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Some Integral Inequalities of Hermite-Hadamard Type for Multiplicatively s-Preinvex Functions

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Abstract. In this paper, we establish integral inequalities of Hermite-Hadamard type for multiplicatively s-preinvex functions. We also obtain some new inequalities involving multiplicative integrals by using some properties of multiplicatively s-preinvex and preinvex functions.

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1. Introduction

Let $I \subset \mathbb{R}$ be an interval with $a_1, a_2 \in I$ and $a_1 < a_2$, and let $f : I \to \mathbb{R}$ be a convex function. The double inequality

$$f\left(\frac{a_1+a_2}{2}\right) \leqslant \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx \leqslant \frac{f(a_1)+f(a_2)}{2}$$

is known in the literature as Hermite-Hadamard integral inequality for convex functions. Both the inequalities hold in the reversed direction if f is concave. In recent years, several generalizations and extensions have been considered for classical convexity. One of the most important generalizations of the concept of convex function

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is that of preinvex function introduced by Hanson [6]. Ben-Israel and Mond [5] introduced the concepts of invex set and preinvex function. Weir and Mond [20], Noor [14] and Yang and Li [21] have studied the basic properties of the preinvex functions. For recent generalizations and extensions of the preinvex functions, see [2, 3, 7–10, 13, 17, 18].

1.1 *Preinvexity and Hermite-Hadamard inequalities*

Let us recall some definitions and known results concerning invexity and preinvexity.

Definition 1.1 [21] A set $\Im \subseteq \mathbb{R}$ is said to be invex if there exist a function $\eta : \Im \times \Im \to \mathbb{R}$ such that

$$a_1 + \mu\eta(a_2, a_1) \in \Im, \ \forall a_1, a_2 \in \Im, \ \mu \in [0,1].$$

The invex set $\Im$ is also called a η-connected set.

Definition 1.2 [20] Let f be a function on the invex set $\Im$. Then, f is said to be preinvex with respect to η, if

$$f(a_1 + \mu\eta(a_2, a_1)) \leqslant (1-\mu) f(a_1) + \mu f(a_2), \quad \forall a_1, a_2 \in \Im, \quad \mu \in [0,1].$$

It is to be noted that every convex function is preinvex with respect to the map $\eta(a_2, a_1) = a_2 - a_1$, but the converse is not true, see for example [20, 22].

To prove some results in this paper, we need the well-known Condition C introduced by Mohan and Neogy in [11].

Condition C Let $\Im \subseteq \mathbb{R}^n$ be an open invex subset with respect to $\eta : \Im \times \Im \to \mathbb{R}$. We say that the bifunction η satisfies the Condition C if for any $a_1, a_2 \in \Im$ and $\mu \in [0,1]$,

$$\eta(a_1, a_1 + \mu\eta(a_2, a_1)) = -\mu\eta(a_2, a_1),$$

$$\eta(a_2, a_1 + \mu\eta(a_2, a_1)) = (1-\mu)\eta(a_2, a_1).$$

Note that for every $a_1, a_2 \in \Im$ and $\mu \in [0,1]$ and from condition C, we have

$$\eta(a_1 + \mu_2\eta(a_2, a_1), a_1 + \mu_1\eta(a_2, a_1)) = (\mu_2 - \mu_1)\eta(a_2, a_1).$$

In [12] Noor has obtained the following Hermite-Hadamard inequalities for the preinvex functions.

Theorem 1.3 *Let $f : \Im = [a_1 + \eta(a_2, a_1)] \to (0,\infty)$ be a preinvex function on the interval of real numbers $\Im^\circ$ and $a_1, a_2 \in \Im^\circ$ with $a_1 < \eta(a_2, a_1)$. Then the following inequality holds:*

$$f\left(\frac{2a_1 + \eta(a_2, a_1)}{2}\right) \leqslant \frac{1}{\eta(a_2, a_1)} \int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))\, dx \leqslant \frac{f(a_1) + f(a_2)}{2}.$$

Definition 1.4 [15] A nonnegative function $f : \Im \subset [0,\infty) \to \mathbb{R}$ is said to be

s-preinvex with respect to η for some fixed $s \in (0,1]$, if

$$f(a_1 + \mu\eta(a_2, a_1)) \leqslant (1-\mu)^s f(a_1) + \mu^s f(a_2)$$

for all $a_1, a_2 \in \Im$, $\mu \in [0,1]$.

