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(m_1, m_2)-AG-Convex Functions and Some New Inequalities

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Abstract. In this manuscript, we introduce concepts of (m_1, m_2)-logarithmically convex (AG-convex) functions and establish some Hermite-Hadamard type inequalities of these classes of functions.

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1. Introduction

A function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t) f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. One of the most important inequalities in convex theory is the Hermite-Hadamard integral inequality. This inequality is given below.

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Let $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function f is concave. The above inequalities was firstly discovered by the famous scientist Charles Hermite. In recent years, readers can find more information in [3, 5–12, 14–16] for different convex classes and related Hermite-Hadamard integral inequalities.

Definition 1.1 [13]A function $f : [0, b] \to \mathbb{R}$ is said to be m-convex for $m \in (0, 1]$ if the inequality

$$f(\alpha x + m(1-\alpha)y) \leqslant \alpha f(x) + m(1-\alpha)f(y)$$

holds for all $x, y \in [0, b]$ and $\alpha \in [0, 1]$.

Definition 1.2 [9]The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (m_1, m_2)-convex, if

$$f(m_1 tx + m_2(1-t)y) \leqslant m_1 t f(x) + m_2(1-t)f(y)$$

for all $x, y \in I$, $t \in [0, 1]$ and $(m_1, m_2) \in (0, 1]^2$.

Definition 1.3 [10]$f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m_1, m_2)-convex function, if

$$f(m_1 tx + m_2(1-t)y) \leqslant m_1 t^\alpha f(x) + m_2(1-t^\alpha)f(y)$$

for all $x, y \in I$, $t \in [0, 1]$ and $(\alpha, m_1, m_2) \in (0, 1]^3$.

Definition 1.4 [1, 17]If a function $f : I \subseteq \mathbb{R} \to (0, \infty)$ satisfies

$$f(\lambda x + m(1-\lambda)y) \leqslant [f(x)]^\lambda [f(y)]^{1-\lambda}$$

for all $x, y \in I$, $\lambda \in [0, 1]$, the function f is called logarithmically convex on I. If this inequality reverses, the function f is called logarithmically concave on I.

Definition 1.5 [2]A function $f : [0, b] \to (0, \infty)$ is said to be m-logarithmically convex if the inequality

$$f(tx + m(1-t)y) \leqslant [f(x)]^t [f(y)]^{m(1-t)}$$

holds for all $x, y \in [0, b]$, $m \in (0, 1]$, and $t \in [0, 1]$.

The main purpose of this paper is to introduce the concept of (m_1, m_2)-arithmetic geometrically (AG) or (m_1, m_2)-logarithmically convex functions and then establish some results connected with new inequalities similar to the Hermite-hadamard integral inequality for these classes of functions.

2. Main results

In this section, we introduce a new concept, which is called (m_1, m_2)-AG convex (logarithmicaly convex) functions and we give by setting some algebraic properties for the (m_1, m_2)-AG convex functions, as follows:

Definition 2.1 A function $f : [0, b] \to (0, \infty)$ is said to be (m_1, m_2)-arithmetic geometrically convex (or logarithmically convex) if the inequality

$$f(m_1 tx + m_2(1-t)y) \leqslant [f(x)]^{m_1 t} [f(y)]^{m_2(1-t)}$$

holds for all $x, y \in [0, b]$, $(m_1, m_2) \in (0, 1]^2$, and $t \in [0, 1]$.

We discuss some connections between the class of the (m_1, m_2)-arithmetic geometrically convex functions and other classes of generalized convex functions.

Remark 2.2 When $m_1 = m_2 = 1$, the (m_1, m_2)-arithmetic geometrically convex (concave) function becomes a arithmetic geometrically convex (concave) function in defined [1, 17].

Remark 2.3 When $m_1 = 1$, $m_2 = m$, the (m_1, m_2)-arithmetic geometrically convex (concave) function becomes the m-arithmetic geometrically convex (concave) function defined in [2].

