,b])$, then we get*

$$f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) - \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx \tag{18}$$
$$= \frac{\varphi(b)-\varphi(a)}{4}\left[\int_0^1 \begin{array}{c} t\left(\varphi^{-1}\right)'\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right) \\ f'\left(\varphi^{-1}\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right)dt \end{array}\right.$$
$$\left. - \int_0^1 \begin{array}{c} t\left(\varphi^{-1}\right)'\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right) \\ f'\left(\varphi^{-1}\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right)dt \end{array}\right].$$

***Proof*** Firstly, we take the integral and calculating as follows

$$I_2 = \int_0^1 \begin{array}{c} t\left(\varphi^{-1}\right)'\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right) \\ f'\left(\varphi^{-1}\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right) \end{array} dt \tag{19}$$
$$= t\frac{2}{\varphi(b)-\varphi(a)} f\left(\varphi^{-1}\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right)\Bigg|_0^1$$
$$-\frac{2}{\varphi(b)-\varphi(a)}\int_0^1 f\left(\varphi^{-1}\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right)dt$$
$$= \frac{2}{\varphi(b)-\varphi(a)} f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)$$
$$-\frac{2}{\varphi(b)-\varphi(a)}\int_0^1 f\left(\varphi^{-1}\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right)dt.$$

Changing $x = \varphi^{-1}\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)$, we get

$$I_2 = \frac{2}{\varphi(b)-\varphi(a)} f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \tag{20}$$
$$-\frac{4}{\left(\varphi(b)-\varphi(a)\right)^2}\int_a^{\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)} f(x)\varphi'(x)dx.$$

Calculating the other integral with the same method, we get

$$I_1 = \int_0^1 \begin{array}{c} t\left(\varphi^{-1}\right)'\left(\frac{t}{2}\varphi(a) + \left(1-\frac{t}{2}\right)\varphi(b)\right) \\ f'\left(\varphi^{-1}\left(\frac{t}{2}\varphi(a) + \left(1-\frac{t}{2}\right)\varphi(b)\right)\right) dt \end{array} \tag{21}$$
$$= \frac{2}{\varphi(a)-\varphi(b)} f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)$$
$$-\frac{4}{\left(\varphi(a)-\varphi(b)\right)^2} \int_b^{\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)} f(x)\varphi'(x)dx.$$

By summing $I_2$ with $-I_1$, we obtain (18). ■

In other words, the lemma we have obtained can be expressed as follows:

**Remark 2.12**

(1) If it is taken $\varphi(x) = x$ in (18), then we get

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx$$
$$= \frac{b-a}{4}\left[\int_0^1 tf'\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b\right)dt - \int_0^1 tf'\left(\left(1-\frac{t}{2}\right)b + \frac{t}{2}a\right)dt\right].$$

(2) I it is taken $\varphi(x) = lnx$ in (18), then we get [7, Lemma 2.1] as follow:

$$f\left(\sqrt{(ab)}\right) - \frac{1}{lnb - lna}\int_a^b \frac{f(x)}{x}dx$$
$$= \frac{lnb-lna}{4}\left[\int_0^1 ta^{1-\frac{t}{2}}b^{\frac{t}{2}}f'\left(a^{1-\frac{t}{2}}b^{\frac{t}{2}}\right)dt - \int_0^1 ta^{\frac{t}{2}}b^{1-\frac{t}{2}}f'\left(a^{\frac{t}{2}}b^{1-\frac{t}{2}}\right)dt\right].$$

(3) If it is taken $\varphi(x) = \frac{1}{x}$ in (18), then we get

$$f\left(\frac{a+b}{2ab}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$
$$= \frac{b-a}{4ab}\left[\int_0^1 t\frac{1}{\left(1-\frac{t}{2}\right)\frac{1}{a} + \frac{t}{2}\frac{1}{b}} f'\left(\frac{1}{\left(1-\frac{t}{2}\right)\frac{1}{a} + \frac{t}{2}\frac{1}{b}}\right)dt\right.$$
$$\left. - \int_0^1 t\frac{1}{\left(1-\frac{t}{2}\right)\frac{1}{b} + \frac{t}{2}\frac{1}{a}} f'\left(\frac{1}{\left(1-\frac{t}{2}\right)\frac{1}{b} + \frac{t}{2}\frac{1}{a}}\right)dt\right].$$

