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Solving Fuzzy Impulsive Fractional Differential Equations by Reproducing Kernel Hilbert Space Method

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Abstract. The aim of this paper is to use the Reproducing kernel Hilbert Space Method (RKHSM) to solve the linear and nonlinear fuzzy impulsive fractional differential equations. Finding the numerical solutions of this class of equations are a difficult topic to analyze. In this study, convergence analysis, estimations error and bounds errors are discussed in detail under some hypotheses which provide the theoretical basis of the proposed algorithm. Some numerical examples indicate that this method is an efficient one to solve the mentioned equations.

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1. Introduction

Fractional calculus is a new powerful tool which has been recently employed to model complex biological systems with non-linear behavior and long-term memory. One of the important branches of this theory is impulsive fractional differential equations. The idea of the theory of impulsive fractional differential equations has emerged as an effective tool area of investigation in recent years (see [8, 10, 11]). J.

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Wang, W. Wei and Y. Yang [36], solved impulsive fractional differential equations in banach spaces. Q. Wang, D. Lu and Y. Fang [37], showed that the impulsive fractional differential is stable. Also, [10], showed that the impulsive fractional differential equations exist and has unique solutions. The concept of the fuzzy set theory was first proposed by Zadeh, Zimmerman and Kaleva (see[20, 39, 40]). As a result, many things happening in the real world have fuzzy meanings. Therefore, the fuzzy set theory is a significant tool for modeling unknown problems and can be found in many branches of regional, physical, mathematical and engineering sciences. One of the very important branches of the fuzzy theory is fuzzy impulsive fractional differential equations. In recent years, there has been a growing interest in the fuzzy impulsive fractional differential equations which are a combination of impulsive differential equations and fractional differential equations.The fuzzy impulsive fractional differential equation zqas play an important role in characterizing many social, biological, physical and engineering sciences (for more details see [25] and references cited therein). Fuzzy impulsive fractional differential equations are usually hard to solve analytically and the exact solution is rather difficult to be obtained. But the idea of fuzzy impulsive fractional differential equations has been studied by scientists and engineers like [26, 27]. In this paper, we defined the generalized fractional derivatives of fuzzy-valued functions in the spaces of absolute differentiable continuous and differentiable continuous of fuzzy-valued functions to solve the fuzzy impulsive fractional differential equations. Since Caputo derivatives better describe some physical problems involving memory effect, we defined the Caputo version of the generalized fractional derivatives. We believe that this Caputo version of the generalized fractional derivative would be useful for researchers working on modeling real world phenomena described by fractional operators. Also, we will use combination of RKHSM to solve fuzzy impulsive fractional differential equations with the help of the concept of generalized Hukuhara differentiability.

$$ {}_cD^{\frac{1}{\alpha}}y(t) = f(t, y(t)), \quad t \in [0, T], \quad t \neq t_k, \ m-1 < \frac{1}{\alpha} < m, \quad m \in \mathbb{N}, \tag{1} $$

$$ \Delta y|_{t=t_k} = I_k(y(t_k^-(t))), \tag{2} $$

$$ y(0) = y_0. \tag{3} $$

In this paper the set of all fuzzy numbers is denoted as R_F. Where $k = 1, 2, \ldots, m$, ${}_cD^{\frac{1}{\alpha}}$ denotes the Caputo fractional generalized derivative of order $\frac{1}{\alpha}$, $y(t)$ is an unknown fuzzy function of crisp variable t and $f : [0, T] \times \mathbb{R}_F \to \mathbb{R}_F$, is a continuous fuzzy function, $I_k : \mathbb{R}_{\mathbb{F}} \to \mathbb{R}_{\mathbb{F}}$, $y_0 \in \mathbb{R}_F, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta|_{t=t_k} = y(t_k^+) \ominus_{gH} y(t_k^-)$, $y(t_k^+) = lim_{h\to 0^+} y(t_k + h)$, and $y(t_k^-) = lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$. The fractional differential transform is a numerical method for solving differential equations. Reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics, learning theory and so on [4–6, 12–16, 21, 23, 38]. Also the idea of reproducing kernel theory has been studied by scientists and engineers such as S. Abbasbandy, et al [2, 3, 17, 34]. They considered a new method for solving initial value problems, singular integral equations, nonlinear partial differential equations and operator equations with the help of the concept of reproducing kernel Hilbert space method. The rest of this paper is organized as follows:

In Section 2, we present the basic notions of requirements in this article. Fuzzy impulsive fractional differential equations is introduced in Section 3. The application of RKHSM for solveing the problems. (1)-(3) is explained in Section 4. We introduc error estimations and error bounds in Section 5. The numerical examples are presented in Section 6. The conclusions is brought in Section 7.

2. Basic preliminaries

Definition 2.1 ([22, 30]) We represent an arbitrary fuzzy number by an ordered pair function $(\underline{u}(r), \overline{u}(r))$, which satisfies the following requirements:

L_1: $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
L_2: $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,
L_3: $\underline{u}(r) \leqslant \overline{u}(r)$, $0 \leqslant r \leqslant 1$.

Definition 2.2 ([24]) A crisp number θ is simply represented by $\underline{\theta}(t,r) = \overline{\theta}(t,r) = \theta$, $0 \leqslant r \leqslant 1$. We recall that for $a < b < c$ which $a, b, c \in \mathbb{R}$, the triangular fuzzy number $u = (a, b, c)$ is determined by a, b, c such that $\underline{u}(t,r) = a + (b-c)r$ and $\overline{u}(t,r) = c - (c-b)r$ are left branch and right branch, $\forall\, r \in [0,1]$.

Definition 2.3 ([1]) Let $u, v \in R_F$. If there exists $w \in R_F$ such that $u = v \oplus w$, then w is called the H-difference of u and v, and it is denoted by $w = u \ominus_{gH} v$.

