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(m_1, m_2)-Convexity and Some New Hermite-Hadamard Type Inequalities

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Abstract. In this manuscript, a new class of extended (m_1, m_2)-convex and concave functions is introduced. After some properties of (m_1, m_2)-convex functions have been given, the inequalities obtained with Hölder and Hölder-İşcan and power-mean and improwed power-mean integral inequalities have been compared and it has been shown that the inequality with Hölder-İşcan inequality gives a better approach than with Hölder integral inequality and improwed power-mean inequality gives a better approach than with power-mean inequality.

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1. Introduction

Definition 1.1 A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then the function f is said to be concave on interval $I \neq \emptyset$.

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Definition 1.2 $f:[a,b]\to\mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{f(a)+f(b)}{2}$$

is known as the Hermite-Hadamard inequality.

Convexity theory has appeared as a powerful technique to study a wide class of related problems in pure and applied sciences. Some improvements and refinements of the H-H inequality on convex functions have been extensively investigated by a number of authors (e.g., [1, 5]) and the authors obtained a new refinement of the H-H inequality for convex functions. Also, in recent years, readers can find different convex functions classes and Hermite-Hadamard type inequalities obtained for these classes in ([1, 2, 7, 8, 10, 11, 15, 19]) and references therein.

Definition 1.3 ([18]) The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be m-convex function, where $m\in[0,1]$; if we have

$$f\left(tx+m(1-t)y\right)\leqslant tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$. We say that the function f is m-concave function if $(-f)$ is m-convex.

Obviously, for $m=1$ the above definition recaptures the concept of standard convex functions on $[a,b]$; and for $m=0$ the concept star-shaped functions. For many papers connected with m-convex and (α,m)-convex functions see ([2, 4, 12–14, 16, 17, 20]) and the references therein.

In [6], İşcan gave a refinement of the Hölder integral inequality as follows:

Theorem 1.4 (Hölder-İşcan integral inequality [6]) *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b|f(x)g(x)|\,dx\leqslant\frac{1}{b-a}\left\{\left(\int_a^b(b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b(b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_a^b(x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b(x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}$$

An refinement of power-mean integral inequality as a different version of the Hölder-İşcan integral inequality can be given as follows:

Theorem 1.5 (Improved power-mean integral inequality [9]) *Let $q\geqslant1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b|f(x)g(x)|\,dx\leqslant\frac{1}{b-a}\left\{\left(\int_a^b(b-x)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b(b-x)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_a^b(x-a)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b(x-a)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}$$

2. Definition of (m_1, m_2)-convex function

In this section, we begin by setting some algebraic properties for (m_1, m_2)-convex functions.

Definition 2.1 The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be (m_1, m_2)-convex, if

$$f(m_1tx + m_2(1-t)y) \leqslant m_1tf(x) + m_2(1-t)f(y)$$

for all $x, y \in I$, $t \in [0,1]$ and $(m_1, m_2) \in (0,1]^2$.

We will denote by $K_{m_1,m_2}(b)$ the class of all (m_1, m_2)-convex functions on interval I for which $f(0) \leqslant 0$.

Definition 2.2 Let $f : [0, b] \to \mathbb{R}$. If $f(tx) \leqslant tf(x)$ is valid for all $x \in [0, b]$, then we say that $f(x)$ is an starshaped function on $[0, b]$.

Definition 2.3 Let $f : [0, b] \to \mathbb{R}$ and $m_1 \in (0, 1]$. If $f(m_1tx) \leqslant m_1tf(x)$ is valid for all $x \in [0, b]$ and $t \in [0, 1]$, then we say that the function $f(x)$ is an m_1-starshaped function on $[0, b]$. Specially, for $m_1 = 1$, we have $f(tx) \leqslant tf(x)$.

Remark 2.4 In Definition 2.1, if we choose $m_2 = 0$, we get the concept of (α, m_1)-starshaped functions on $[0, b]$.