(4) If it is taken $\varphi(x)=x^p$ in (18), $p\in\mathbb{R}\backslash\{0\}$, the we get

$$f\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx$$
$$=\frac{b^p-a^p}{4p}\left[\int_a^b t\frac{f'\left(\left(\left(1-\frac{t}{2}\right)a^p+\frac{t}{2}b^p\right)^{\frac{1}{p}}\right)}{\left(\left(1-\frac{t}{2}\right)a^p+\frac{t}{2}b^p\right)^{1-\frac{1}{p}}}dt-\int_0^1 t\frac{f'\left(\left(\left(1-\frac{t}{2}\right)b^p+\frac{t}{2}a^p\right)^{\frac{1}{p}}\right)}{\left(\left(1-\frac{t}{2}\right)b^p+\frac{t}{2}a^p\right)^{1-\frac{1}{p}}}dt\right].$$

**Remark 2.13** In ((1)) equation of Remark 2.12,if we use equation of $\left(1-\frac{1}{t}\right)a+\frac{t}{2}b=(1-t)a+t\left(\frac{a+b}{2}\right)$ in the first integration and we use the changing of variable with $-t=u-1$,$dt=-du$ and equation of $\frac{1-u}{2}a+\frac{1+u}{2}b=ub+(1-u)\frac{a+b}{2}$ in the second integration, then we get [6, Lemma 2.1] as follow:

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx \tag{22}$$
$$=\frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2}+(1-t)a\right)dt+\int_0^1(t-1)f'\left(tb+(1-t)\frac{a+b}{2}\right)dt\right].$$

**Theorem 2.14** *$f:I\subseteq[0,\infty)\to\mathbb{R}$ be differentiable on $I^o$ and $a,b\in I^o$ with $a<b$, $\varphi:I\to\mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1}:\varphi(I^o)\to I^o$ is continuously differentiable functions. If $|f'|$ is a $M_\varphi A$-convex functions, we have*

$$\left|f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \tag{23}$$
$$\leqslant\frac{\varphi(b)-\varphi(a)}{4}\left[A_1(t;a,b)\left|f'(a)\right|+A_2(t;a,b)\left|f'(b)\right|\right]$$

*where*

$$A_1(t;a,b)=\int_0^1\left[\begin{array}{c}\left(\varphi^{-1}\right)'\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\frac{t^2}{2}\\+\left(\varphi^{-1}\right)'\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\left(t-\frac{t^2}{2}\right)\end{array}\right]dt,$$

$$A_2(t;a,b)=\int_0^1\left[\begin{array}{c}\left(\varphi^{-1}\right)'\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\left(t-\frac{t^2}{2}\right)\\+\left(\varphi^{-1}\right)'\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\frac{t^2}{2}\end{array}\right]dt.$$

***Proof*** By using the $M_{\varphi}A$ convexity of $|f'|$ on $[a,b]$ in (18), we get

$$\left| f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) - \frac{1}{\varphi(b)-\varphi(a)} \int_a^b f(x)\varphi'(x)dx \right| \tag{24}$$

$$\leqslant \frac{\varphi(b)-\varphi(a)}{4}\left[\int_0^1 t\left(\varphi^{-1}\right)'\left(\tfrac{t}{2}\varphi(a)+\left(1-\tfrac{t}{2}\right)\varphi(b)\right)\left(\tfrac{t}{2}\left|f'(a)\right|+\left(1-\tfrac{t}{2}\right)\left|f'(b)\right|\right)dt\right.$$

$$\left. + \int_0^1 t\left(\varphi^{-1}\right)'\left(\left(1-\tfrac{t}{2}\right)\varphi(a)+\tfrac{t}{2}\varphi(b)\right)\left(\left(1-\tfrac{t}{2}\right)\left|f'(a)\right|+\tfrac{t}{2}\left|f'(b)\right|\right)dt\right]$$

$$= \frac{\varphi(b)-\varphi(a)}{4}\left[\int_0^1 \left(\varphi^{-1}\right)'\left(\tfrac{t}{2}\varphi(a)+\left(1-\tfrac{t}{2}\right)\varphi(b)\right)\left(\tfrac{t^2}{2}\left|f'(a)\right|+\left(t-\tfrac{t^2}{2}\right)\left|f'(b)\right|\right)dt\right.$$

$$\left. + \int_0^1 \left(\varphi^{-1}\right)'\left(\left(1-\tfrac{t}{2}\right)\varphi(a)+\tfrac{t}{2}\varphi(b)\right)\left(\left(t-\tfrac{t^2}{2}\right)\left|f'(a)\right|+\tfrac{t^2}{2}\left|f'(b)\right|\right)dt\right].$$

