International Journal of
Mathematical Modelling & Computations
Vol. 09, No. 02, Spring 2019, 149- 154

A Consistent and Accurate Numerical Method for Approximate Numerical Solution of Two Point Boundary Value Problems

P. K. Pandey*

Department of Mathematics, Dyal Singh College (Univ. of Delhi), Lodhi Road, New Delhi-110003, India.

Abstract. In this article we have proposed an accurate finite difference method for approximate numerical solution of second order boundary value problem with Dirichlet boundary conditions. There are numerous numerical methods for solving these boundary value problems. Some these methods are more efficient and accurate than others with some advantages and disadvantages. The results in experiment on model problems show an improved and good approximation to the solution of considered problems.

Received: 01 March 2019, Revised: 22 May 2019, Accepted: 23 June 2019.

Keywords: Boundary value problem; Dirichlet boundary condition; Finite difference method; Non-linear equation; Unconventional difference method.

AMS Subject Classification: 45J05, 65L10, 65L12.

Index to information contained in this paper

1. Introduction

In this article we consider following boundary value problem

$$y''(x) - \alpha y'(x) = f(x, y), \quad a < x < b, \tag{1}$$

subject to the boundary conditions

$$y(a) = \beta, \quad y(b) = \gamma$$

*Corresponding author. Email: pramod_10p@hotmail.com

where α, β and γ are real constants. The function $f(x, y)$ is known.

In this article we have assumed the exitance and uniqueness of the solution of the problem (1). However the specific assumption on concept of existence and uniqueness of the solution of the problem (1) can be found in [1] and references therein. Thus the emphasis in this article will be on the development of finite difference method for approximate numerical solution of the problem (1).

Ordinary differential equations come in many area of studies of engineering and sciences. It contains many terms and it can be described and classified in many different ways by considering these terms. Thus these terms describe different properties of the ODEs. Generally it is not possible to get an accurate and exact continuous solution of these ODEs and corresponding boundary value problems. It is desirous to achieve a good numerical approximation to the continuous solution of these ODEs. To get an approximate solution we transform continuous problems into a discrete problems. The transformation of these differential equations and corresponding boundary value problems into a discrete problems at discrete points must reflect properties of continuous problems in some approximate manner.

There are several methods for finding approximate solution of these problems. A finite difference method one of the conventional method used for computation of approximate solution of these problem. Last few decades have seen substantial progress in the development of unconventional difference method [4–6], improved difference methods [3] and references therein. These finite difference methods generated an impressive numerical result with high accuracy. Hence, the purpose of this article is to describe a finite difference method similar to [3] for numerical solution of the second-order boundary value problem (1) with Dirichlet boundary conditions.

We have presented our work in this article as follows. In the next section we proposed a finite difference method. We have discussed derivation of the proposed method in Section 3. The application of the proposed method on the model problems and numerical results so produced in Section 4. Some words and conclusion on the performance of the proposed method are presented in Section 5.

2. The finite difference method

The finite difference method presented in this article is new but derivation depends on the work in [3]. We substitute domain $[a, b]]$ by a discrete set of points and we wish to determine the numerical solution at these points. Thus we define N finite numbers of $a = x_0 < x_1 < x_2 ...... < x_{N+1} = b$ nodal points in the domain of $[a,b]$ using a uniform step length h such that $x_i = a + ih, \quad i = 0, 1, 2, ..., N + 1$. We wish to determine the numerical approximation of the theoretical solution $y(x)$ of the problem (1) at the nodal points $x_i, \quad i = 1, 2, ..., N$. We denote the numerical approximation of $y(x)$ at node $x = x_i$ as y_i, $i = 1, 2, .., N$. Let us denote f_i as the approximation of the theoretical value of the source function $f(x, y(x))$ at node $x = x_i, \quad i = 0, 1, 2, ..., N + 1$. Thus, the finite difference method reduces the problem (1) to the following discrete problem at node $x = x_i$

$$y''_i - \alpha y'_i = f_i, \quad a < x_i < b, \tag{2}$$

subject to the boundary conditions

$$y_0 = \beta, \quad y_{N+1} = \gamma$$

Following improved finite difference method for the numerical solution of the problem (1) when source function $f(x, y) \equiv 0$ is given in [3] as,

$$y_{i+1} - (1 + \exp(h\alpha))y_i + \exp(h\alpha)y_{i-1} = 0 \tag{3}$$

We propose following finite difference method for the problem (1) at nodal point x_i as,

$$y_{i+1} - (1 + \exp(h\alpha))y_i + \exp(h\alpha)y_{i-1} = h^2 f_i, \quad i = 1, 2, .., N. \tag{4}$$

Thus difference equation (4) is system of linear or nonlinear equations in unknown $y_i, i = 1, 2, .., N$ which depends on the source function $f(x, y)$.

3. Derivation and local truncation error

In this section we shall outline the derivation of the proposed finite difference method (4) and estimate local truncation error associated with the difference equation (4). Let us define

$$y_{i+1} - (1 + \exp(h\alpha))y_i + \exp(h\alpha)y_{i-1} - h^2 a_0 f_i = 0 \tag{5}$$

where a_0 is real constant. To determine a_0, we replace the terms $\exp(h\alpha)$ by $1 + h\alpha$ in (5). Expand each term in (5) in a Taylor series about node x_i. Using (2) and compare the coefficients of $h^p, p = 0, 1, 2$, we obtained following:

$$h^2(1 - a_0)f_i + \frac{h^3}{2}\alpha y_i'' + R_i = 0 \tag{6}$$

Thus, we find our proposed finite difference method (4) if $a_0 = 1$. Finally, the local truncation error associated with (4) at the interior nodes $x = x_i$ using exact arithmetic may be defined as:

$$T_i = \frac{y_{i+1} - (1 + \exp(h\alpha))y_i + \exp(h\alpha)y_{i-1}}{h^2} - f_i \tag{7}$$

Using (6) in (7), we obtained

$$T_i = \frac{h}{2}\alpha y_i'' + \frac{1}{h^2}R_i \tag{8}$$

Thus, we conclude from (8) that the order of a local truncation error associated with finite difference method (4) at each node $x = x_i$ is at least $O(h)$.