Definition 2.4 ([33]) The generalized Hukuhara difference of two fuzzy numbers $u, v \in \mathbb{R}_F$ is defined as follows:

$$u \ominus_{gH} v = \phi \iff \begin{cases} (i) \qquad u = v \oplus \phi, \\ or \\ (ii) \;\; v = u \oplus (-1)\phi. \end{cases}$$

The condition $u \ominus_{gH} v \in \mathbb{R}_F$ is given in [33]. Please note that a function $f : [a,b] \to \mathbb{R}_F$ so called fuzzy-valued function. The r-level representation of fuzzy-valued function f is expressed by $f_r(t) = [\underline{f}(t,r), \overline{f}(t,r)], t \in [a,b], r \in [0,1]$. We will denote $\mathbb{R}_F$ the set of fuzzy numbers, i.e. normal, fuzzy convex, upper semi continuous and compactly supported fuzzy sets dened over the real line. Fundamental concepts in fuzzy sets theory are the support, the level-sets (or level-cuts) and the core of a fuzzy number.

Definition 2.5 ([9]) Let $u \in \mathbb{R}_F$ be a fuzzy number. For $r \in (0,1]$, the r-level set of u (or simply the r-cut) defined by $[u]_r = \{t \in \mathbb{R} | u(t) \geqslant r\} = [\underline{u}(r), \overline{u}(r)]$ and for $r = 0$ by the closure of the support $[u]_0 = cl\{t | t \in \mathbb{R}, u(t) > 0\}$ where cl denotes the closure of a subset. The addition $u + v$ and the scale multiplication ku are defined as

$$[u \oplus v]_r = [u]_r \oplus [v]_r = \{x + y | x \in [u]_r, y \in [v]_r\},$$

$$[k \odot u]_r = k.[u]_r = \{kx | x \in [u]_r\}, [0] = \{0\}, \;\forall\, r \in [0,1].$$

The subtraction of fuzzy numbers $u - v$ is defined as the addition $u + (-1)v$, if $v = [\underline{v}, \overline{v}]$ where $(-1)v = [-\overline{v}, -\underline{v}]$.

Definition 2.6 The Hausdorff distances between fuzzy numbers is given by d :

$\mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$ as in [7].

$$d(u,v) = \sup_{0<r\leq 1} \max\Big(|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)| \Big).$$

Consider $u, v, w, z \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, then the following properties are well-known for metric d

1): $d(u \oplus w, v \oplus w) = d(u,v)$,
2): $d(\lambda u, \lambda v) = |\lambda| d(u,v)$,
3): $d(u \oplus v, w \oplus z) \leq d(u,w) + d(v,z)$,
4): $d(u \ominus_{gH} v, w \ominus_{gH} z) \leq d(u,w) + d(v,z)$.

as long as $u \ominus_{gH} v$ and $w \ominus_{gH} z$ exist, where $u, v, w, z \in \mathbb{R}_F$. Where, $\ominus$ is the Hukuhara difference.

Theorem 2.7 ([31])*Let $f : [a,b] \to R_F$ be fuzzy continuous. Then $\int_a^b f(x)dx$ exists and belongs to $\mathbb{R}_F$, furthermore it holds*

$$\int_a^b f(x,r)dx = \Big(\int_a^b \underline{f}(x,r)dx, \int_a^b \overline{f}(x,r)dx \Big).$$

Definition 2.8 ([18]) let $f : [a,b] \longrightarrow \mathbb{R}_F$ is called fuzzy continuous if for arbitrary fixed $x_0 \in \mathbb{R}_F$ and $\xi > 0$, there exists an $\delta > 0$, such that if

$$|x - x_0| < \delta, then \ \ d(f(x), f(x_0)) < \xi$$

Lemma 2.9 *For all $\alpha > 0$ and $\gamma > -1$*

$$\int_0^t (t-s)^{\frac{1}{\alpha}-1} s^\gamma ds = \frac{\Gamma(\frac{1}{\alpha})\Gamma(\gamma+1)}{\Gamma(\frac{1}{\alpha}+\gamma+1)} t^{\frac{k}{\alpha}+\gamma},$$

where Γ is the gamma function and defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Proof *Lemma 2.6 [36].* ■

Definition 2.10 ([7]) The generalized Hukuhara derivative of a fuzzy-valued function $f : (a,b) \to \mathbb{R}_F$ at x_0 is defined

$$f'_{gH}(x_0) = \lim_{h\to 0} \frac{f(x_0+h) \ominus_{gH} f(x_0)}{h} \tag{4}$$

If $f'_{gH}(x_0) \in \mathbb{R}_F$ satisfying (7) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at $x0$. Also we say that f is [(i)-gH]-diferentiable at x_0 if

$$(i) \ \ f'_{gH}(x_0) = [\underline{f}'(x_0,r), \ \overline{f}'(x_0,r)] \tag{5}$$

and if f is [(ii)-gH]-diferentiable at x_0 if

$$(ii) \ \ f'_{gH}(x_0) = [\overline{f}'(x_0,r), \underline{f}'(x_0,r)] \tag{6}$$

Throughout this paper, we denote the space of all Lebesgue integrable fuzzy-valued function on the bounded interval $[a, b] \subset \mathbb{R}$ by $L^F[a, b]$. Also, we denote $C^F[a, b]$ as the space of all continuous fuzzy-valued function on $[a, b]$, Moreover we suppose that the generalized Hukuhara difference of any two fuzzy numbers exist.

Definition 2.11 ([7]) Let $f : (a, b) \to \mathbb{R}_F$. We say that f is gH- differentiable of the n^{th}order at x_0 whenever the function f is gH-differentiable of the order $j, j = 1, 2, \ldots, n-1$, at x_0 provided that gH-differentiable type has no change, then there exist $(f)^n_{gH}(x_0) \in \mathbb{R}_F$ such that

$$f^{(n)}_{gH}(x_0) = \lim_{h\to 0} \frac{f^{(n-1)}(x_0 + h) \ominus_{gH} f^{(n-1)}(x_0)}{h} \tag{7}$$

Definition 2.12 ([27]) Let $f : [a, b] \to \mathbb{R}_F$, the fuzzy Riemann-Liouville integral of fuzzy -valued f is defined as follows:

$$(I^{\alpha}_{a|t}f)(t, r) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(t, r)}{(t-s)^{1-\alpha}} \mathrm{d}s, \ t > a,$$

where

$$[I^{\alpha}_{a|t}f](t, r) = \left[\tfrac{1}{\Gamma(\alpha)} \int_a^t \tfrac{\underline{f}(t,r)}{(t-s)^{1-\alpha}} \mathrm{d}s, \ \tfrac{1}{\Gamma(\alpha)} \int_a^t \tfrac{\overline{f}(t,r)}{(t-s)^{1-\alpha}} \mathrm{d}s\right],$$

for $a \le s \le t$ and $0 < \alpha \le 1$.