This proof is completed. ■

**Corollary 2.15**

*i i. If we take $\varphi(x)=mx+n$ to (23), we get*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leqslant \frac{b-a}{8}\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \tag{25}$$

*ii ii. If we take $\varphi(x)=lnx$ to (23), we get*

$$\left| f\left(\sqrt{a}b\right) - \frac{1}{lnb-lna}\int_a^b f(x)dx \right| \leqslant \frac{lnb-lna}{4}\left[A_1(t;a,b)\left|f'(a)\right|+A_2(t;a,b)\left|f'(b)\right|\right] \tag{26}$$

*where*

$$A_1(t;a,b) = \int_0^1 \left[\frac{t^2}{2}a^{\frac{t}{2}}b^{1-\frac{t}{2}} + \left(t-\frac{t^2}{2}\right)b^{\frac{t}{2}}a^{1-\frac{t}{2}}\right]dt$$

$$A_2(t;a,b) = \int_0^1 \left[\left(t-\frac{t^2}{2}\right)a^{\frac{t}{2}}b^{1-\frac{t}{2}} + \frac{t^2}{2}b^{\frac{t}{2}}a^{1-\frac{t}{2}}\right]dt$$

*iii iii. If we take* $\varphi(x)=x^{-1}$ *to* (23)*, we get*

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}dx\right| \leqslant \frac{b-a}{4ab}\left[A_1(t;a,b)\left|f'(a)\right|+A_2(t;a,b)\left|f'(b)\right|\right] \tag{27}$$

*where*

$$A_1(t;a,b)=\int\limits_0^1\left[\frac{t^2}{2\left(\frac{t}{2}\frac{1}{a}+\left(1-\frac{t}{2}\right)\frac{1}{b}\right)^2}.\frac{t-\frac{t^2}{2}}{\left(\left(1-\frac{t}{2}\right)\frac{1}{a}+\frac{t}{2}\frac{1}{b}\right)^2}\right]dt,$$

$$A_2(t;a,b)=\int\limits_0^1\left[\frac{t-\frac{t^2}{2}}{2\left(\frac{t}{2}\frac{1}{a}+\left(1-\frac{t}{2}\right)\frac{1}{b}\right)^2}\frac{t^2}{\left(\left(1-\frac{t}{2}\right)\frac{1}{a}+\frac{t}{2}\frac{1}{b}\right)^2}\right]dt.$$

*iv iv. If we take* $\varphi(x)=x^p$,$p\in\mathbb{R}\setminus\{0\}$, *to* (23)*, we get*

$$\left|f\left(\left(\frac{a^p+b^p}{2}\right)^{1/p}\right)-\frac{p}{b^p-a^p}\int\limits_a^b\frac{f(x)}{x^{1-p}}dx\right|\leqslant\frac{b^p-a^p}{4p}\left[A_1(t;a,b)\left|f'(a)\right|+A_2(t;a,b)\left|f'(b)\right|\right] \tag{28}$$

*where*

$$A_1(t;a,b)=\int\limits_a^b\left[\left(\frac{t}{2}a^p+\left(1-\frac{t}{2}\right)b^p\right)^{\frac{1}{p}-1}\frac{t^2}{2}+\left(\left(1-\frac{t}{2}\right)a^p+\frac{t}{2}b^p\right)^{\frac{1}{p}-1}\left(t-\frac{t^2}{2}\right)\right]dt$$

$$A_2(t;a,b)=\int\limits_a^b\left[\left(\frac{t}{2}a^p+\left(1-\frac{t}{2}\right)b^p\right)^{\frac{1}{p}-1}\left(t-\frac{t^2}{2}\right)+\frac{t^2}{2}\left(\left(1-\frac{t}{2}\right)a^p+\frac{t}{2}b^p\right)^{\frac{1}{p}-1}\right]dt$$