Proposition 2.4 *The function $f : I \subset (0, \infty) \to \mathbb{R}$ is (m_1, m_2)-AG convex function on I if and only if $\ln \circ f : (0, \infty) \to \mathbb{R}$ is (m_1, m_2)-convex function on the interval $(0, \infty)$.*

Proof $(\Rightarrow)$ Let $f : I \subset (0, \infty) \to \mathbb{R}$ (m_1, m_2)-AG convex function. Then, we have

$$\begin{aligned}(\ln \circ f)(m_1 ta + m_2(1-t)b) &\leqslant \ln\left([f(a)]^{m_1 t} [f(b)]^{m_2(1-t)}\right) \\ &= m_1 t f(\ln a) + m_2(1-t) f(\ln b).\end{aligned}$$

Therefore, the function $\ln \circ f$ is (m_1, m_2)-convex function on the interval $(0, \infty)$.

$(\Leftarrow)$ Let $\ln \circ f : (0, \infty) \to \mathbb{R}$, (m_1, m_2)-convex function on the interval $(0, \infty)$. Then, we get

$$\begin{aligned}(\ln \circ f)(m_1 ta + m_2(1-t)b) &\leqslant m_1 t f(\ln a) + m_2(1-t) f(\ln b) \\ e^{(\ln \circ f)(m_1 ta + m_2(1-t)b)} &\leqslant e^{m_1 t f(\ln a) + m_2(1-t) f(\ln b)}\end{aligned}$$

which means that the function $f(x)$ (m_1, m_2)-AG convex function on I. ■

Theorem 2.5 *If $f : I \to J$ is a (m_1, m_2)-convex and $g : J \to \mathbb{R}$ is a (m_1, m_2)-arithmetic geometrically convex function and nondecreasing, then $g \circ f : I \to \mathbb{R}$ is a (m_1, m_2)-AG convex function.*

Proof For $a, b \in I$ and $t \in [0, 1]$, we get

$$\begin{aligned}(g \circ f)(m_1 ta + m_2(1-t)b) &= g(f(m_1 ta + m_2(1-t)b)) \\ &\leqslant g(m_1 t f(a) + m_2(1-t) f(b)) \\ &\leqslant [g(f(a))]^{m_1 t} [g(f(a))]^{m_2(1-t)} \\ &\leqslant [(g \circ f)(a)]^{m_1 t} [(g \circ f)(b)]^{m_2(1-t)}.\end{aligned}$$

This completes the proof of theorem. ■

Theorem 2.6 *Let $b > 0$ and $f_\alpha : [a, b] \to \mathbb{R}$ be an arbitrary family of (m_1, m_2)-arithmetic geometrically convex functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J =$*

$\{u \in [a,b] : f(u) < \infty\}$ is nonempty, then J is an interval and f is an (m_1, m_2)-arithmetic geometrically convex function on J.

Proof Let $t \in [0,1]$ and $a, b \in J$ be arbitrary. Then

$$\begin{aligned} f\left(ta+(1-t)b\right) &= \sup_{\alpha} f_{\alpha}\left(m_1 t \frac{a}{m_1} + m_2(1-t)\frac{b}{m_2}\right) \\ &\leqslant \sup_{\alpha}\left(\left[f_{\alpha}\left(\frac{a}{m_1}\right)\right]^{m_1 t}\left[f_{\alpha}\left(\frac{b}{m_2}\right)\right]^{m_2(1-t)}\right) \\ &\leqslant \left[\sup_{\alpha} f_{\alpha}\left(\frac{a}{m_1}\right)\right]^{m_1 t}\left[\sup_{\alpha} f_{\alpha}\left(\frac{b}{m_2}\right)\right]^{m_2(1-t)} \\ &= \left[f\left(\frac{a}{m_1}\right)\right]^{m_1 t}\left[f\left(\frac{b}{m_2}\right)\right]^{m_2(1-t)} < \infty. \end{aligned}$$