Proposition 2.5 *If the function f is in the class $K_{m_1,m_2}(b)$, then it is m_1-starshaped.*

Proof For any $x \in [0, b]$, $t \in [0, 1]$ and $(m_1, m_2) \in (0, 1]^2$, we have

$$f(m_1tx) = f(m_1tx + m_2(1-t).0) \leqslant m_1tf(x) + m_2(1-t)f(0) \leqslant m_1tf(x).$$

Specially, for $m_1 = 1$, we have $f(tx) \leqslant tf(x)$. Lemma is proved. ■

Following theorems can be proved easily.

Theorem 2.6 *Let $f, g : [0, b] \to \mathbb{R}$. If f and g are (m_1, m_2)-convex, then $f + g$ is (m_1, m_2)-convex and for $c \in \mathbb{R}$ $(c \geqslant 0)$ cf is (m_1, m_2)-convex.*

Theorem 2.7 *Let f be a (m_1, m_2)-convex function. If the function g is an (m_1, m_2)-convex and increasing, then the function gof is an (m_1, m_2)-convex.*

Theorem 2.8 *Let $f, g : [0, b] \to \mathbb{R}$ are both nonnegative and monotone (increasing or decreasing). If f and g are (m_1, m_2)-convex function, then fg is (m_1, m_2)-convex function under the condition $[f(x) - f(y)][g(y) - g(x)] \leqslant 0$.*

Theorem 2.9 *Let $m_1, m_2 \in [0, 1]$,$b > 0$ and $f_\alpha : [0, b] \to \mathbb{R}$ be an arbitrary family of (m_1, m_2)-convex functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If*

$$J = \left\{ u \in [0, b] : \frac{u}{m_1}, \frac{u}{m_2} \in [0, b] \ \textit{and}\ f(u), f\left(\frac{u}{m_1}\right), f\left(\frac{u}{m_2}\right) < \infty \right\}$$

is nonempty, then J is an interval and f is (m_1, m_2)-convex on J.

Theorem 2.10 *Let $f : [0, b^*] \to \mathbb{R}$ be a finite function on $\frac{a}{m_1}, \frac{b}{m_2} \in [0, b^*]$, (m_1, m_2)-convex with $m_1, m_2 \in (0, 1]$. Then the function f is on bounded any closed interval $[a, b]$.*

3. Hermite-Hadamard inequality for (m_1, m_2)-convex functions

The goal of this paper is to develop concepts of the (m_1, m_2)-convex functions and to establish some inequalities of H-H type for these classes of functions.

Theorem 3.1 *Let the function $f : [0, b^*] \to \mathbb{R}$, $b^* > 0$, be an (m_1, m_2)-convex functions with $m_1, m_2 \in (0, 1]^2$. If $0 \leqslant a < b < b^*$ and $f \in L[a, b]$, then the following inequalities holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leqslant \min\left\{\frac{m_1 f\left(\frac{a}{m_1}\right) + m_2 f\left(\frac{b}{m_2}\right)}{2}, \frac{m_1 f\left(\frac{b}{m_1}\right) + m_2 f\left(\frac{a}{m_2}\right)}{2}\right\}.$$

Proof By using (m_1, m_2)-convexity of the function f, if the variable is changed as $u = ta + (1-t)b$, then

$$\begin{aligned} I = \int_0^1 f(ta + (1-t)b)\, dt = \frac{1}{b-a}\int_a^b f(u)du \\ \leqslant \int_0^1 \left[tm_1 f\left(\frac{a}{m_1}\right) + (1-t)m_2 f\left(\frac{b}{m_2}\right)\right] dt \\ = \frac{m_1 f\left(\frac{a}{m_1}\right) + m_2 f\left(\frac{b}{m_2}\right)}{2} \end{aligned}$$

and similarly for $z = tb + (1-t)a$, then

$$\begin{aligned} I = \int_0^1 f(tb + (1-t)a)\, dt = \frac{1}{b-a}\int_a^b f(z)dz \leqslant \int_0^1 [tf(b) + (1-t)f(a)]\, dt \\ = \frac{m_1 f\left(\frac{b}{m_1}\right) + m_2 f\left(\frac{a}{m_2}\right)}{2}. \end{aligned}$$