Definition 2.13 ([27]) The Caputo generalized Hukuhara differentiability of fuzzy-valued function $f([gH]-$differentiability for short), where $x > a$, is defned as following:

$$_cD^{\alpha}_{a|t} f_{gH}(t, r) = \frac{1}{\Gamma(m-\alpha} \int_a^t \frac{f^{(m)}_{gH}(t, r)\mathrm{d}s}{(t-s)^{\alpha-m+1}}, \quad 0 < \alpha < 1,$$

We suppose that any order of differentiability of fuzzy function f exist in the sense of gH. Moreover we say that f is $[(i) - gH]$-differentiable at t if

$$[_cD^{\alpha}_{a|t} f_{i.gH}](t, r) = [_cD^{\alpha}_{a|t}\underline{f}(t, r), \ _cD^{\alpha}_{a|t}\overline{f}(t, r)],$$

as well as f is $[(ii) - gH]$-differentiable at t if

$$[_cD^{\alpha}_{a|t} f_{ii.gH}](t, r) = [_cD^{\alpha}_{a|t}\overline{f}(t, r), \ _cD^{\alpha}_{a|t}\underline{f}(t, r)],$$

Definition 2.14 ([3, 17]) A Hilbert space is a complete infinite-dimensional inner-product space. The elements of this space can be functions defined on a set T. In particular, the abstract reproducing kernel Hilbert space (RKHS), H, is a Hilbert space of functions defined on a set T such that there exists a unique function, $R(t, y)$, defined on $T \times T$ with the following properties:

$$\begin{array}{ll} (I). & R_y(t) = R(t, y) \in H \quad for\ all\ t \in T \\ (II). & \langle f(t), R_y(t)\rangle = f(y) \quad for\ all\ t \in T \quad for\ all\ f \in H \end{array}$$

The function $R(t, y)$ is called the reproducing kernel of the abstract RKHS.

Definition 2.15 ([17, 29]) Let ϕ be a mapping from T into the space H such that $\phi = R(t,.)$. A function $R : T \times T \to \mathbb{R}$ such that $R_y = R(t,y) = \langle \phi(t), \phi(y) \rangle$,for all $t, y \in T$ is called a kernel.

Definition 2.16 ([12, 21]) $W_2^m[0,1] = \{u^{(m-1)}(t)$ is an absolutely continuous real value function, $u^{(m)}(t) \in L^2[0,1]$,$u(0) = 0,\ u(1) = 0\}$. Her $L^2[a,b] = \{z| \int_a^b z^2 dt < \infty$. The inner product and norm in $W_2^m[0,1]$ are given respectively by

$$\langle u, v \rangle = \sum_{i=0}^{m-1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(m)}(t)v^{(m)}(t)\mathrm{d}x$$

and

$$\|u\|_W = \sqrt{\langle u, u \rangle}_W, \qquad u, v \in W^m[0,1].$$

$W^m[0,1]$ is a reproducing kernel space and its reproducing kernel $R(t,y)$ can be obtained In [12].

Definition 2.17 ([2]) $W_2^1[a,b]$=$\{u(t)|u(t)$ is an absolutely continuous real value function, on $[a,b]$ and $u, u^{'} \in L^2[a,b]\}$. The inner product and norm in $W_2^1[a,b]$ are given respectively by

$$\langle u, v \rangle = \int_a^b \Big(u(t)v(t) + u^{'}(t)v^{'}(t) \Big)\mathrm{d}t$$

$$\|u\|_W = \sqrt{\langle u, u \rangle}_W, \qquad u \in W_2^1[a,b].$$

Cui and Lin defined a reproducing kernel space $W_2^1[0,1]$ and gave its reproducing kernel

$$\bar{R}(t,y) = \begin{cases} 1 + y & y \le t \\ 1 + t & y > t \end{cases}$$

Definition 2.18 ([2, 3]) $W_2^2[0,1]$=$\{u : u(t), u^{'}(t)$ is an absolutely continuous real value function, on $[0,1]$ and $u, u^{'}, u^{''} \in L^2[0,1]$ and $u(0) = 0\}$. The inner product and norm in $W_2^2[0,1]$ are given respectively by

$$\langle u, v \rangle = u(0)v(0) + u^{'}(0)v^{'}(0) + \int_0^1 u^{''}(t)v^{''}(t)\mathrm{d}t$$

and

$$\|u\|_W = \sqrt{\langle u, u \rangle}_W, \qquad u \in W_2^2[0,1].$$

Using Mathematica 8.0 software package, the representation of the reproducing kernel function $R_t(y)$ is provided by

$$R_t(y) = \begin{cases} \frac{1}{6}(y-a)(2a^2 - y^2 + 3t(2+y) - a(6+3t+y)), & y \le t, \\ \frac{1}{6}(t-a)(2a^2 - t^2 + 3y(2+t) - a(6+3y+t)), & y > t. \end{cases} \tag{8}$$

3. Fuzzy impulsive fractional differential equations

In this section, we are going to introduce fuzzy integral equations method expansion for solving fuzzy impulsive fractional differential equations by using concept of generalized Hukuhara differentiability.

$$ {}_cD^{\frac{1}{\alpha}}y(t) = f(t, y(t)), \quad t \in [0, T], \quad t \neq t_k, \ m-1 < \frac{1}{\alpha} < m, \quad m \in \mathbb{N}, \tag{9} $$

$$ \Delta y(t)|_{t=t_k} = I_k(y(t_k^-(t))), \tag{10} $$

$$ y(0) = y_0. \tag{11} $$

where $k = 1, 2, \dots, m$, ${}_cD^{\frac{1}{\alpha}}$ denotes the Caputo fractional generalized derivative of order $\frac{1}{\alpha}$, $y(t)$ is an unknown fuzzy function of crisp variable t and $f : [0, T] \times \mathbb{R}_F \to \mathbb{R}_F$, is continuous fuzzy function, $I_k : \mathbb{R}_{\mathbb{F}} \to \mathbb{R}_{\mathbb{F}}$, $y_0 \in \mathbb{R}_F, 0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta|_{t=t_k} = y(t_k^+) \ominus_{gH} y(t_k^-)$, $y(t_k^+) = lim_{h\to 0^+} y(t_k + h)$, and $y(t_k^-) = lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