So, this shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is an (m_1, m_2)-arithmetic geometrically convex function on J. This completes the proof of theorem. ■

Theorem 2.7 *Let $f : [0, b^*] \to \mathbb{R}$ a finite function on $\frac{a}{m_1}, \frac{b}{m_2} \in [0, b^*]$, (m_1, m_2)-arithmetic geometrically convex function with $m_1, m_2 \in (0,1]$. Then the function f is bounded on any closed interval $[a,b]$.*

Proof Let

$$M = \max\left\{f\left(\frac{a}{m_1}\right), f\left(\frac{b}{m_2}\right)\right\},$$

and $x \in [a,b]$ is an arbitrary point. Then there exist a $t \in [0,1]$ such that $x = ta + (1-t)b$. Thus, since $m_1 t + m_2(1-t) \leqslant 1$ we have

$$\begin{aligned} f(x) &= f\left(ta+(1-t)b\right) \\ &= f\left(m_1 t \frac{a}{m_1} + m_2(1-t)\frac{b}{m_2}\right) \\ &\leqslant \left[f\left(\frac{a}{m_1}\right)\right]^{m_1 t}\left[f\left(\frac{b}{m_2}\right)\right]^{m_2(1-t)} \\ &\leqslant M. \end{aligned}$$

Thus, the function f is upper bounded in interval $[a,b]$. Now we notice that any $z \in [a,b]$ can be written as $\frac{a+b}{2} + t$ for $|t| \leqslant \frac{b-a}{2}$, hence

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}\left(\frac{a+b}{2}+t\right) + \frac{1}{2}\left(\frac{a+b}{2}-t\right)\right) \\ &= f\left(\frac{m_1}{2}\left(\frac{\frac{a+b}{2}+t}{m_1}\right) + \frac{m_2}{2}\left(\frac{\frac{a+b}{2}-t}{m_2}\right)\right) \\ &\leqslant \left[f\left(\frac{\frac{a+b}{2}+t}{m_1}\right)\right]^{\frac{m_1}{2}}\left[f\left(\frac{\frac{a+b}{2}-t}{m_2}\right)\right]^{\frac{m_2}{2}}. \end{aligned}$$

In other word, we get

$$f\left(\frac{\frac{a+b}{2}+t}{m_1}\right) \geqslant \left\{\frac{f\left(\frac{a+b}{2}\right)}{\left[f\left(\frac{\frac{a+b}{2}-t}{m_2}\right)\right]^{\frac{m_2}{2}}}\right\}^{\frac{2}{m_1}} \geqslant \frac{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{2}{m_1}}}{M^{\frac{m_2}{m_1}}}$$

and similarly

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}\left(\frac{a+b}{2}+t\right)+\frac{1}{2}\left(\frac{a+b}{2}-t\right)\right) \\ &= f\left(\frac{m_2}{2}\left(\frac{\frac{a+b}{2}+t}{m_2}\right)+\frac{m_1}{2}\left(\frac{\frac{a+b}{2}-t}{m_1}\right)\right) \\ &\leqslant \left[f\left(\frac{\frac{a+b}{2}+t}{m_2}\right)\right]^{\frac{m_2}{2}}\left[f\left(\frac{\frac{a+b}{2}-t}{m_1}\right)\right]^{\frac{m_1}{2}}, \end{aligned}$$

hence, we get

$$f\left(\frac{\frac{a+b}{2}+t}{m_2}\right) \geqslant \left\{\frac{f\left(\frac{a+b}{2}\right)}{\left[f\left(\frac{\frac{a+b}{2}-t}{m_1}\right)\right]^{\frac{m_1}{2}}}\right\}^{\frac{2}{m_2}} \geqslant \frac{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{2}{m_2}}}{M^{\frac{m_1}{m_2}}}$$

and since $\frac{a+b}{2}+t$ is arbitrary in $[a,b]$, the function f is also bounded below in $[a,b]$. This completes the proof of theorem. ■

3. Hermite-Hadamard inequality for (m_1, m_2)-AG convex function

The goal of this section is to establish some inequalities of Hermite-Hadamard type integral inequalities for (m_1, m_2)-arithmetic geometrically convex functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on the interval $[a,b]$.