So, we have

$$\frac{1}{b-a}\int_a^b f(x)dx \leqslant \min\left\{\frac{m_1 f\left(\frac{a}{m_1}\right) + m_2 f\left(\frac{b}{m_2}\right)}{2}, \frac{m_1 f\left(\frac{b}{m_1}\right) + m_2 f\left(\frac{a}{m_2}\right)}{2}\right\}.$$

This completes the proof of theorem. ■

Remark 3.2 Under the conditions of Theorem 3.1, if $m_1 = 1,\ m_2 = m$, then, the following inequality holds:

$$\frac{1}{b-a}\int_a^b f(x)dx \leqslant \min\left\{\frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2}\right\}.$$

This inequality is the Hermite-Hadamard inequality for the m-convex functions [1].

Remark 3.3 Under the conditions of Theorem 3.1, if $m_1 = m_2 = 1$, then, the following inequality holds:

$$\frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a) + f(b)}{2}.$$

This inequality is the Hermite-Hadamard inequality for the convex functions [2].

Theorem 3.4 *Let the function* $f : [0, b^*] \to \mathbb{R},\ b^* > 0$,*be an* (m_1, m_2)*-convex functions with* $m_1, m_2 \in (0,1]^2$*. If* $m = \min\{m_1, m_2\}$, $0 \leqslant a < b < \frac{b}{m} < b^*$ *and* $f \in L\left[a, \frac{b}{m}\right]$*, then the following inequalities holds:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2(b-a)} \int_a^b \left[m_1 f\left(\frac{x}{m_1}\right) + m_2 f\left(\frac{x}{m_2}\right)\right] dx.$$

Proof By the (m_1, m_2)-convexity of the function f, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{\left[m_1 t \frac{a}{m_1} + m_2(1-t)\frac{b}{m_2}\right] + \left[m_1(1-t)\frac{a}{m_1} + m_2 t \frac{b}{m_2}\right]}{2}\right) \\ &= f\left(\frac{1}{2}m_1\left[t\frac{a}{m_1} + \frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right] + \frac{1}{2}m_2\left[t\frac{b}{m_2} + \frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right]\right) \\ &\leqslant \frac{1}{2}m_1 f\left(\frac{ta+(1-t)b}{m_1}\right) + \frac{1}{2}m_2 f\left(\frac{tb+(1-t)a}{m_2}\right). \end{aligned}$$

Now, if we take integral the last inequality on $t \in [0,1]$ and choose $x = ta + (1-t)b$ and $y = tb + (1-t)a$, we deduce

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant \frac{1}{2}\frac{m_1}{b-a}\int_a^b f\left(\frac{x}{m_1}\right)dx + \frac{1}{2}\frac{m_2}{b-a}\int_a^b f\left(\frac{y}{m_2}\right)dy \\ &= \frac{1}{2}\frac{1}{b-a}\int_a^b \left[m_1 f\left(\frac{x}{m_1}\right) + m_2 f\left(\frac{x}{m_2}\right)\right] dx. \end{aligned}$$

This completes the proof of theorem. ■

Remark 3.5 Under the conditions of Theorem 3.4, if $m_1 = m_2 = 1$, then, the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx$$

This inequality is the left hand side of Hermite-Hadamard inequality for the convex functions [17].

4. Some new inequalities for (m_1, m_2)-convexity

Dragomir and Agarwal in [3] used the following lemma to prove Theorems.

Lemma 4.1 **([3])** *Let* $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable mapping on* I°, $a, b \in I^\circ$ *with* $a < b$*. If* $f' \in L[a, b]$*, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t) f'(ta + (1-t)b)\, dt.$$

The main purpose of this section is to establish new estimations and refinements of the Hermite-Hadamard inequality for functions whose first derivatives in absolute value are (m_1, m_2)-convex. Also, the inequalities obtained with Hölder and Hölder-İşcan integral inequalities have been compared and it has been shown that the inequality with Hölder-İşcan inequality gives a better approach than with Hölder integral inequality. Similarly, it will be shown that the result obtained with improwed power-mean integral inequalities gives a better approach than power-mean integral inequalities. For this, we will use the following lemma.