Lemma 3.1 *([8]) The initial value problem* (9)*under the conditions* (10) *and* (11) *is equivalent to one of the following integral equations:*

$$ y(t) = y_0 \oplus \frac{1}{\Gamma(\frac{1}{\alpha})} \int_0^t (t-s)^{\frac{1}{\alpha}-1} f(s, y(s))ds, \quad t \in [0, t_1] \tag{12} $$

whenever $y(t)$ *as* $[(i) - gH]$*-differentiable,*

$$ y(t) = y_0 \ominus (-1) \frac{1}{\Gamma(\frac{1}{\alpha})} \int_0^t (t-s)^{\frac{1}{\alpha}-1} f(s, y(s))ds, \quad t \in [0, t_1] \tag{13} $$

whenever $y(t)$ *as* $[(ii) - gH]$*-differentiable,*

$$ y(t) = \begin{cases} y_0 \oplus \frac{1}{\Gamma(\frac{1}{\alpha})} \int_0^t (t-s)^{\frac{1}{\alpha}-1} f(s, y(s))ds, t \in [0, t_1] \\ y_0 \ominus (-1) \frac{1}{\Gamma(\frac{1}{\alpha})} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (t_k - s)^{\frac{1}{\alpha}-1} f(s, y(s))ds \ominus (-1) \\ \frac{1}{\Gamma(\frac{1}{\alpha})} \int_{t_k}^t (t-s)^{\frac{1}{\alpha}-1} f(s, y(s))ds \ominus (-1) \sum_{k=1}^m I_k(y(t_k^-)), if \ t \in (t_1, t_{k+1}] \end{cases} \tag{14} $$

if there exists a point $t_1 \in (0, t_{k+1})$ *such that* $y(t)$ *is* $[(i) - gH]-$*differentiable on* $[0, t_1]$ *and* $[(ii) - gH]-$ *differentiable on* (t_1, t_{k+1}).

Theorem 3.2 ([26])*Assume that*
$*(H_1)$ *There exists a constant* $0 \le l$ *such that* $d(f(t, y), f(t, \overline{y})) \le ld(y, \overline{y})$, *for each* $t \in [0, T]$, *and each* $u, \overline{u} \in R_F$
(H_2) *There exists a constant* $0 \le l^*$ *such that* $d(I_k(y), I_k(\overline{y})) \le l^* d(y, \overline{y})$, *for each* $y, \overline{y} \in R_F$, *and* $k = 1, 2, \dots, m$. *if*

$$ [\frac{T^{\frac{1}{\alpha}} l(m+1)}{\Gamma(\frac{1}{\alpha}+1)} + ml^*] < 1 $$

Such that T is very small numbers therefore, Eqs. (9)-(11) *has a unique solution on* $[0, T]$.

4. Solving fuzzy impulsive fractional differential equation in $W_2^2[a, b]$

Using Lemma (3.1), the solution of Eqs. (9)-(11) is equivalent to solution of Eqs. (12)-(14). We show how RKHSM applied to solve integral equation Eq. (14). Thus

$$y(t) = \begin{cases} y_0 \oplus \frac{1}{\Gamma(\frac{1}{\alpha})} \int_0^t (t-s)^{\frac{1}{\alpha}-1} f(s, y(s)) ds, \ t \in [0, t_1], \\ \\ y_0 \ominus (-1) \frac{1}{\Gamma(\frac{1}{\alpha})} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (t_k - s)^{\frac{1}{\alpha}-1} f(s, y(s)) ds \ominus (-1) \\ \frac{1}{\Gamma(\frac{1}{\alpha})} \int_{t_k}^t (t-s)^{\frac{1}{\alpha}-1} f(s, y(s)) ds \ominus (-1) \sum_{k=1}^m I_k(y(t_k^-)), \ t \in (t_1, t_{k+1}], \end{cases} \tag{15}$$

By impulsive effect, we have:

$$\Delta y(t) = y(0^+) \ominus_{gH} y(0^-) = I_0(y(0^-))$$

$$\Longrightarrow \begin{cases} y(0^+) = I_0(y(0^-)) \oplus y(0^-), \ t \in [0, t_1], \\ y(0^-) = y(0^+) \oplus (-1) I_0(y(0^-)), \ t \in (t_1, t_{k+1}]. \end{cases} \tag{16}$$

By substituting Eq. (16) into Eq. (15) we have

$$y(t) = \begin{cases} y(0^-) \oplus I_0(y(0^-)) \oplus \frac{1}{\Gamma(\frac{1}{\alpha})} \int_0^t (t-s)^{\frac{1}{\alpha}-1} f(s, y(s)) ds, \ t \in [0, t_1], \\ \\ y(0^-) \ominus (-1) I_0(y(0^-)) \ominus (-1) \frac{1}{\Gamma(\frac{1}{\alpha})} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (t_k - s)^{\frac{1}{\alpha}-1} f(s, y(s)) ds \ominus (-1) \\ \frac{1}{\Gamma(\frac{1}{\alpha})} \int_{t_k}^t (t-s)^{\frac{1}{\alpha}-1} f(s, y(s)) ds \ominus (-1) \sum_{k=1}^m I_k(y(t_k)), \ t \in (t_1, t_{k+1}], \end{cases} \tag{17}$$

In Eq. (17), we define $y(t^-) = y(t)$. Thus

$$y(t) = \begin{cases} I_0(y(0)) \oplus y(0) \oplus \frac{1}{\Gamma(\frac{1}{\alpha})} \int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s)) ds, \ t \in [0, t_1], \\ \\ y(0) \ominus (-1) I_0(y(0)) \ominus (-1) \frac{1}{\Gamma(\frac{1}{\alpha})} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (t_k - s)^{\frac{1}{\alpha}-1} f(s, y(s)) ds \ominus \\ (-1) \frac{1}{\Gamma(\frac{1}{\alpha})} \int_{t_k}^t (t-s)^{\frac{1}{\alpha}-1} f(s, y(s)) ds \ominus (-1) \sum_{k=1}^m I_k(y(t_k)), \ t \in (t_1, t_{k+1}], \end{cases} \tag{18}$$