**Theorem 2.16** *Let* $f:I\subseteq[0,\infty)\to\mathbb{R}$ *be differentiable on* $I^o$ *and* $a,b\in I^o$ *with* $a<b$, $\varphi:I\to\mathbb{R}$ *be a continuous and strictly monotonic function such that* $\varphi^{-1}:\varphi(I^o)\to I^o$ *is continuously differentiable functions. If* $|f'|^q$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *is* $M_\varphi A$*- convex function on* $[a,b]$ *then we get*

$$\left|f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)-\frac{1}{\varphi(b)-\varphi(a)}\int\limits_a^b f(x)\varphi'(x)dx\right| \leqslant\frac{\varphi(b)-\varphi(a)}{4}\left\{(B_1(t;a,b))^{1/p}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{1/q} + (B_2(t;a,b))^{1/p}\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{1/q}\right\} \tag{29}$$

*where*

$$B_1(t;a,b)=\int_0^1 t^p\left|\left(\varphi^{-1}\right)'\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right|^p dt; \tag{30}$$

$$B_2(t;a,b)=\int_0^1 t^p\left|\left(\varphi^{-1}\right)'\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right|^p dt.$$

***Proof*** By taking (18) equality with absolute value and using Hölder inequality, we get

$$\left|f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \tag{31}$$
$$\leqslant\left[\begin{array}{c}\left(\int_0^1 t^p\left|\left(\varphi^{-1}\right)'\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right|^p dt\right)^{1/p}\\ \left(\int_0^1\left|f'\left(\varphi^{-1}\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right)\right|^q dt\right)^{1/q}\end{array}\right]$$
$$+\left[\begin{array}{c}\left(\int_0^1 t^p\left|\left(\varphi^{-1}\right)'\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right|^p dt\right)^{1/p}\\ \left(\int_0^1\left|f'\left(\varphi^{-1}\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right)\right|^q dt\right)^{1/q}\end{array}\right]$$

Since $|f'|^q$ is $M_\varphi A$-convex function, we have

$$\left|f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \tag{32}$$
$$\leqslant\left[\begin{array}{c}\left(\int_0^1 t^p\left|\left(\varphi^{-1}\right)'\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right|^p dt\right)^{1/p}\\ \left(\int_0^1\left(\frac{t}{2}|f'(a)|^q+\left(1-\frac{t}{2}\right)|f'(b)|^q\right)dt\right)^{1/q}\end{array}\right]$$
$$+\left[\begin{array}{c}\left(\int_0^1 t^p\left|\left(\varphi^{-1}\right)'\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right|^p dt\right)^{1/p}\\ \left(\int_0^1\left(\left(1-\frac{t}{2}\right)|f'(a)|^q+\frac{t}{2}|f'(b)|^q\right)dt\right)^{1/q}\end{array}\right]$$
$$=\left(\int_0^1 t^p\left|\left(\varphi^{-1}\right)'\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right|^p dt\right)^{1/p}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{1/q}$$
$$+\left(\int_0^1 t^p\left|\left(\varphi^{-1}\right)'\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right|^p dt\right)^{1/p}\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{1/q}.$$

The proof is completed. ■

**Corollary 2.17**

*i i. If we take $\varphi(x) = mx + n$ to (29), we obtain*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leqslant \frac{b-a}{4(p+1)^{1/p}}\left[\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{1/q} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{1/q}\right] \tag{33}$$

*ii ii. If we take $\varphi(x) = lnx$ to (29), we obtain*

$$\left| f\left(\sqrt{a}b\right) - \frac{1}{lnb - lna}\int_a^b \frac{f(x)}{x}dx \right| \leqslant \frac{lnb-lna}{4}\left[(B_1(t;a,b))^{1/p}\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{1/q} + (B_2(t;a,b))^{1/p}\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{1/q}\right] \tag{34}$$

*where*

$$B_1(t;a,b) = \int_0^1 t^p a^{\frac{t}{2}} b^{1-\frac{t}{2}} dt,$$
$$B_2(t;a,b) = \int_0^1 t^p a^{1-\frac{t}{2}} b^{\frac{t}{2}} dt.$$