Theorem 3.1 *Let $f : [a,b] \to \mathbb{R}$ be an (m_1, m_2)-arithmetic geometrically convex function. If $a < b$ and $f \in L[a,b]$, then the following Hermite-Hadamard type integral inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leqslant \exp\left\{\frac{m_1^2}{2(b-a)}\int_a^b \ln f(m_1 x)dx + \frac{m_2^2}{2(b-a)}\int_a^b \ln f(m_2 y)dy\right\}$$

and

$$\frac{1}{m_2 b - m_1 a}\int_{m_1 a}^{m_2 b} f(x)dx = L\left([f(a)]^{m_1}, [f(b)]^{m_2}\right) \leqslant A\left([f(a)]^{m_1}, [f(b)]^{m_2}\right).$$

Proof Firstly, from the property of the (m_1, m_2)-arithmetic geometrically convex function of f, we write

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{\left[m_1 t\frac{a}{m_1}+m_2(1-t)\frac{b}{m_2}\right]+\left[m_1(1-t)\frac{a}{m_1}+m_2 t\frac{b}{m_2}\right]}{2}\right) \\ &= f\left(\frac{m_1}{2}\left[t\frac{a}{m_1}+\frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right]+\frac{m_2}{2}\left[t\frac{b}{m_2}+\frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right]\right) \\ &\leqslant \left[f\left(t\frac{a}{m_1}+\frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right)\right]^{\frac{m_1}{2}}\left[f\left(t\frac{b}{m_2}+\frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right)\right]^{\frac{m_2}{2}}. \end{aligned}$$

By taking the logarithm on the both sides of the above inequality we get

$$\begin{aligned} \ln f\left(\frac{a+b}{2}\right) &\leqslant \ln\left\{\left[f\left(t\frac{a}{m_1}+\frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right)\right]^{\frac{m_1}{2}}\left[f\left(t\frac{b}{m_2}+\frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right)\right]^{\frac{m_2}{2}}\right\} \\ &= \frac{m_1}{2}\ln f\left(t\frac{a}{m_1}+\frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right)+\frac{m_2}{2}\ln f\left(t\frac{b}{m_2}+\frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right). \end{aligned}$$

Now, if we take integral in the last inequality with respect to $t\in[0,1]$ and choose $m_1x = ta+(1-t)b$ and $m_2y = tb+(1-t)a$, we deduce that

$$\begin{aligned} \ln f\left(\frac{a+b}{2}\right) &\leqslant \frac{m_1}{2}\int_0^1 \ln f\left(t\frac{a}{m_1}+\frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right)dt \\ &\quad +\frac{m_2}{2}\int_0^1 \ln f\left(t\frac{b}{m_2}+\frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right)dt \\ &= \frac{m_1^2}{2(b-a)}\int_a^b \ln f(m_1x)dx+\frac{m_2^2}{2(b-a)}\int_a^b \ln f(m_2y)dy \\ f\left(\frac{a+b}{2}\right) &\leqslant \exp\left\{\frac{m_1^2}{2(b-a)}\int_a^b \ln f(m_1x)dx+\frac{m_2^2}{2(b-a)}\int_a^b \ln f(m_2y)dy\right\}. \end{aligned}$$

Secondly, by using the property of the (m_1,m_2)-arithmetic geometrically convex function of f, if the variable is changed as $u=[f(a)]^{m_1t}[f(b)]^{m_2(1-t)}$, then

$$\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx = L\left([f(a)]^{m_1},[f(b)]^{m_2}\right)\leqslant A\left([f(a)]^{m_1},[f(b)]^{m_2}\right).$$

This completes the proof of theorem. ■

Corollary 3.2 *If we take $m_1=m_2=1$ in Theorem 3.1, we get*

$$\ln\ f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b \ln f(x)dx\leqslant L\left([f(a)],[f(b)]\right)\leqslant A\left([f(a)],[f(b)]\right).$$

This inequality coincides with the inequality in [4].