Definition 1.5 [16] A function $f : \Im \to (0,\infty)$ is said to be multiplicatively (or logarithmically) s-preinvex for $s \in (0,1)$with respect to η, if

$$f(a_1 + \mu\eta(a_2, a_1)) \leqslant [f(a_1)]^{(1-\mu)^s} [f(a_2)]^{\mu^s}, \ a_1, a_2 \in \Im, \ \mu \in [0,1].$$

From the above definition, we have

$$\begin{aligned}
\ln f(a_1 + \mu\eta(a_2, a_1)) &\leqslant \ln\left\{[f(a_1)]^{(1-\mu)^s} [f(a_2)]^{\mu^s}\right\} \\
&= \ln[f(a_1)]^{(1-\mu)^s} + \ln[f(a_2)]^{\mu^s} \\
&= (1-\mu)^s \ln f(a_1) + \mu^s \ln f(a_2).
\end{aligned}$$

1.2 *Multiplicative calculus*

Recall that the notion of multiplicative integral is denoted by $\int_u^v (f(x))^{dx}$ while the ordinary integral is denoted by $\int_u^v (f(x))\,dx$. This comes from the fact that the sum of the terms of product is used in the definition of a classical Riemann integral of f on $[u,v]$, the product of terms raised to certain powers is used in the definition of multiplicative integral of f on $[u,v]$.

There is the following relation between Riemann integral and multiplicative integral [4].

Proposition 1.6 *If f is Riemann integrable on $[u,v]$, then f is multiplicative integrable on $[u,v]$ and*

$$\int_u^v (f(x))^{dx} = e^{\int_u^v \ln(f(x))dx}.$$

In [4], Bashirov et al. show that multiplicative integral has the following results:

Proposition 1.7 *If f is positive and Riemann integrable on $[u,v]$, then f is multiplicative integrable on $[u,v]$ and*

(1) $\int_u^v ((f(x))^r)^{dx} = \int_u^v \left((f(x))^{dx}\right)^r$,
(2) $\int_u^v (f(x)g(x))^{dx} = \int_u^v (f(x))^{dx} \cdot \int_u^v (g(x))^{dx}$,
(3) $\int_u^v \left(\frac{f(x)}{g(x)}\right)^{dx} = \frac{\int_u^v (f(x))^{dx}}{\int_u^v (g(x))^{dx}}$,
(4) $\int_u^v (f(x))^{dx} = \int_u^w (f(x))^{dx} \cdot \int_w^v (f(x))^{dx}, \ u \leqslant w \leqslant v.$
(5) $\int_u^u (f(x))^{dx} = 1$ and $\int_u^v (f(x))^{dx} = \left(\int_v^u (f(x))^{dx}\right)^{-1}$.

2. Main results

In this section we establish some Hermite-Hadamard type inequalities for multiplicatively s-preinvex functions. We also obtain integral inequalities of Hermite-

Hadamard type for product and quotient of preinvex and multiplicatively s-preinvex functions.

Theorem 2.1 *Let $\Im \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \Im \times \Im \to \mathbb{R}$ and $a_1, a_2 \in \Im$ with $a_1 < a_1 + \eta(a_2, a_1)$. If f is a positive and multiplicatively s-preinvex function on the interval $[a_1, a_1 + \eta(a_2, a_1)]$ and η satisfies Condition C, then*

$$\left[f\left(\frac{2a_1 + \eta(a_2, a_1)}{2}\right)\right]^{2^{s-1}} \leqslant \left(\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}} \leqslant [f(a_1) f(a_2)]^{1/(s+1)}. \tag{1}$$

Proof Since f is a multiplicatively s-preinvex function, we have for every $u, v \in [a_1, a_1 + \eta(a_2, a_1)]$ with $\mu = \frac{1}{2}$

$$f\left(\frac{2u + \eta(v, u)}{2}\right) = f\left(u + \frac{\eta(v, u)}{2}\right) \leqslant (f(u))^{1/2^s} (f(v))^{1/2^s}.$$