*iii iii. If we take $\varphi(x) = x^p$, $p \in \mathbb{R}\setminus\{0\}$, to (29), we obtain*

$$\left| f\left(\left(\frac{a^p+b^p}{2}\right)^{1/p}\right) - \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \right| \leqslant \frac{b^p-a^p}{4p}\left[(B_1(t;a,b))^{1/p}\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{1/q} + (B_2(t;a,b))^{1/p}\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{1/q}\right] \tag{35}$$

*where*

$$B_1(t;a,b)=\int_0^1 t^p\left(\frac{t}{2}a^p+\left(1-\frac{t}{2}\right)b^p\right)^{\frac{1}{p}-1}dt,$$

$$B_2(t;a,b)=\int_0^1 t^p\left(\left(1-\frac{t}{2}\right)a^p+\frac{t}{2}b^p\right)^{\frac{1}{p}-1}dt.$$

**Theorem 2.18** *Let $f:I\subseteq[0,\infty)\to\mathbb{R}$ be differentiable on $I^o$ and $a,b\in I^o$ with $a<b$, $\varphi:I\to\mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1}:\varphi(I^o)\to I^o$ is continuously differentiable functions. If $|f'|^q$, $q\geqslant 1$, is $M_\varphi A$-convex function on $[a,b]$ then we get*

$$\left|f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \tag{36}$$
$$\leqslant\frac{\varphi(b)-\varphi(a)}{2^{3-\frac{1}{q}}}\left[\left(C_1(t;a,b)\left|f'(a)\right|^q+C_2(t;a,b)\left|f'(b)\right|^q\right)^{1/q}\right.$$
$$\left.+\left(C_3(t;a,b)\left|f'(a)\right|^q+C_4(t;a,b)\left|f'(b)\right|^q\right)^{1/q}\right]$$

*where*

$$C_1(t;a,b)=\int_0^1\frac{t^2}{2}\left|\varphi^{-1}\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right|^q dt,$$

$$C_2(t;a,b)=\int_0^1\left(t-\frac{t^2}{2}\right)\left|\varphi^{-1}\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right|^q dt,$$

$$C_3(t;a,b)=\int_0^1\left(t-\frac{t^2}{2}\right)\left|\varphi^{-1}\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right|^q dt,$$

$$C_3(t;a,b)=\int_0^1\frac{t^2}{2}\left|\varphi^{-1}\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right|^q dt.$$

***Proof*** we use the power mean inequality in (18) and the $|f'|^q$, $q\geqslant 1$, is $M_\varphi A$-

convex function then we get

$$\left| f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) - \frac{1}{\varphi(b)-\varphi(a)}\int\limits_a^b f(x)\varphi'(x)dx \right| \tag{37}$$

$$\leqslant \frac{\varphi(b)-\varphi(a)}{4}\left[\left(\int\limits_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int\limits_0^1 \begin{array}{c} t\left|\varphi^{-1}\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right. \\ f'\left(\varphi^{-1}\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right)\Big|^q dt \end{array}\right)^{1/q}\right.$$

$$\left. + \left(\int\limits_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int\limits_0^1 \begin{array}{c} t\left|\varphi^{-1}\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right. \\ f'\left(\varphi^{-1}\left(\left(1-\frac{t}{2}\right)\varphi(a)+\frac{t}{2}\varphi(b)\right)\right)\Big|^q dt \end{array}\right)^{1/q}\right]$$

$$\leqslant \frac{\varphi(b)-\varphi(a)}{2^{3-\frac{1}{q}}}\left[\left(\int\limits_0^1 \frac{t^2}{2}\left|\varphi^{-1}\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right|^q |f'(a)|^q dt\right.\right.$$

$$\left. + \int\limits_0^1 \left(t-\frac{t^2}{2}\right)\left|\varphi^{-1}\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right|^q |f'(b)|^q dt\right)^{1/q}$$

$$+ \left(\int\limits_0^1 \left(t-\frac{t^2}{2}\right)\left|\varphi^{-1}\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right|^q |f'(a)|^q dt\right.$$

$$\left.\left. + \int\limits_0^1 \frac{t^2}{2}\left|\varphi^{-1}\left(\frac{t}{2}\varphi(a)+\left(1-\frac{t}{2}\right)\varphi(b)\right)\right|^q |f'(b)|^q dt\right)^{1/q}\right].$$

This completes the proof. ■

## 3. Conclusions

In this study, we have defined a new and general convex function class. We have given the properties of this convex function. We obtained Hermite-Hadamard inequality for the convex function we achieved, and in special cases we showed that it was reduced to different convex classes.