To solve Eq. (18); we define operator

$$L = W_2^2[a, b] \to W_2^1[a, b], \quad t \in [0, t_{k+1}]$$

as follows:

$$Ly(t)=\begin{cases} y(t)\ominus I_0(y(0))\ominus y(0)\ominus \frac{1}{\Gamma(\frac{1}{\alpha})}\int_0^t (t-s)^{\frac{k}{\alpha}-1}f(s,y(s))ds,\ t\in[0,t_1],\\ y(t)\oplus(-1)I_0(y(0))\oplus y(0)\oplus(-1)\frac{1}{\Gamma(\frac{1}{\alpha})}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\frac{1}{\alpha}-1}f(s,y(s))ds\oplus\\ (-1)\frac{1}{\Gamma(\frac{1}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{1}{\alpha}-1}f(s,y(s))ds\oplus(-1)\sum_{k=1}^{m}I_k(y(t_k)),\ t\in(t_1,t_{k+1}],\end{cases} \tag{19}$$

It is clear that L is a bounded linear operator. Model problem Eq. (19) changes to the following problems:

$$Ly(t)=\begin{cases} F(t,y(t),T_1y(t)),\ t\in[0,t_1],\\ F(t,y(t),Ty(t),Sy(t)),\ t\in(t_1,t_{k+1}]\end{cases} \tag{20}$$

Such that

$$F(t,y(t),T_1y(t))=y(t)\ominus I_0(y(0))\ominus y(0)\oplus(-1)T_1y(t),$$
$$T_1y(t)=\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}f(s,y(s))ds,$$

and

$$F(t,y(t),Ty(t),Sy(t))=y(t)\ominus I_0(y(0))\ominus y(0)\oplus(-1)\sum_{k=1}^{k}I_k(y(t_k))\ominus Ty(t)\oplus(-1)Sy(t),$$
$$Ty(x)=\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\frac{k}{\alpha}-1}f(s,y(s))ds,$$
$$Sy(x)=\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,y(s))ds.$$

Where $y(t)\in W_2^2[a,b]$, $F(t,y(t),T_1y(t))$ and $F(t,y(t),Ty(t),Sy(t))\in W_2^1[a,b]$ for $t\in[a,b]$. Where $y(t)\in W_2^2[a,b]$ and $Ly(t)\in W_2^1[a,b]$ for $t\in[a,b]$. Using the Eq. (8) put $\phi_i(t)=R(t,t_i)$ and $\psi_i(t)=L^*\phi_i(t)$ where $\{t_i\}_{i=1}^{\infty}$ in $[a,b]$ and L^* is the adjoint operator of L. It is easy to see that

$$\psi_i(t)=[L_yR_t(y)](t_i)=\begin{cases} R(t,t_i)\ominus I_0(R(0,t_i)))\ominus R(0,t_i)\ominus\frac{1}{\Gamma(\frac{1}{\alpha})}\int_0^t(t-s)^{\frac{1}{\alpha}-1}f(s,R(s,t_i))ds,\ t\in[0,t_1],\\ R(t,t_i)\oplus(-1)I_0(R(0,t_i))\oplus R(0,t_i)\oplus(-1)\sum_{k=1}^{m}I_k(R(t_k,t_i))\oplus(-1)\\ \frac{1}{\Gamma(\frac{1}{\alpha})}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\frac{1}{\alpha}-1}f(s,R(s,t_i))ds\\ \oplus(-1)\frac{1}{\Gamma(\frac{1}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{1}{\alpha}-1}f(s,R(s,t_i))ds,\ t\in(t_1,t_{k+1}]\end{cases}$$

Theorem 4.1 *If $\{t_i\}_{i=1}^{\infty}$ is dense on $[a,b]$ then $\{\psi_i\}_{i=1}^{\infty}$ is the complete function system of the space $W_2^2[a,b]$ and $\psi_i(t)=[L_yR_t(y)](t_i)$, where the subscript t in the operator L indicates that the operator L applies to the function of t.*

Proof We have

$$\psi_i(t)=\langle(L^*\phi_i)),R_y(t)\rangle_{W_2^2}=\langle(\phi_i)),L_tR_y(t)\rangle_{W_2^2}=L_tR_y(t)|_{t=y_i},t\in[0,t_{k+1}]$$

Clearly $\psi_i\in W_2^2[a,b]$.For each fixed $y(t)\in W_2^2[a,b]$, let

$$\langle y(t),\psi_i(t)\rangle_{W_2^2[a,b]}=0,\ t\in[0,t_{k+1}]$$

$i=1,2,\ldots$, it means

$$\langle y(t),L^*\phi_i(t)\rangle_{W_2^2[a,b]}=\langle Ly(t),\phi_i(t)\rangle_{W_2^2[a,b]}=Ly(t_i)=0,\ t\in[0,t_{k+1}]$$

Assume that $\{t_i\}_{i=1}^{\infty}$ is dense on $[a, b]$ and so $Ly(t) = 0$. It follows that $y = 0$ from the existence of L^1. Now, the theorem is proved. ■

Definition 4.2 The orthonormal system $\{\hat{\psi}_i(t)\}_{i=1}^{\infty}$ of $W^2[a, b]$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(t)_{i=1}^{\infty}\}$,

$$\hat{\psi}_i(t) = \sum_{k=1}^{i} \beta_{ik}\psi_k(t),\ t \in [0, t_{k+1}], \tag{21}$$

where $\beta_{ik}\{(i = 1, 2, \ldots), (k = 1, 2, \ldots)\}$ are coefficients of Gram-Schmidt orthonormalizarion and $\{\hat{\psi}_i(t)\}_{i=1}^{\infty}$ is an orthonormal system, could be determined by solving the following equations.

$$B_{ik} = \langle \psi_i, \hat{\psi}_i \rangle = \psi_i(a)\hat{\psi}_i(a) + \psi_i'(a)\hat{\psi}_i'(a) + \int_a^b \psi_i''(t)\hat{\psi}_i''(t)\mathrm{d}t,\ t \in [0, t_{k+1}],$$

$$\beta_{ii} = 1/\Big(\sqrt{[(\psi_i(a))^2 + (\psi_i'(a))^2 + \int_a^b (\psi_i''(t))^2\mathrm{d}t - \sum_{k=1}^{i-1} B_{ik}^2]}\Big),\ t \in [0, t_{k+1}],$$

$$\beta_{ij} = \beta_{ii} * (-\sum_{k=j}^{i-1} B_{ik} * \beta_{kj})\ (i = 1, 2, \ldots),\ (j = 1, 2, \ldots, i-1),\ (k = 1, 2, \ldots, i-1).$$