Now let $u = a_1 + (1-\mu)\eta(a_2, a_1)$ and $v = a_1 + \mu\eta(a_2, a_1)$. From Condition C, we have

$$\begin{aligned} & f\left(a_1 + (1-\mu)\eta(a_2, a_1) + \frac{\eta(a_1 + \mu\eta(a_2, a_1), a_1 + (1-\mu)\eta(a_2, a_1))}{2}\right) \\ &= f\left(a_1 + (1-\mu)\eta(a_2, a_1) + \frac{(2\mu - 1)\eta(a_2, a_1)}{2}\right) \\ &= f\left(\frac{2a_1 + \eta(a_2, a_1)}{2}\right) \\ &\leqslant (f(a_1 + \mu\eta(a_2, a_1)))^{1/2^s} (f(a_1 + (1-\mu)\eta(a_2, a_1)))^{1/2^s}. \end{aligned}$$

Taking logarithms of both sides of the above inequality leads to

$$\begin{aligned} \ln f\left(\frac{2a_1 + \eta(a_2, a_1)}{2}\right) &\leqslant \ln\left((f(a_1 + \mu\eta(a_2, a_1)))^{1/2^s} (f(a_1 + (1-\mu)\eta(a_2, a_1)))^{1/2^s}\right) \\ &= \frac{1}{2^s}\ln(f(a_1 + \mu\eta(a_2, a_1))) + \frac{1}{2^s}\ln(f(a_1 + (1-\mu)\eta(a_2, a_1)))^{1/2^s}. \end{aligned}$$

Integrating the above inequality with respect to μ on $[0,1]$, we have

$$\begin{aligned}
&\ln f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\\
&\leqslant \frac{1}{2^s}\int_0^1 \ln\left(f\left(a_1+\mu\eta(a_2,a_1)\right)\right)d\mu+\frac{1}{2^s}\int_0^1 \ln\left(f\left(a_1+(1-\mu)\eta(\beta,\alpha)\right)\right)d\mu\\
&=\frac{1}{2^s}\left[\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))\,dx-\frac{1}{\eta(a_2,a_1)}\int_{a_1+\eta(a_2,a_1)}^{a_1}\ln(f(x))\,dx\right]\\
&=\frac{1}{2^s}\left[\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))\,dx+\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))\,dx\right]\\
&=\frac{1}{2^{s-1}}.\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))\,dx,
\end{aligned}$$

which implies that

$$2^{s-1}\ln f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\leqslant\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))\,dx.$$

Thus, we have

$$\begin{aligned}
f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)^{2^{s-1}} &\leqslant e^{\left(\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx\right)}\\
&=\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}(f(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}}.
\end{aligned}$$

Hence, we obtain

$$f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)^{2^{s-1}}\leqslant\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}(f(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}}, \tag{2}$$

which completes the proof of the left hand side of (1). Now consider the right hand side of (1).

$$\begin{aligned}
\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}(f(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}} &= \left(e^{\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx\right)}\right)^{\frac{1}{\eta(a_2,a_1)}}\\
&= e^{\frac{1}{\eta(a_2,a_1)}\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx\right)}\\
&= e^{\int_0^1\ln(f(a_1+\mu\eta(a_2,a_1)))d\mu}\\
&\leqslant e^{\int_0^1\ln\left((f(a_1))^{(1-\mu)^s}(f(a_2))^{\mu^s}\right)d\mu}\\
&= e^{\int_0^1((1-\mu)^s\ln f(a_1)+\mu^s\ln f(a_2))d\mu}\\
&= e^{\left(\ln(f(a_1)f(a_2))\int_0^1\mu^s d\mu\right)}\\
&= \left[f(a_1)f(a_2)\right]^{1/(s+1)}.
\end{aligned}$$

Hence, we get the inequality

$$\left(\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}} \leqslant [f(a_1) f(a_2)]^{1/(s+1)}. \tag{3}$$

Combining (2) and (3) gives the desired result. ■

Remark 2.2 If we choose $s = 1$, then Theorem 2.1 reduces to Theorem 3.1 in [18].