4. Some new inequalities for (m_1,m_2)-AG convex functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is (m_1, m_2)-AG convex function. Will use the following lemma to obtain our main results.

Lemma 4.1 [10]*Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$. If $f' \in L[m_1a, m_2b]$, then the following equality*

$$\frac{1}{m_2b - m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a + m_2b}{2}\right)$$
$$= (m_2b - m_1a)\left[\int_0^{\frac{1}{2}} tf'(m_1ta + m_2(1-t)b)\,dt + \int_{\frac{1}{2}}^1 (t-1)f'(m_1ta + m_2(1-t)b)\,dt\right]$$

holds for $t \in [0,1]$ and $m_1, m_2 \in (0,1]^2$.

Theorem 4.2 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$ and $f' \in L[m_1a, m_2b]$. If $|f'|$ is (m_1, m_2)-AG convex on the interval $[m_1a, m_2b]$, then the following equality*

$$\left|\frac{1}{m_2b - m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a + m_2b}{2}\right)\right|$$
$$\leqslant \frac{2(m_2b - m_1a)}{\left(\ln|f'(b)|^{m_2} - \ln|f'(a)|^{m_1}\right)^2}\left[A\left(|f'(a)|^{m_1}, |f'(b)|^{m_2}\right) - G\left(|f'(a)|^{m_1}, |f'(b)|^{m_2}\right)\right]$$

holds for $t \in [0,1]$ and $m_1, m_2 \in (0,1]^2$.

Proof Using Lemma 4.1 and the following inequality

$$\left|f'(m_1ta + m_2(1-t)b)\right| \leqslant |f'(a)|^{m_1t}\,|f'(b)|^{m_2(1-t)},$$

we get

$$\left|\frac{1}{m_2b - m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a + m_2b}{2}\right)\right|$$
$$\leqslant \left|(m_2b - m_1a)\left[\int_0^{\frac{1}{2}} tf'(m_1ta + m_2(1-t)b)\,dt + \int_{\frac{1}{2}}^1 (t-1)f'(m_1ta + m_2(1-t)b)\,dt\right]\right|$$
$$\leqslant (m_2b - m_1a)\left[\int_0^{\frac{1}{2}} t\left|f'(m_1ta + m_2(1-t)b)\right|dt + \int_{\frac{1}{2}}^1 |t-1|\left|f'(m_1ta + m_2(1-t)b)\right|dt\right]$$
$$\leqslant (m_2b - m_1a)\left[\int_0^{\frac{1}{2}} t\,|f'(a)|^{m_1t}\,|f'(b)|^{m_2(1-t)}\,dt + \int_{\frac{1}{2}}^1 |t-1|\,|f'(a)|^{m_1t}\,|f'(b)|^{m_2(1-t)}\,dt\right]$$
$$= (m_2b - m_1a)\left[\frac{|f'(a)|^{m_1} + |f'(b)|^{m_2} - 2\sqrt{|f'(a)|^{m_1}|f'(b)|^{m_2}}}{\left(\ln|f'(b)|^{m_2} - \ln|f'(a)|^{m_1}\right)^2}\right]$$
$$= \frac{2(m_2b - m_1a)}{\left(\ln|f'(b)|^{m_2} - \ln|f'(a)|^{m_1}\right)^2}\left[A\left(|f'(a)|^{m_1}, |f'(b)|^{m_2}\right) - G\left(|f'(a)|^{m_1}, |f'(b)|^{m_2}\right)\right].$$