Lemma 4.2 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$. If $f' \in L[m_1a, m_2b]$, then the following equality*

$$\begin{aligned} &\frac{f(m_1a)+f(m_2b)}{2} - \frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx \\ &= \frac{m_2b-m_1a}{2}\int_0^1 (1-2t)f'\left(m_1ta+m_2(1-t)b\right)dt \end{aligned}$$

holds for $t \in [0,1]$ and $m_1, m_2 \in (0,1]^2$.

Proof Integrating by parts, it can be seen easily that

$$\begin{aligned} &\frac{m_2b-m_1a}{2}\int_0^1 (1-2t)f'\left(m_1ta+m_2(1-t)b\right)dt \\ &= \frac{m_2b-m_1a}{2}\left[-(1-2t)\frac{f\left(m_1ta+m_2(1-t)b\right)}{m_2b-m_1a}\Big|_0^1 - 2\int_0^1 \frac{f\left(m_1ta+m_2(1-t)b\right)}{m_2b-m_1a}dt\right] \\ &= \frac{f(m_1a)+f(m_2b)}{2} - \frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx. \end{aligned}$$

■

Theorem 4.3 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$ and $f' \in L[m_1a, m_2b]$. If $|f'|$ is (m_1, m_2)-convex on interval $[m_1a, m_2b]$, then the following inequality*

$$\frac{f(m_1a)+f(m_2b)}{2} - \frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx \leqslant \frac{(m_2b-m_1a)\left[m_1|f'(a)|+m_2|f'(b)|\right]}{8}$$

holds for $t \in [0,1]$ and $m_1, m_2 \in (0,1]^2$.

Proof Using Lemma 4.2 and the inequality

$$f'\left(m_1ta+m_2(1-t)b\right) \leqslant m_1tf'(a)+m_2(1-t)f'(b),$$

we get

$$\begin{aligned}
&\left|\frac{f(m_1a)+f(m_2b)}{2}-\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b}f(x)dx\right| \\
&\leqslant \left|\frac{m_2b-m_1a}{2}\int_0^1(1-2t)f'\left(m_1ta+m_2(1-t)b\right)dt\right| \\
&\leqslant \frac{m_2b-m_1a}{2}\int_0^1|1-2t|\left|f'\left(m_1ta+m_2(1-t)b\right)\right|dt \\
&\leqslant \frac{m_2b-m_1a}{2}\int_0^1|1-2t|\left[m_1t\left|f'(a)\right|+m_2(1-t)\left|f'(b)\right|\right]dt \\
&= \frac{m_2b-m_1a}{2}\left[m_1\left|f'(a)\right|\int_0^1 t\,|1-2t|\,dt+m_2\left|f'(b)\right|\int_0^1(1-t)\,|1-2t|\,dt\right] \\
&= \frac{m_2b-m_1a}{2}\left[\frac{m_1|f'(a)|}{4}+\frac{m_2|f'(b)|}{4}\right] \\
&= \frac{(m_2b-m_1a)\left[m_1|f'(a)|+m_2|f'(b)|\right]}{8}.
\end{aligned}$$

■

Remark 4.4 When $m_1 = m_2 = 1$, our result coincides with [3].

Theorem 4.5 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$ and let $q > 1$. If the mapping $|f'|^q$ is (m_1, m_2)-convex on interval $[m_1a, m_2b]$, then the following inequality*

$$\begin{aligned}
&\frac{f(m_1a)+f(m_2b)}{2}-\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b}f(x)dx \\
&\leqslant \frac{m_2b-m_1a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{m_1|f'(a)|^q+m_2|f'(b)|^q}{2} \qquad (1)
\end{aligned}$$

holds for $t \in [0,1]$ and $m_1, m_2 \in (0,1]^2$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof Using Lemma 4.2, Hölder's integral inequality and inequality

$$\left|f'\left(m_1ta+m_2(1-t)b\right)\right|^q \leqslant m_1t\left|f'(a)\right|^q+m_2(1-t)\left|f'(b)\right|^q$$

which is the (m_1, m_2)-concexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(m_1a)+f(m_2b)}{2}-\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b}f(x)dx\right| \\
&\leqslant \frac{m_2b-m_1a}{2}\left(\int_0^1|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(m_1ta+m_2(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leqslant \frac{m_2b-m_1a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[m_1t\left|f'(a)\right|^q+m_2(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&= \frac{m_2b-m_1a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{m_1|f'(a)|^q+m_2|f'(b)|^q}{2}.
\end{aligned}$$

■

Remark 4.6 When $m_1 = m_2 = 1$, our result coincides with [3].