Remark 2.3 If we choose $\eta(a_2, a_1) = b - a$ and $s = 1$, then Theorem 2.1 reduces to Theorem 5 in [1].

Theorem 2.4 *Let $\Im \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \Im \times \Im \to \mathbb{R}$ and $a_1, a_2 \in \Im$ with $a_1 < a_1 + \eta(a_2, a_1)$. If f and g are positive and multiplicatively s-preinvex functions on the interval $[a_1, a_1 + \eta(a_2, a_1)]$ and η satisfies Condition C, then*

$$\begin{aligned}
&\left[f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right) g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\right]^{2^{s-1}} \\
&\leqslant \left(\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx} \cdot \int_{a_1}^{a_1+\eta(a_2,a_1)} (g(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}} \\
&\leqslant [(f(a_1) f(a_2)) . (g(a_1) g(a_2))]^{1/(s+1)}.
\end{aligned} \tag{4}$$

Proof Since f and g are positive and multiplicatively s-preinvex functions and η satisfies Condition C, we have

$$\begin{aligned}
&\ln\left(f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right) g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\right) \\
&= \ln\left(f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\right) + \ln\left(g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\right) \\
&\leqslant \ln\left((f(a_1+\mu\eta(a_2,a_1)))^{1/2^s} . (f(a_1+(1-\mu)\eta(a_2,a_1)))^{1/2^s}\right) \\
&\quad + \ln\left((g(a_1+\mu\eta(a_2,a_1)))^{1/2^s} . (g(a_1+(1-\mu)\eta(a_2,a_1)))^{1/2^s}\right) \\
&= \frac{1}{2^s}\left[\ln(f(a_1+\mu\eta(a_2,a_1))) + \ln(f(a_1+(1-\mu)\eta(a_2,a_1)))\right] \\
&\quad + \frac{1}{2^s}\left[\ln(g(a_1+\mu\eta(a_2,a_1))) + \ln(g(a_1+(1-\mu)\eta(a_2,a_1)))\right].
\end{aligned}$$

Integrating the above inequality with respect to μ on $[0,1]$, we have

$$\begin{aligned}
&\ln\left(f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\right)\\
&\leqslant \int_0^1\left[\frac{1}{2^s}\ln\left(f\left(a_1+\mu\eta(a_2,a_1)\right)\right)+\ln\left(f\left(a_1+(1-\mu)\eta(a_2,a_1)\right)\right)\right]d\mu\\
&\quad+\int_0^1\left[\frac{1}{2^s}\ln\left(g\left(a_1+\mu\eta(a_2,a_1)\right)\right)+\frac{1}{2^s}\ln\left(g\left(a_1+(1-\mu)\eta(a_2,a_1)\right)\right)\right]d\mu\\
&=\frac{1}{2^s}\left[\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))\,dx-\frac{1}{\eta(a_2,a_1)}\int_{a_1+\eta(a_2,a_1)}^{a_1}\ln(f(x))\,dx\right]\\
&\quad+\frac{1}{2^s}\left[\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))\,dx-\frac{1}{\eta(a_2,a_1)}\int_{a_1+\eta(a_2,a_1)}^{a_1}\ln(g(x))\,dx\right]\\
&=\frac{1}{2^{s-1}}\left[\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))\,dx+\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))\,dx\right],
\end{aligned}$$

which implies that

$$\begin{aligned}
&2^{s-1}\ln\left(f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\right)\\
&\leqslant\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))\,dx+\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))\,dx.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left(f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\right)^{2^{s-1}}\\
&\leqslant e^{\left(\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx+\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))dx\right)}\\
&=e^{\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx+\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))dx\right)^{\frac{1}{\eta(a_2,a_1)}}}\\
&=\left(e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx}.e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))dx}\right)^{\frac{1}{\eta(a_2,a_1)}}\\
&=\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}(f(x))^{dx}.\int_{a_1}^{a_1+\eta(a_2,a_1)}(g(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}}.
\end{aligned}$$

Hence, we attain

$$\begin{aligned}
&\left(f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\right)^{2^{s-1}}\\
&\leqslant\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}(f(x))^{dx}.\int_{a_1}^{a_1+\eta(a_2,a_1)}(g(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}}. \qquad (5)
\end{aligned}$$