This completes the proof of theorem. ■

Corollary 4.3 *Under the conditions of Theorem 4.2, If we take $m_1 = m_2 = 1$, then we get the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$
$$\leqslant \frac{2(b-a)}{(\ln|f'(b)| - \ln|f'(a)|)^2}\left[A\left(\left|f'(a)\right|, \left|f'(b)\right|\right) - G\left(\left|f'(a)\right|, \left|f'(b)\right|\right)\right].$$

Theorem 4.4 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$ and $f' \in L[m_1a, m_2b]$, and let $q > 1$. If $|f'|$ is (m_1, m_2)-AG convex on the interval $[m_1a, m_2b]$, then the following equality*

$$\left|\frac{1}{m_2b - m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right|$$
$$\leqslant (m_2b - m_1a)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left\{\left(\frac{G\left(|f'(a)|^{qm_1}, |f'(b)|^{qm_2}\right) - |f'(b)|^{qm_2}}{(\ln|f'(b)|^{qm_2} - \ln|f'(a)|^{qm_1})^2}\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{|f'(a)|^{qm_1} - G\left(|f'(a)|^{qm_1}, |f'(b)|^{qm_2}\right)}{(\ln|f'(b)|^{qm_2} - \ln|f'(a)|^{qm_1})^2}\right)^{\frac{1}{q}}\right\}$$

holds for $t \in [0, 1]$ and $m_1, m_2 \in (0, 1]^2$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Using Lemma 4.1, Hölder's integral inequality and the inequality

$$\left|f'\left(m_1ta + m_2(1-t)b\right)\right|^q \leqslant \left|f'(a)\right|^{qm_1t}\left|f'(b)\right|^{qm_2(1-t)},$$

we obtain

$$\left|\frac{1}{m_2b - m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right|$$
$$\leqslant \left|(m_2b - m_1a)\left[\int_0^{\frac{1}{2}} tf'\left(m_1ta + m_2(1-t)b\right)dt + \int_{\frac{1}{2}}^1 (t-1)f'\left(m_1ta + m_2(1-t)b\right)dt\right]\right|$$
$$\leqslant (m_2b - m_1a)\left[\int_0^{\frac{1}{2}} t\left|f'\left(m_1ta + m_2(1-t)b\right)\right|dt + \int_{\frac{1}{2}}^1 |t-1|\left|f'\left(m_1ta + m_2(1-t)b\right)\right|dt\right]$$
$$\leqslant (m_2b - m_1a)\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'\left(m_1ta + m_2(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_{\frac{1}{2}}^1 |t-1|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'\left(m_1ta + m_2(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leqslant (m_2b - m_1a)\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left|f'(a)\right|^{qm_1t}\left|f'(b)\right|^{qm_2(1-t)} dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^1 |t-1|^p\, dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left|f'(a)\right|^{qm_1t}\left|f'(b)\right|^{qm_2(1-t)} dt\right)^{\frac{1}{q}}\right]$$

$$= (m_2b - m_1a)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left\{\left(\frac{G\left(|f'(a)|^{qm_1}, |f'(b)|^{qm_2}\right) - |f'(b)|^{qm_2}}{(\ln|f'(b)|^{qm_2} - \ln|f'(a)|^{qm_1})^2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{|f'(a)|^{qm_1} - G\left(|f'(a)|^{qm_1}, |f'(b)|^{qm_2}\right)}{(\ln|f'(b)|^{qm_2} - \ln|f'(a)|^{qm_1})^2}\right)^{\frac{1}{q}}\right\},$$

where

$$\int_0^{\frac{1}{2}} t^p dt = \int_{\frac{1}{2}}^1 |t-1|^p\, dt = \frac{1}{(p+1)2^{p+1}}.$$