Theorem 4.7 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$ and let $f' \in L[m_1a, m_2b]$. If the mapping $|f'|^q$ is (m_1, m_2)-convex on interval $[m_1a, m_2b]$, for $q \geqslant 1$, then the following inequality*

$$\frac{f(m_1a)+f(m_2b)}{2} - \frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx \leqslant \frac{m_2b-m_1a}{4}A^{\frac{1}{q}}\left(m_1\left|f'(a)\right|^q, m_2\left|f'(b)\right|^q\right) \tag{2}$$

holds for $t \in [0,1]$ and $m_1, m_2 \in (0,1]^2$, where $\frac{1}{p} + \frac{1}{q} = 1$ and A is the arithmetic mean.

Proof From Lemma 4.2, well known power-mean integral inequality and (m_1, m_2)-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{f(m_1a)+f(m_2b)}{2} - \frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx\right| \\
&\leqslant \frac{m_2b-m_1a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left|f'\left(m_1ta+m_2(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leqslant \frac{m_2b-m_1a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left[m_1t\left|f'(a)\right|^q + m_2(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&= \frac{m_2b-m_1a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(m_1\left|f'(a)\right|^q\int_0^1 t\,|1-2t|\,dt + m_2\left|f'(b)\right|^q\int_0^1 (1-t)\,|1-2t|\,dt\right)^{\frac{1}{q}} \\
&= \frac{m_2b-m_1a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{m_1\left|f'(a)\right|^q}{4} + \frac{m_2\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}} \\
&= \frac{m_2b-m_1a}{4}A^{\frac{1}{q}}\left(m_1\left|f'(a)\right|^q, m_2\left|f'(b)\right|^q\right).
\end{aligned}$$

■

Corollary 4.8 *Under the assumption of Theorem 4.7 with $q = 1$, we get*

$$\frac{f(m_1a)+f(m_2b)}{2} - \frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx \leqslant \frac{m_2b-m_1a}{4}A\left(m_1\left|f'(a)\right|, m_2\left|f'(b)\right|\right). \tag{3}$$

Corollary 4.9 *Under the assumption of Theorem 4.7 with $m_1 = m_2 = 1$, we get*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{b-a}{4}A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right).$$

Corollary 4.10 *Under the assumption of Theorem 4.7 with $m_1 = m_2 = q = 1$, we get*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{b-a}{4}A\left(\left|f'(a)\right|, \left|f'(b)\right|\right). \tag{4}$$

Theorem 4.11 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$ and let $q > 1$. If the mapping $|f'|^q$ is (m_1, m_2)-convex on* *interval* $[m_1a, m_2b]$*, then the following inequality*

$$\begin{aligned}
&\frac{f(m_1a)+f(m_2b)}{2}-\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b}f(x)dx \qquad (5)\\
&\leqslant \frac{m_2b-m_1a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{m_1\left|f'(a)\right|^q}{6}+\frac{2m_2\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}\\
&+\frac{m_2b-m_1a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{2m_1\left|f'(a)\right|^q}{6}+\frac{m_2\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}
\end{aligned}$$

holds for $t \in [0, 1]$ *and* $m_1, m_2 \in (0, 1]^2$*, where* $\frac{1}{p}+\frac{1}{q}=1$.