Theorem 4.3 *If $\{t_i\}$ is dense on $[a, b]$ and the solution of Eq.* (20) *is unique, then the solution of Eq.* (20) *is*

$$u(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} y(t_k)\hat{\psi}_i(t),\ t \in [0, t_{k+1}] \tag{22}$$

Put throughout this article $t_1 = 1$

Proof Using Eq. (21), we have

$$\begin{aligned} u(t) &= \textstyle\sum_{i=1}^{\infty} \langle y(t), \hat{\psi}_i(t)\rangle_{W_2^1} \hat{\psi}_i(t) \\ &= \textstyle\sum_{i=1}^{\infty} \langle y(t), \sum_{k=1}^{i} \beta_{ik}\psi_k(t)\rangle_{W_2^1} \hat{\psi}_i(t) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle y(t), \psi_k(t)\rangle_{W_2^1} \hat{\psi}_i(t) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle y(t), L^*\phi_k(t)\rangle_{W_2^1} \hat{\psi}_i(t) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle Ly(t), \phi_k(t)\rangle_{W_2^1} \hat{\psi}_i(t) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} Ly(t_k)\hat{\psi}_i(t) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} y(t_k)\hat{\psi}_i(t),\ t \in [0, t_{k+1}] \end{aligned}$$

On the other hand, $y(t) \in W_2^2[a, b]$ and

$$y(t) = \sum_{i=1}^{\infty} a_i\hat{\psi}_i(t),\ t \in [0, t_{k+1}],$$

where

$$a_i = \langle y(t), \hat{\psi}_i(t)\rangle,\ t \in [0, t_{k+1}]$$

are the Fourier series expansion about normal orthogonal system $\{\hat{\psi}_i(t)\}_{i=1}^{\infty}$ and $W_2^2[a,b]$ is the Hilbert space. Thus the series $\sum_{i=1}^{\infty} a_i\hat{\psi}_i(t), t \in [0, t_{k+1}]$. is convergent in the sense of $||.||_{W_2^2}$ and the proof would be complete. ■

Now the approximate solution $y_N(t)$ can be obtained by the N-term intercept of the exact solution $y(t)$ and

$$y_N(t) = \sum_{i=1}^{N}\sum_{k=1}^{i} \beta_{ik} y(t_k)\hat{\psi}_i(t),\ t \in [0, t_{k+1}] \tag{23}$$

In the sequel, a new iterative method to achieve the solution of Eq. (20) is presented. If

$$A_i = \sum_{k=1}^{i} \beta_{ik} y(t_k),\ t \in [0, t_{k+1}]$$

then Eq. (22) can be written as

$$y(t) = \sum_{k=1}^{\infty} A_i\hat{\psi}_i(t),\ t \in [0, t_{k+1}].$$

Now suppose, for some $t_i, y(t_i)$ is known. There is no problem if we assume $i = 1$.We put $y_0(t_1) = y(t_1)$ and define the N-term approximation to $y(t)$ by

$$y_N(t) = \sum_{k=1}^{N} c_i\hat{\psi}_i(t),\ t \in [0, t_{k+1}], \tag{24}$$

where

$$c_i = \sum_{i=1}^{k} \beta_{ik} y_{k-1}(t_k),\ t \in [0, t_{k+1}]. \tag{25}$$

In the following, it would be proven that the approximate solution $y_N(t)$ in the iterative Eq.(24) is convergent to the exact solution of Eq. (20) uniformly.

Theorem 4.4 *Suppose that $||y_N(t)||_{W_2^2}$ is bounded in Eq. (24). If $\{t_i\}_{i=1}^{\infty}$ is dense on $[a,b]$ then N-term approximate solution $y_N(t)$ in the iterative Eq. (24) converges to the exact solution $y(t)$ of Eq. (20) and*

$$y(t) = \lim_{n\to\infty} \sum_{i=1}^{n} c_i^N \psi_i(t),\ t \in [0, t_{k+1}],$$

where as c_i are give by Eq. (25).

Proof First of all,the convergence of $u_N(t)$ from Eq. (24) will be proved. We infer

$$y_{n+1}(t) = y_n + c_{n+1}\hat{\psi}_{n+1}(t),\ t \in [0, t_{k+1}]$$

It is obvious that the sequence $||y_n(t)||_{W_2^2}$ is monotonically increasing. Because $||y_n(t)||_{W_2^2}$ is bounded and $||y_n(t)||_{W_2^2}$ is convergent, then $\sum_{i=1}^{\infty} c_i^2$ is bounded and this implies that $\{c_i\}_{i=1}^{\infty} \in L^2$. If $m > n$ then

$$||y_m - y_n||^2_{W_2^2[a,b]} = ||\sum_{i=m}^{n+1}(y_i - y_{i-1})||^2_{W_2^2[a,b]} = \sum_{i=m}^{n+1}||(y_i - y_{i-1})||^2_{W_2^2[a,b]}.$$

So $||(y_i - y_{i-1})||^2_{W_2^2[a,b]} = c_i^2$. Consequently $||y_m - y_n||^2_{W_2^2[a,b]} = \sum_{i=1}^{\infty} c_i^2 \to 0$ as $n \to \infty$. To prove the completeness of $W_2^2[a,b]$ it requires $\hat{y}$, where $\hat{y} \in W_2^2[a,b]$ that $y_n \to \hat{y}$ as $n \to \infty$. Now we can prove $\hat{y}$ is the solution of Eq. (20). If we take limit from Eq. (24), we will have $\hat{y}(t) = \sum_{i=1}^{N} c_i\hat{\psi}_i$, so $L\hat{y}(t) = \sum_{i=1}^{N} c_i L\hat{\psi}_i$. Let $t_l \in \{t_i\}_{i=1}^{\infty}$, then

$$\begin{aligned} L\hat{y}_l(t) &= \textstyle\sum_{i=1}^{\infty} c_i\hat{\psi}_i(t_l) \\ &= \textstyle\sum_{i=1}^{\infty} c_i\langle L\hat{\psi}_i(t), \phi_l(t)\rangle_{W_2^2[a,b]} \\ &= \textstyle\sum_{i=1}^{\infty} c_i\langle \hat{\psi}_i(t), L^*\phi_l(t)\rangle_{W_2^2[a,b]} \\ &= \textstyle\sum_{i=1}^{\infty} c_i\left\langle \hat{\psi}_i(t), \hat{\psi}_l(t)\right\rangle_{W_2^2[a,b]} \\ &= \textstyle\sum_{i=1}^{\infty} c_i\left\langle \hat{\psi}_i(t), \sum_{l=1}^{i} c_{il}\psi_l(t)\right\rangle_{W_2^2[a,b]}, \ t \in (t_1, t_{k+1}] \end{aligned}$$