Consider the second inequality in (4):

$$\begin{aligned}
&\left(\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx} \cdot \int_{a_1}^{a_1+\eta(a_2,a_1)} (g(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}} \\
&= e^{\left(\int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(f(x))dx + \int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(g(x))dx\right)\frac{1}{\eta(a_2,a_1)}} \\
&= \left(e^{\eta(a_2,a_1)\left(\int_0^1 \ln(f(a_1+\mu\eta(a_2,a_1)))d\mu + \int_0^1 \ln(g(a_1+\mu\eta(a_2,a_1)))d\mu\right)}\right)^{\frac{1}{\eta(a_2,a_1)}} \\
&= e^{\int_0^1 \ln(f(a_1+\mu\eta(a_2,a_1)))d\mu + \int_0^1 \ln(g(a_1+\mu\eta(a_2,a_1)))d\mu} \\
&\leqslant e^{\int_0^1 \ln\left((f(a_1))^{(1-\mu)^s}(f(a_2))^{\mu^s}\right)d\mu + \int_0^1 \ln\left((g(a_1))^{(1-\mu)^s}(g(a_2))^{\mu^s}\right)d\mu} \\
&= e^{\int_0^1 ((1-\mu)^s \ln f(a_1) + \mu^s \ln f(a_2))d\mu + \int_0^1 ((1-\mu)^s \ln g(a_1) + \mu^s \ln g(a_2))d\mu} \\
&= e^{\ln(f(a_1)f(a_2))\int_0^1 \mu^s d\mu + \ln(g(a_1)g(a_2))\int_0^1 \mu^s d\mu} \\
&= \left[(f(a_1) f(a_2)) \cdot (g(a_1) g(a_2))\right]^{1/(s+1)}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\left(\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx} \cdot \int_{a_1}^{a_1+\eta(a_2,a_1)} (g(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}} \\
&\leqslant \left[(f(a_1) f(a_2)) \cdot (g(a_1) g(a_2))\right]^{1/(s+1)}. \qquad (6)
\end{aligned}$$

From the inequalities (5) and (6), we get the inequality (4). ■

Remark 2.5 If we choose $s = 1$, then Theorem 2.4 reduces to Theorem 3.2 in [18].

Remark 2.6 If we choose $\eta(a_2, a_1) = b - a$ and $s = 1$, then Theorem 2.4 reduces to Theorem 7 in [1].

Theorem 2.7 *Let $\Im \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \Im \times \Im \to \mathbb{R}$ and $a_1, a_2 \in \Im$ with $a_1 < a_1 + \eta(a_2, a_1)$. If f and g are positive and multiplicatively s-preinvex functions on the interval $[a_1, a_1 + \eta(a_2, a_1)]$ and η satisfies Condition C, then*

$$\begin{aligned}
\left[\frac{f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)}{g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)}\right]^{2^{s-1}} &\leqslant \left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)} (g(x))^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}} \\
&\leqslant \left[\frac{f(a_1) f(a_2)}{g(a_1) g(a_2)}\right]^{\frac{1}{s+1}}. \qquad (7)
\end{aligned}$$

Proof Since f and g are positive and multiplicatively s-preinvex functions and η

satisfies Condition C, we can write

$$\begin{aligned}
&\ln \frac{f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)}{g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)} \\
&= \ln\left(f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right) - g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\right) \\
&\leqslant \ln\left((f(a_1+\mu\eta(a_2,a_1)))^{1/2^s} . (f(a_1+(1-\mu)\eta(a_1,a_2)))^{1/2^s}\right) \\
&\quad - \ln\left((g(a_1+\mu\eta(a_2,a_1)))^{1/2^s} . (g(a_1+(1-\mu)\eta(a_1,a_2)))^{1/2^s}\right) \\
&= \frac{1}{2^s}\left[\ln(f(a_1+\mu\eta(a_2,a_1))) + \ln(f(a_1+(1-\mu)\eta(a_1,a_2)))\right] \\
&\quad - \frac{1}{2^s}\left[\ln(g(a_1+\mu\eta(a_2,a_1))) + \ln(g(a_1+(1-\mu)\eta(a_1,a_2)))\right]
\end{aligned}$$