Proof Using Lemma 4.2, Hölder-İşcan integral inequality and the (m_1, m_2)-concexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(m_1a)+f(m_2b)}{2}-\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b}f(x)dx\right|\\
&\leqslant \frac{m_2b-m_1a}{2}\int_0^1|1-2t|\left|f'\left(m_1ta+m_2(1-t)b\right)\right|dt\\
&\leqslant \frac{m_2b-m_1a}{2}\left(\int_0^1(1-t)\left|1-2t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left|f'\left(m_1ta+m_2(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+\frac{m_2b-m_1a}{2}\left(\int_0^1 t\left|1-2t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(m_1ta+m_2(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\leqslant \frac{m_2b-m_1a}{2}\left(\int_0^1(1-t)\left|1-2t\right|^p dt\right)^{\frac{1}{p}}\left(m_1\left|f'(a)\right|^q\int_0^1 t(1-t)dt+m_2\left|f'(b)\right|^q\int_0^1(1-t)^2dt\right)^{\frac{1}{q}}\\
&\quad+\frac{m_2b-m_1a}{2}\left(\int_0^1 t\left|1-2t\right|^p dt\right)^{\frac{1}{p}}\left(m_1\left|f'(a)\right|^q\int_0^1 t^2dt+m_2\left|f'(b)\right|^q\int_0^1 t(1-t)dt\right)^{\frac{1}{q}}\\
&=\frac{m_2b-m_1a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{m_1\left|f'(a)\right|^q}{6}+\frac{2m_2\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}\\
&\quad+\frac{m_2b-m_1a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{2m_1\left|f'(a)\right|^q}{6}+\frac{m_2\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^1(1-t)\left|1-2t\right|^p dt=\int_0^1 t\left|1-2t\right|^p dt=\frac{1}{2(p+1)}\\
&\int_0^1 t(1-t)dt=\frac{1}{6},\quad \int_0^1(1-t)^2dt=\int_0^1 t^2dt=\frac{1}{3}.
\end{aligned}$$

■

Remark 4.12 When $m_1 = m_2 = 1$, we get the following inequality:

$$\begin{aligned}
\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}&\left[\left(\frac{|f'(a)|^q}{6}+\frac{2|f'(b)|^q}{6}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{2|f'(a)|^q}{6}+\frac{|f'(b)|^q}{6}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 4.13 The inequality (5) gives better result than (1). Let us show that

$$\begin{aligned}
&\frac{m_2b-m_1a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{m_1\left|f'(a)\right|^q}{6}+\frac{2m_2\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}\\
&+\frac{m_2b-m_1a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{2m_1\left|f'(a)\right|^q}{6}+\frac{m_2\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}\\
&\leqslant\frac{m_2b-m_1a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{m_1\left|f'(a)\right|^q+m_2\left|f'(b)\right|^q}{2}.
\end{aligned}$$

If we use the concavity of $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^{\lambda}, 0<\lambda\leqslant 1$, we get

$$\begin{aligned}
&\frac{m_2b-m_1a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{m_1\left|f'(a)\right|^q}{6}+\frac{2m_2\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}\\
&+\frac{m_2b-m_1a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{2m_1\left|f'(a)\right|^q}{6}+\frac{m_2\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}\\
&\leqslant 2\frac{m_2b-m_1a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\\
&=\frac{m_2b-m_1a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{m_1\left|f'(a)\right|^q+m_2\left|f'(b)\right|^q}{2},
\end{aligned}$$

which completes the proof of remark.

Theorem 4.14 *Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b\in I^{\circ}$ with $m_1a<m_2b$ and let $f'\in L[m_1a,m_2b]$. If the mapping $|f'|^q$ is (m_1,m_2)-convex on interval $[m_1a,m_2b]$, for $q\geqslant 1$, then the following inequality*

$$\begin{aligned}
\frac{f(m_1a)+f(m_2b)}{2}-\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b}f(x)dx\leqslant\frac{m_2b-m_1a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left[\left(\frac{m_1\left|f'(a)\right|^q}{16}\right.\right.\\
\left.+\frac{3m_2\left|f'(b)\right|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{3m_1\left|f'(a)\right|^q}{16}+\frac{m_2\left|f'(b)\right|^q}{16}\right)^{\frac{1}{q}}\Bigg] \qquad (6)
\end{aligned}$$