4. Numerical example

To illustrate the computational efficiency of our method, we have considered model problems. We took uniform step size h in our numerical experiments. We have computed MAY, the maximum absolute error in the solution of the problems (1) for different values of N and α using following formula

$$MAY = \max_{1 \leqslant i \leqslant N} |Y(x_i) - y_i|,$$

where $Y(x_i)$ and y_i are respectively exact and approximate solutions. The numerical results presented in Tables for considered model problems. We have used an iterative method to solve system of equations arise from equation (4). All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compilers (2.95 of gcc) on Intel Core i3-2330M, 2.20 GHz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than $10^{(-10)}$ or the number of iterations reached 10^3.

Problem 1. The model linear problem given by

$$y''(x) - 2y'(x) = y(x) + f(x), \quad 0 < x < 1$$

subject to boundary conditions

$$y(0) = 1, \quad \text{and} \quad y(1) = \exp(2).$$

The approximate analytical solution of the problem is $y(x) = \exp(2x)$. For shake of comparison we have computed MAY using conventional finite difference method in [2] and proposed method (4) for different values of N and presented in Table 1.

Problem 2. The model nonlinear problem given by

$$y''(x) - 8y'(x) = (x^2 - 64)y^2 + f(x) \quad 1.5 < x < 2.5$$

subject to boundary conditions

$$y(1.5) = \exp(-4), \quad \text{and} \quad y(2.5) = \exp(4).$$

The approximate analytical solution of the problem is $y(x) = \exp(8(x-2))$. We computed MAY for different values of N and presented in Table 2.

Problem 3. The model linear problem given by

$$y''(x) + 4y'(x) = (x^2 - 16)y(x) + f(x) \quad 0 < x < 1$$

subject to boundary conditions

$$y(0) = \exp(-4), \quad \text{and} \quad y(1) = \exp(-8).$$

The approximate analytical solution of the problem is $y(x) = \exp(-4(x+1))$. We computed MAY for different values of N and presented in Table 3.

Table 1. Maximum absolute error (Problem 1).

Method	Maximum absolute error				
	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$
(4)	0.11920929(-6)	0.23841858(-6)	0.47683716(-6)	0.47683716(-6)	0.47683716(-6)
[2]	0.59604645(-5)	0.47683716(-6)	0.47683716(-6)	0.95367432(-6)	0.95367432(-6)

Table 2. Maximum absolute error (Problem 2).

Method	Maximum absolute error				
	$N=128$	$N=256$	$N=512$	$N=1024$	$N=2048$
(4)	0.38146973(-5)	0.38146973(-5)	0.38146973(-5)	0.38146973(-5)	0.38146973(-5)
[2]	0.38146973(-5)	0.38146973(-5)	0.38146973(-5)	0.38146973(-5)	0.38146973(-5)

Table 3. Maximum absolute error (Problem 3).

Method	Maximum absolute error				
	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
(4)	0.29103830(-10)	0.93132257(-9)	0.93132257(-9)	0.93132257(-9)	0.93132257(-9)
[2]	0.18626451(-8)	0.18626451(-8)	0.27939677(-8)	0.55879354(-8)	0.55879354(-8)

We have performed experiment with proposed numerical method (4) for numerical solution boundary value problem on ODEs. Three model problems considered to illustrate the preciseness and effectiveness of the proposed method. Numerical results for model problems presented in tables for different values of N approve the performance of method (4). Numerical results verify the convergence of the method (4). It is an advantage of the proposed finite difference method over existing method [2].

5. Conclusion

An improved finite difference method for the approximate numerical solution of boundary value problems presented. The present article extends the idea developed in [3]. An important observation about our method is that the discretization of the differential equation at an interior node is based on single evaluation of source function $f(x, y)$. The propose method produces good approximate numerical solution for considered model problems. For considered model problems our method out performed existing method in literature. The idea presented in this article may be extended to develop numerical method for approximate solution of other problems in ODEs/PDEs. Works in these directions are in progress.

References

[1] R. P. Agarwal, Boundary Value Problems for Higher Order Differential Equations, World Scientific, Singapore, (1986).

[2] M. M. Chawla, A fourth-order tridiagonal finite difference method for general two-point boundary value problems with non-linear boundary conditions, Journal of the Institute of Mathematics and its Applications, **22** (1978) 89-97.

[3] W. Lick and T. Gaskins, A consistent and accurate procedure for obtaining difference equations from differential equation, International Journal for Numerical Methods in Engineering, **20** (1984) 1433-1441.

[4] R. E. Mickens, Nonstandard Finite Difference Models of Differential equations, World Scientific, Singapore, (1994).

[5] P. K. Pandey, A non-classical finite difference method for solving two point boundary value problems, Pacific Journal of Science and Technology, **14 (2)** (2013) 147-152.

[6] H. Ramos, A non-standard explicit integration scheme for initial value problems, Applied Mathematics and Computation, **189 (1)** (2007) 710-718.