Integrating the above inequality with respect to μ on $[0,1]$, we have

$$\begin{aligned}
&\ln \frac{f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)}{g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)} \\
&\leqslant \int_0^1 \frac{1}{2^s}\left[\ln(f(a_1+\mu\eta(a_2,a_1))) + \ln(f(a_1+(1-\mu)\eta(a_1,a_2)))\right] d\mu \\
&\quad - \int_0^1 \frac{1}{2^s}\left[\ln(g(a_1+\mu\eta(a_2,a_1))) + \ln(g(a_1+(1-\mu)\eta(a_1,a_2)))\right] d\mu \\
&= \frac{1}{2^s}\left[\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(f(x))\,dx + \frac{1}{\eta(a_2,a_1)}\int_{a_1+\eta(a_2,a_1)}^{a_1} \ln(f(x))\,dx\right] \\
&\quad - \frac{1}{2^s}\left[\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(g(x))\,dx + \frac{1}{\eta(a_2,a_1)}\int_{a_1+\eta(a_2,a_1)}^{a_1} \ln(g(x))\,dx\right] \\
&= \frac{1}{2^{s-1}}\left[\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(f(x))\,dx - \frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(g(x))\,dx\right]
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&2^{s-1}\ln \frac{f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)}{g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)} \\
&\leqslant \frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(f(x))\,dx - \frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(g(x))\,dx.
\end{aligned}$$

Thus, we have

$$\begin{aligned}&\left[\frac{f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)}{g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)}\right]^{2^{s-1}}\\ &\leqslant e^{\left(\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx-\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))dx\right)}\\ &=\left(e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx-\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))dx}\right)^{\frac{1}{\eta(a_2,a_1)}}\\ &=\left(\frac{e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx}}{e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}\\ &=\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f\left(x\right)\right)^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(g\left(x\right)\right)^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}.\end{aligned}$$

Hence,

$$\left[\frac{f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)}{g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)}\right]^{2^{s-1}}\leqslant\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f\left(x\right)\right)^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(g\left(x\right)\right)^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}. \tag{8}$$

Now, consider the second inequality in (7):

$$\begin{aligned}&\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f\left(x\right)\right)^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(g\left(x\right)\right)^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}\\ &=\left(\frac{e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx}}{e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}\\ &=\left(e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx-\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))dx}\right)^{\frac{1}{\eta(a_2,a_1)}}\\ &=\left(e^{\left(\int_0^1\ln(f(a_1+\mu\eta(a_2,a_1)))d\mu-\int_0^1\ln(g(a_1+\mu\eta(a_2,a_1)))d\mu\right)}\right)^{\frac{1}{\eta(a_2,a_1)}}\\ &=e^{\int_0^1\ln(f(a_1+\mu\eta(a_2,a_1)))d\mu-\int_0^1\ln(g(a_1+\mu\eta(a_2,a_1)))d\mu}\\ &\leqslant e^{\int_0^1\ln\left((f(a_1))^{(1-\mu)^s}(f(a_2))^{\mu^s}\right)d\mu-\int_0^1\ln\left((g(a_1))^{(1-\mu)^s}(g(a_2))^{\mu^s}\right)d\mu}\\ &=e^{\int_0^1((1-\mu)^s\ln f(a_1)+\mu^s\ln f(a_2))d\mu-\int_0^1((1-\mu)^s\ln g(a_1)+\mu^s\ln g(a_2))d\mu}\\ &=e^{\ln(f(a_1)f(a_2))\int_0^1\mu^s d\mu-\ln(g(a_1)g(a_2))\int_0^1\mu^s d\mu}\\ &=\left[\frac{f\left(a_1\right)f\left(a_2\right)}{g\left(a_1\right)g\left(a_2\right)}\right]^{\frac{1}{s+1}}.\end{aligned}$$

Consequently,

$$\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f\left(x\right)\right)^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(g\left(x\right)\right)^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}\leqslant\left[\frac{f\left(a_1\right)f\left(a_2\right)}{g\left(a_1\right)g\left(a_2\right)}\right]^{\frac{1}{s+1}}. \tag{9}$$

By using the inequalities (8) and (9), we get the inequality (7) which is the required result. ∎

Remark 2.8 If we choose $s = 1$, then Theorem 2.7 reduces to Theorem 3.3 in [18].