## References


[1] J. Aczél, The notion of mean values, Norske Vid. Selsk. Forhdl., Trondhjem **19** (1947) 83–86.

[2] J. Aczél, A generalization of the notion of convex functions, Norske Vid. Selsk. Forhd., Trondhjem **19 (24)** (1947), 87–90.

[3] G. Aumann, Konvexe Funktionen und Induktion bei Ungleichungen zwischen Mittelverten, Bayer. Akad. Wiss.Math.-Natur. Kl. Abh., Math. Ann. **109** (1933) 405–413.

[4] M. Avci, H. Kavurmaci and M. E. Ozdemir, "New inequalities of Hermite-Hadamard type via $s$-convex functions in the second sense with applications", Applied Mathematics and Computation, **217** (2011) 5171–5176.

[5] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, Journal of Mathematical Analysis and Applications, **335 (2)** (2007) 1294-1308.

[6] M. W. Alomari, M. Darus and U. Kirmaci, Some inequalities of Hermite-Hadamard type for $s$-convex functions, Acta Mathematica Scentia, **31B (4)** (2011) 1643–1652.

[7] X. Bo-Yan and F. Qi, Hermite-Hadamard type inequalities for functions whose derivatives are of convexities, Nonlinear Functional Analysis and Applications, **18 (2)** (2013) 163–176.

[8] S. S. Dragomir and R. P. Agarwal, Two Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Trapezoidal Formula, Applied Mathematics Letters, **11 (5)** (1998) 91-95.

[9] S. S. Dragomir - S. Fitzpatrick, The Hadamard's inequality for $s$-convex functions in the second sense, Demonstr. Math., 32 (4) (1999), 687–696.

[10] M. Kunt and İ. İşcan, On new inequalities of Hermite-Hadamard-Fejer type for GA-s-convex functions via fractional integrals, Konuralp journal of Mathematics, **4 (1)** (2016) 130-139.

[11] İ. İşcan, A new generalization of some integral inequalities for -convex functions, Mathematical Sciences, **7 (22)** (2013) 1-8.

[12] İ. İşcan, New estimates on generalization of some integral inequalities for s-convex functions and their applications, International Journal of Pure and Applied Mathematics, **86 (4)** (2013) 727-746.

[13] İ. İşcan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe Journal of Mathematics and Statistics, **43 (6)** (2014) 935–942.

[14] İ. İşcan, Some new general integral inequalities for h-convex and h-concave functions, Advances in Pure and Applied Mathematics, **5 (1)** (2014) 21-29.

[15] İ. İşcan, Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals, Studia Universitatis Babeş-Bolyai Mathematica, **60 (3)** (2015) 355-366.

[16] İ. İşcan, Ostrowski type inequalities for $p$-convex functions, New Trends in Mathematical Sciences, **4 (3)** (2016) 140-140.

[17] İ. İşcan, Hermite-Hadamard type inequalities for $p$-convex functions, International Journal of Analysis and Applications, **11 (2)** (2016) 137-145.

[18] U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula,Applied Mathematics and Computation, **147 (1)** (2004) 137-146.

[19] J. Matkowski, Convex functions with respect to a mean and a characterization of quasi-arithmetic means, Real Analysis Exchange, **29 (1)** (2003) 229–246.

[20] C. P. Niculescu, Convexity according to the geometric mean, Mathematical Inequalities and Applications, **3 (2)** (2000) 155-167.

[21] C. P. Niculescu, Convexity according to means, Mathematical Inequalities and Applications, **6** (2003) 571–579.

[22] W. Orlicz, A note on modular spaces I, Bulletin L'Académie Polonaise des Science, Série des Sciences Mathématiques, Astronomiques et Physiques, **9** (1961) 157-162.