From Eq. (24),it is concluded that $L\hat{y}(t_l) = \hat{y}(t_l)$. $\{t_i\}_{i=1}^{\infty}$ is dense on $[a,b]$. For each $t \in [a,b], \{t_{n_i}\}_{i=1}^{\infty}$ subsequence exists that $t_{n_i} \to t$, as $i \to \infty$. Hence, when $i \to \infty$, we have $ly(t) = y(t)$ which indicates that $\hat{y}$ is the solution of Eq. (20). ■

Lemma 4.5 *If $y(t) \in W_2^2[a,b]$, then there exists a constant C such that $\mid y(t) \mid \leqslant C \parallel y(t) \parallel_W$, $\mid y^{'}(t) \mid \leqslant C \parallel y(t) \parallel_W$.*

Proof.

$$\mid y(t) \mid = \mid \langle y(y), R(t,y)\rangle \mid \leqslant \parallel y(y) \parallel_W \parallel k(t,y) \parallel_W$$

there exists a constant c_0 such that

$$c_0 = \parallel k(t,y) \parallel_W \in W_2^2[a,b], \quad \mid y(x) \mid \leqslant c_0 \parallel y(t) \parallel_W .$$

Note that

$$\begin{aligned} \mid y^{(i)}(t) \mid = \mid \langle y(t), \frac{\partial^i k(t,y)}{\partial t^i}\rangle_W \mid &\leqslant \parallel y(t) \parallel_W \parallel \frac{\partial^i k(t,y)}{\partial t^i} \parallel_W \\ &\leqslant c_i \parallel y \parallel_W \end{aligned}$$

where c_i are constants. Putting $C = \max c_i$ and the proof of the lemma is complete.

Theorem 4.6 *The approximate solution $y_n(t)$ and its derivatives $y_n^{'}(t)$, are all uniformly convergent.*

Proof. We know

$$|y_n(t) - y(t)| = |\langle y_n(t) - y(t), R(t,y)\rangle| \leq \|y_n - y\| \|R(t,y)\|_{W^2} \leq M\|y_n - y\|.$$

Also

$$\begin{aligned} y_n^{'}(t) - y^{'}(t) &= (y_n(t) - y((t))^{'} \\ &= \tfrac{\partial}{\partial t}\langle (y_n(t) - y(t)), k(t,y)\rangle_{W_2^2} \\ &= \langle y_n(t) - y(t), \tfrac{\partial}{\partial t}k(t,y)\rangle_{W_2^2} \\ &\leq \|(y_n(t) - y(t))\| \parallel \tfrac{\partial}{\partial t}k(t,y) \parallel, \ \tfrac{\partial}{\partial t}k(t,y) \in W_2^2[a,b] \end{aligned}$$

one obtains

$$\mid y_n^{'}(t) - y^{'}(t) \mid \leqslant \parallel y_n(y) - y(y) \parallel_{W_2^2} \parallel \frac{\partial}{\partial t}k(t,y) \parallel .$$

Also $\parallel \frac{\partial}{\partial x}k(x,y) \parallel_{W_2^2}$ is continuous with respect to x in $[a,b]$, then

$$\mid y_n^{'}(t) - y^{'}(t) \mid \leqslant M \parallel y_n(t) - y(y) \parallel_{W_2^2}$$

where M is a positive number. So that

$$\lim_{x\to n} y_n(t) = y(t) \Rightarrow \lim_{t\to n} y_n^{'}(t) = y^{'}(t).$$

5. Error estimations and error bounds

Lemma 5.1 *Suppose that $y_j \in C^m[a,b]$ and $y_j^{(m+1)} \in L^2[a,b]$, for some $1 \leq m$. If y_j vanishes at M with $N \leq m+1$,then $u_j \in W_2^2[a,b]$ and there is a constant A_j such that*

$$||y_j||_{W_2^2} \leq A_j h^m \max\left|y_j^{(m+1)}(t)\right|, \ t \in [0, t_{k+1}],$$

where $j = 1, 2, \ldots, n$. We mention here that for the next results the hypotheses of Lemma 5.1 are hold.

Proof The proof similar Lemma 3 [5]. ■

Theorem 5.2 *If $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$, then there is a constant $D = (D_1, D_2, \ldots, D_n)$ such that*

$$||y||_H \leq h^m ||D||_2 \Big|\Big| \max|y^{(m+1)}(t)| \Big|\Big|_2, \ t \in [0, t_{k+1}]$$

Proof Considering Denition 2.17, one can write

$$\begin{aligned} ||y||_H &= \left(\textstyle\sum_{j=1}^n ||y_j||^2_{W_2^2}\right)^{\frac{1}{2}} \\ &\leq \sqrt{\textstyle\sum_{j=1}^n \left(h^m D_j \Big| \max|y^{(m+1)}(t)| \Big|\right)^2} \\ &= h^m \sqrt{\textstyle\sum_{j=1}^n D_j^2 \left(\Big| \max|y^{(m+1)}(t)| \Big|\right)^2} \\ &\leq h^m \sqrt{\textstyle\sum_{j=1}^n D_j^2 \sum_{j=1}^n \left(\Big| \max|y^{(m+1)}(t)| \Big|\right)^2} \\ &= h^m ||D||_2 || \max|y^{(m+1)}(t)| ||_2, \ t \in [0, t_{k+1}] \end{aligned}$$

■

Definition 5.3 we may use a norm that represents the maximum value at each function. That means, if $y(t) = (y_1(t), y_2(t), \dots, y_n(t))$, then

$$||y(t,r)||_\infty = \max_j \left(\sup |y_j| \right),\ t \in (0, t_{k+1}]$$

and $j = 1, 2, \dots n$.