holds for $t\in[0,1]$ and $m_1,m_2\in(0,1]^2$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof From Lemma 4.2, improwed power-mean integral inequality and (m_1,m_2)-

convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{f(m_1a)+f(m_2b)}{2}-\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b}f(x)dx\right| \\
&\leqslant \frac{m_2b-m_1a}{2}\left(\int_0^1(1-t)\left|1-2t\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)\left|1-2t\right|\left|f'\left(m_1ta+m_2(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad+\frac{m_2b-m_1a}{2}\left(\int_0^1 t\left|1-2t\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|1-2t\right|\left|f'\left(m_1ta+m_2(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leqslant \frac{m_2b-m_1a}{2}\left(\int_0^1(1-t)\left|1-2t\right|dt\right)^{1-\frac{1}{q}}\left(m_1\left|f'(a)\right|^q\int_0^1 t(1-t)\left|1-2t\right|dt\right. \\
&\quad\left.+m_2\left|f'(b)\right|^q\int_0^1(1-t)^2\left|1-2t\right|dt\right)^{\frac{1}{q}}+\frac{m_2b-m_1a}{2}\left(\int_0^1 t\left|1-2t\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(m_1\left|f'(a)\right|^q\int_0^1 t^2\left|1-2t\right|dt+m_2\left|f'(b)\right|^q\int_0^1 t(1-t)\left|1-2t\right|dt\right)^{\frac{1}{q}} \\
&=\frac{m_2b-m_1a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left[\left(\frac{m_1\left|f'(a)\right|^q}{16}+\frac{3m_2\left|f'(b)\right|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{3m_1\left|f'(a)\right|^q}{16}+\frac{m_2\left|f'(b)\right|^q}{16}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^1(1-t)\left|1-2t\right|dt=\int_0^1 t\left|1-2t\right|dt=\frac{1}{4}, \\
&\int_0^1 t(1-t)\left|1-2t\right|dt=\frac{1}{16}, \\
&\int_0^1 t^2\left|1-2t\right|dt=\int_0^1(1-t)^2\left|1-2t\right|dt=\frac{3}{16}.
\end{aligned}$$

■

Corollary 4.15 *Under the assumption of Theorem 4.14 with $q=1$, we get*

$$\frac{f(m_1a)+f(m_2b)}{2}-\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b}f(x)dx\leqslant\frac{m_2b-m_1a}{4}A\left(m_1\left|f'(a)\right|,m_2\left|f'(b)\right|\right),$$

where A is the arithmetic mean. This inequality coincides with the inequality (3).

Corollary 4.16 *Under the assumption of Theorem 4.14 with $m_1=m_2=1$, we get*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx \\
&\leqslant\frac{b-a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left[\left(\frac{\left|f'(a)\right|^q}{16}+\frac{3\left|f'(b)\right|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'(a)\right|^q}{16}+\frac{\left|f'(b)\right|^q}{16}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 4.17 *Under the assumption of Theorem 4.14 with $m_1=m_2=q=1$, we get*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{b-a}{4}A\left(\left|f'(a)\right|,\left|f'(b)\right|\right),$$

where A is the arithmetic mean. This inequality coincides with the inequality (4).

Remark 4.18 The inequality (6) gives better result than the inequality (2). That is,

$$\frac{m_2 b - m_1 a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left[\left(\frac{m_1\left|f'(a)\right|^q}{16}+\frac{3m_2\left|f'(b)\right|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{3m_1\left|f'(a)\right|^q}{16}+\frac{m_2\left|f'(b)\right|^q}{16}\right)^{\frac{1}{q}}\right]$$
$$\leqslant \frac{m_2 b - m_1 a}{4} A^{\frac{1}{q}}\left(m_1\left|f'(a)\right|^q, m_2\left|f'(b)\right|^q\right).$$

This remark's proof can be done in a similar way to the Reamark 4.13.

5. Conclusion

In this article, the inequalities obtained with Hölder and Hölder-İşcan and power-mean and improwed power-mean integral inequalities are compared and results giving better approach are obtained. This method can be applied to different classes of convexity.

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