Remark 2.9 If we choose $\eta(a_2, a_1) = b - a$ and $s = 1$, then Theorem 2.7 reduces to Theorem 9 in [1].

Theorem 2.10 *Let $\Im \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \Im \times \Im \to \mathbb{R}$ and $a_1, a_2 \in \Im$ with $a_1 < a_1 + \eta(a_2, a_1)$. Let f and g be preinvex and multiplicatively s-preinvex positive functions, respectively, on the interval $[a_1, a_1 + \eta(a_2, a_1)]$. Then, we have*

$$\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)} (g(x))^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}} \leqslant \frac{\left(\frac{(f(a_2))^{f(a_2)}}{(f(a_1))^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}}}{e.\,(g(a_1)\,g(a_2))^{1/(s+1)}}.$$

Proof Note that,

$$\begin{aligned}
&\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)} (g(x))^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}} \\
&= \left(\frac{e^{\int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(f(x))dx}}{e^{\int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(g(x))dx}}\right)^{\frac{1}{\eta(a_2,a_1)}} \\
&= \left(e^{\int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(f(x))dx - \int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(g(x))dx}\right)^{\frac{1}{\eta(a_2,a_1)}} \\
&= e^{\int_0^1 \ln(f(a_1+\mu\eta(a_2,a_1)))d\mu - \int_0^1 \ln(g(a_1+\mu\eta(a_2,a_1)))d\mu} \\
&\leqslant e^{\int_0^1 \ln(f(a_1)+\mu(f(a_2)-f(a_1)))d\mu - \int_0^1 \ln\left((g(a_1))^{(1-\mu)^s}(g(a_2))^{\mu^s}\right)d\mu} \\
&= e^{\ln\left(\left(\frac{(f(a_2))^{f(a_2)}}{(f(a_1))^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}}\right) - 1 - \ln(g(a_1)g(a_2))^{\int_0^1 \mu^s d\mu}} \\
&= \frac{\left(\frac{(f(a_2))^{f(a_2)}}{(f(a_1))^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}}}{e.\,(g(a_1)\,g(a_2))^{1/(s+1)}}.
\end{aligned}$$

So, the proof is completed. ∎

Remark 2.11 If we choose $s = 1$, then Theorem 2.10 reduces to Theorem 3.4 in [18].

Remark 2.12 If we choose $\eta(a_2, a_1) = b - a$ and $s = 1$, then Theorem 2.10 reduces to Theorem 11 in [1].

Theorem 2.13 *Let $\Im \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \Im \times \Im \to \mathbb{R}$ and $a_1, a_2 \in \Im$ with $a_1 < a_1 + \eta(a_2, a_1)$. Let f and g be multiplicatively s-preinvex and preinvex positive functions, respectively, on the interval $[a_1, a_1 + \eta(a_2, a_1)]$. Then, we have*

$$\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)} (g(x))^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}} \leqslant \frac{e.\,(f(a_1)\,f(a_2))^{1/(s+1)}}{\left(\frac{(g(a_2))^{g(a_2)}}{(g(a_1))^{g(a_1)}}\right)^{\frac{1}{g(a_2)-g(a_1)}}}.$$

Proof Note that

$$\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)}(f(x))^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)}(g(x))^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}$$
$$=\left(\frac{e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx}}{e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}$$
$$=\left(e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx-\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))dx}\right)^{\frac{1}{\eta(a_2,a_1)}}$$
$$=e^{\int_0^1\ln(f(a_1+\mu\eta(a_2,a_1)))d\mu-\int_0^1\ln(g(a_1+\mu\eta(a_2,a_1)))d\mu}$$
$$\leqslant e^{\int_0^1\ln\left((f(a_1))^{(1-\mu)^s}(f(a_2))^{\mu^s}\right)d\mu-\int_0^1\ln(g(a_1)+\mu(g(a_2)-g(a_1)))d\mu}$$
$$=e^{\ln(f(a_1).f(a_2))^{\int_0^1\mu^s d\mu}-\ln\left(\left(\frac{(g(a_2))^{g(a_2)}}{(g(a_1))^{g(a_1)}}\right)^{\frac{1}{g(a_2)-g(a_1)}}\right)+1}$$
$$=\frac{e.\left(f(a_1)f(a_2)\right)^{1/(s+1)}}{\left(\frac{(g(a_2))^{g(a_2)}}{(g(a_1))^{g(a_1)}}\right)^{\frac{1}{g(a_2)-g(a_1)}}}.$$