This completes the proof of theorem. ■

Remark 4.5 Under the conditions of Theorem 4.4, if we take $m_1 = m_2 = 1$, then we get the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leqslant (b-a)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\frac{G\left(|f'(a)|^q, |f'(b)|^q\right) - |f'(b)|^q}{(\ln|f'(b)|^q - \ln|f'(a)|^q)^2}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q - G\left(|f'(a)|^q, |f'(b)|^q\right)}{(\ln|f'(b)|^q - \ln|f'(a)|^q)^2}\right)\right\}^{\frac{1}{q}}$$

Theorem 4.6 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$ and $f' \in L[m_1a, m_2b]$, and let $q \geqslant 1$. If $|f'|$ is (m_1, m_2)-AG convex on the interval $[m_1a, m_2b]$, then the foUnder the conditions of Theoreml-lowing equality*

$$\left|\frac{1}{m_2b - m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a + m_2b}{2}\right)\right|$$
$$\leqslant \frac{(m_2b - m_1a)}{2^{1-\frac{1}{q}}}\left[B_1^{\frac{1}{q}}(a, b, q, f) + B_2^{\frac{1}{q}}(a, b, q, f)\right]$$

holds for $t \in [0, 1]$ and $m_1, m_2 \in (0, 1]^2$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Using Lemma 4.1, well-known power mean inequality and the inequality

$$\left|f'(m_1ta + m_2(1-t)b)\right|^q \leqslant \left|f'(a)\right|^{qm_1t}\left|f'(b)\right|^{qm_2(1-t)},$$

we get

$$\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right|$$
$$\leqslant \left|(m_2b-m_1a)\left[\int_0^{\frac{1}{2}} tf'\left(m_1ta+m_2(1-t)b\right)dt + \int_{\frac{1}{2}}^1 (t-1)f'\left(m_1ta+m_2(1-t)b\right)dt\right]\right| \tag{1}$$

$$\leqslant (m_2b-m_1a)\left[\int_0^{\frac{1}{2}} t\left|f'\left(m_1ta+m_2(1-t)b\right)\right|dt + \int_{\frac{1}{2}}^1 |t-1|\left|f'\left(m_1ta+m_2(1-t)b\right)\right|dt\right]$$
$$\leqslant (m_2b-m_1a)\left[\left(\int_0^{\frac{1}{2}} tdt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^q\left|f'\left(m_1ta+m_2(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1 |t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 |t-1|^q\left|f'\left(m_1ta+m_2(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leqslant (m_2b-m_1a)\left[\left(\int_0^{\frac{1}{2}} tdt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^q\left|f'(a)\right|^{qm_1t}\left|f'(b)\right|^{qm_2(1-t)} dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1 |t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 |t-1|^q\left|f'(a)\right|^{qm_1t}\left|f'(b)\right|^{qm_2(1-t)} dt\right)^{\frac{1}{q}}\right]$$
$$\leqslant (m_2b-m_1a)\left[\left(\frac{1}{8}\right)^{1-\frac{1}{q}} B_1^{\frac{1}{q}}(a,b,q,f) + \left(\frac{1}{8}\right)^{1-\frac{1}{q}} B_2^{\frac{1}{q}}(a,b,q,f)\right]$$
$$= \frac{(m_2b-m_1a)}{8^{1-\frac{1}{q}}}\left[B_1^{\frac{1}{q}}(a,b,q,f) + B_2^{\frac{1}{q}}(a,b,q,f)\right]$$

where

$$\int_0^{\frac{1}{2}} tdt = \int_{\frac{1}{2}}^1 |t-1|\,dt = \frac{1}{8}$$
$$B_1(a,b,q,f) \; : = \int_0^{\frac{1}{2}} t^q\left|f'(a)\right|^{qm_1t}\left|f'(b)\right|^{qm_2(1-t)} dt$$
$$B_2(a,b,q,f) \; : = \int_{\frac{1}{2}}^1 |t-1|^q\left|f'(a)\right|^{qm_1t}\left|f'(b)\right|^{qm_2(1-t)} dt$$

where integrals can be calculated as above. ∎

Corollary 4.7 *Under the conditions of Theorem 4.6, if we take* $m_1 = m_2 = 1$ *, then we get the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{(b-a)}{8^{1-\frac{1}{q}}}\left[B_1^{\frac{1}{q}}(a,b,q,f) + B_2^{\frac{1}{q}}(a,b,q,f)\right]$$