■

Remark 2.14 If we choose $s=1$, then Theorem 2.13 reduces to Theorem 3.5 in [18].

Remark 2.15 If we choose $\eta(a_2,a_1)=b-a$ and $s=1$, then Theorem 2.13 reduces to Theorem 12 in [1].

Theorem 2.16 *Let $\Im\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:\Im\times\Im\to\mathbb{R}$ and $a_1,a_2\in\Im$ with $a_1<a_1+\eta(a_2,a_1)$. Let f and g be preinvex and multiplicatively s-preinvex positive functions, respectively, on the interval $[a_1,a_1+\eta(a_2,a_1)]$. Then, we have*

$$\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}(f(x))^{dx}.\int_{a_1}^{a_1+\eta(a_2,a_1)}(g(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}}$$
$$\leqslant\frac{\left(\frac{(f(a_2))^{f(a_2)}}{(f(a_1))^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}}.\left(g(a_1)g(a_2)\right)^{1/(s+1)}}{e}.$$

Proof Note that

$$\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}(f(x))^{dx}.\int_{a_1}^{a_1+\eta(a_2,a_1)}(g(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}}$$
$$=\left(e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(f(x))dx+\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln(g(x))dx}\right)^{\frac{1}{\eta(a_2,a_1)}}$$
$$=\left(e^{\eta(a_2,a_1)\left(\int_0^1\ln(f(a_1+\mu\eta(a_2,a_1)))d\mu+\int_0^1\ln(a_1+\mu\eta(a_2,a_1))d\mu\right)}\right)^{\frac{1}{\eta(a_2,a_1)}}$$

$$= e^{\int_0^1 \ln(f(a_1+\mu\eta(a_2,a_1)))d\mu+\int_0^1 \ln(g(a_1+\mu\eta(a_2,a_1)))d\mu}$$

$$\leqslant e^{\int_0^1 \ln(f(a_1)+\mu(f(a_2)-f(a_1)))d\mu-\int_0^1 \ln\left((g(a_1))^{(1-\mu)^s}(g(a_2))^{\mu^s}\right)d\mu}$$

$$= e^{\ln\left(\left(\frac{(f(a_2))^{f(a_2)}}{(f(a_1))^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}}\right)-1+\ln(g(a_1)g(a_2))^{\int_0^1 \mu^s d\mu}}$$

$$= \frac{\left(\frac{(f(a_2))^{f(a_2)}}{(f(a_1))^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}} \cdot (g(a_1)\, g(a_2))^{1/(s+1)}}{e}.$$

Consequently,

$$\left(\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx} \cdot \int_{a_1}^{a_1+\eta(a_2,a_1)} (g(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}}$$

$$\leqslant \frac{\left(\frac{(f(a_2))^{f(a_2)}}{(f(a_1))^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}} \cdot (g(a_1)\, g(a_2))^{1/(s+1)}}{e}.$$

This completes the proof. ■

Remark 2.17 If we choose $s = 1$, then Theorem 2.16 reduces to Theorem 3.6 in [18].

Remark 2.18 If we choose $\eta(a_2, a_1) = b-a$ and $s = 1$, then Theorem 2.16 reduces to Theorem 13 in [1].

3. Conclusion

In this paper, integral inequalities of Hermite-Hadamard type for multiplicatively s-preinvex and preinvex functions are established in the setting of multiplicative calculus. Some integral inequalities of Hermite-Hadamard type for product and quotient of multiplicatively s-preinvex and preinvex functions are derived in multiplicative calculus. It has shown that, previously known results can be obtained as special cases from our results. It is expected that idea of this article may attract interested readers.

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