Corollary 4.8 *Under the conditions of Theorem 4.6, if we take $q = 1$, then we get the following inequality:*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ &\leqslant (m_2b - m_1a)\left[B_1(a,b,1,f) + B_2(a,b,1,f)\right] \\ &= \frac{2(m_2b-m_1a)}{\left(\ln|f'(b)|^{m_2} - \ln|f'(a)|^{m_1}\right)^2}\left[A\left(|f'(a)|^{m_1}, |f'(b)|^{m_2}\right) - G\left(|f'(a)|^{m_1}, |f'(b)|^{m_2}\right)\right] \end{aligned}$$

This inequality coincides with the inequality in Theorem 4.2.

Corollary 4.9 *Under the conditions of Theorem 4.6, if we take $m_1 = m_2 = 1$ and $q = 1$, then we get the following inequality:*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ &\leqslant \frac{2(b-a)}{\left(\ln|f'(b)| - \ln|f'(a)|\right)^2}\left[A\left(|f'(a)|, |f'(b)|\right) - G\left(|f'(a)|, |f'(b)|\right)\right]. \end{aligned}$$

Corollary 4.10 *Under the conditions of Theorem 4.6, we can also write the following inequality:*

$$\begin{aligned} &\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right| \\ &\leqslant \frac{m_2b-m_1a}{8^{1-\frac{1}{q}}}\left[\frac{|f'(b)|^{qm_2} - G\left(|f'(a)|^{qm_1}, |f'(b)|^{qm_2}\right)}{\ln|f'(b)|^{qm_2} - \ln|f'(a)|^{qm_1}}\right]^{\frac{1}{q}} \\ &+\frac{m_2b-m_1a}{8^{1-\frac{1}{q}}}\left[\frac{G\left(|f'(a)|^{qm_1}, |f'(b)|^{qm_2}\right) - |f'(a)|^{qm_1}}{\ln|f'(b)|^{qm_2} - \ln|f'(a)|^{qm_1}}\right]^{\frac{1}{q}}, \end{aligned} \tag{2}$$

where G is the geometric mean.

Proof If we use the inequalities $t \leqslant 1$, $|t-1| \leqslant 1$, $|t-1|^q \leqslant 1$ and $t^q \leqslant 1$ in the the inequality (1), we obtain the desired result. ■

Corollary 4.11 *If we take $m_1 = m_2 = 1$ in the inequality (2), we obtain the following inequality:*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ &\leqslant \frac{b-a}{8^{1-\frac{1}{q}}}\left[\frac{|f'(b)|^q - G\left(|f'(a)|^q, |f'(b)|^q\right)}{\ln|f'(b)|^q - \ln|f'(a)|^q}\right]^{\frac{1}{q}} + \frac{b-a}{8^{1-\frac{1}{q}}}\left[\frac{G\left(|f'(a)|^q, |f'(b)|^q\right) - |f'(a)|^q}{\ln|f'(b)|^q - \ln|f'(a)|^q}\right]^{\frac{1}{q}}. \end{aligned}$$

Corollary 4.12 *If we take $m_1 = m_2 = 1$ and $q = 1$ in the inequality (2), we obtain the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{2(b-a)}{\ln|f'(b)| - \ln|f'(a)|}A\left(|f'(a)|, |f'(b)|\right).$$

5. Conclusions

In this article, the inequalities obtained with Hölder and power-mean integral inequalities are obtained (m_1, m_2)-logarithmically convex (AG-convex) functions. This method can be applied to different classes of convexity.

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