Theorem 5.4 *Let* $y(t,r)$ *and* $y_n(t,r)$ *are given by Eqs.* (22) *and* (23)*, respectively, then there is a constant* B *such that*

$$||y^{(i)} - y_n^{(i)}||_\infty \leq = \begin{cases} B_1 h^m,\ i = 0,1 \\ B_2 h^m,\ i = 0,1 \end{cases}$$

Proof The proof will be obtained by mathematical inductionas follows: from Eq $y_n(t) = \sum_{j=1}^N A_j \hat{\psi}_j$,$\forall\ j \leq n$, we see that

$$Ly_N(t) = \sum_{j=1}^N A_j L\hat{\psi}_j$$

and

$$(Ly_N)(t_n) = \sum_{j=1}^N A_j (L\hat{\psi}_j, \phi_n) = \sum_{j=1}^N A_j (\hat{\psi}_j, L^* \phi_n) = \sum_{j=1}^N A_j (\hat{\psi}_j, \hat{\psi}_n)$$

therefore

$$\sum_{i=1}^n \beta_{ni} (Ly_N)(t_i) = \sum_{j=1}^N A_j (L\hat{\psi}_j, \sum_{i=1}^n \beta_{ni} \phi_i) = \sum_{j=1}^N A_j (\hat{\psi}_i, \hat{\psi}_n) = A_n.$$

If $n = 1$, then

$$(Ly_N)(t_1) = \begin{cases} F(t_1, y(t_1), T_1 y(t_1)),\ t \in [0, t_1], \\ F(t_1, y(t_1), Ty(t_1), Sy(t_1)),\ \in (t_1, t_{k+1}]. \end{cases}$$

If $n = 2$, then

$$\beta_{21}(Ly_N)(t_1) + \beta_{22}(Ly_N)(t_2) = \begin{cases} \beta_{21} F(t_1, y(t_1), T_1 y(t_1)),\ t \in [0, t_1], \\ \beta_{21} F(t_1, y(t_1), Ty(t_1), Sy(t_1)),\ t \in (t_1, t_{k+1}]. \end{cases}$$

$$+ \begin{cases} \beta_{22} F(t_2, y(t_2), T_1 y(t_2)),\ t \in [0, t_1], \\ \beta_{22} F(t_2, y(t_2), Ty(t_2), Sy(t_2)),\ t \in (t_1, t_{k+1}]. \end{cases}$$

It is clear that

$$(Ly_N)(t_2) = \begin{cases} F(t_2, y(t_2), T_1 y(t_2)),\ t \in [0, t_1], \\ F(t_2, y(t_2), Ty(t_2), Sy(t_2)),\ t \in (t_1, t_{k+1}]. \end{cases}$$

Moreover, it is easy to see by induction that

$$(Ly_N)(t_j) = \begin{cases} F(t_j, y(t_j), T_1 y(t_j)), \ t \in [0, t_1], \\ F(t_j, y(t_j), Ty(t_j), Sy(t_j)), \ t \in (t_1, t_{k+1}], \end{cases}$$

where $j = 1, 2, \ldots, N$. Clearly $R_N \in C^m[o, t_{k+1}]$ and $R_N^{(m+1)} \in L^2[0, t_{k+1}]$. Therefore, from Theorem 5.2 there is a constant $D = (D_1, D_2, \ldots, D_n)$ such that

$$||R_N|| \leq h^m ||D||_2 \Big|\Big| \max |R_N^{(m+1)}(t)| \Big|\Big|_2, \ t \in (0, t_{k+1}].$$

Recalling that the error function

$$\begin{aligned} R_N(t) &= (Ly_N)(t_j) - F(t_j, y(t_j), Ty(t_j), Sy(t_j)) \\ &= Ly_N(t) - Ly(t) = L(y_N(t) - y(t)), \ t \in (0, t_{k+1}] \end{aligned}$$

Hence, $y_N - y = L^1 R_N$, then there is fixed a E so that

$$\begin{aligned} ||y - y_N||_W &= ||L^{-1} R_N|| \\ &\leq ||L^{-1}|| ||R_N|| \\ &\leq E h^m ||D||_2 \Big|\Big| \max |R_N^{(m+1)}(t)| \Big|\Big|_2, \ t \in (0, t_{k+1}] \end{aligned}$$

Eventually, in view of Lemma 4.5, we can nd that

$$||y^{(i)}(t) - y_N^{(i)}(t)||_\infty \leq K ||y - y_N||_W \leq K E h^m ||D||_2 \Big|\Big| \max |R_N^{(m+1)}(t)| \Big|\Big|_2, \ t \in (0, t_{k+1}]$$

$$= \begin{cases} B_1 h^m \\ B_2 h^m \end{cases}$$

where

$$\begin{aligned} B_1 &= K E ||D||_2 || \max |R_N^{(m+1)}(t)| ||_2, \ t \in [0, t_1], \\ B_2 &= K E ||D||_2 || \max |R_N^{(m+1)}(t)| ||_2, \ t \in (t_1, t_{k+1}]. \end{aligned}$$

■

Lemma 5.5 *Suppose that* $h = \begin{cases} \frac{t_1 - t_0}{N} \\ \frac{t_{k+1} - t_k}{N} \end{cases}$ *is the ll distance for the uniform partition of* $t \in \begin{cases} [0, t_1], \\ (t_1 t_{k+1}] \end{cases}$. *Let* $y(t)$ *and* $y_N(t)$ *are given by Eqs.* (22) *and* (23), *respectively, then*

$$||y^{(i)}(t) - y_N^{(i)}(t)||_\infty = O(N^{-m}), \ i = 0, 1.$$

Proof The proof follows directly from Theorem 5.4. ■

Theorem 5.6 *Let $\epsilon_N = ||y - y_N||_W$,where $y(t)$ and $y_N(t)$ are the exact and the numerical solution, respectively. Then, the sequence of numbers $\{\epsilon_N\}$ are decreasing in the sense of the norm of $W_2^2[a,b]$ and $\epsilon_N \to 0$ as $N \to \infty$.*

Proof Using the expansions form of $y(t)$ and $y_N(t)$ in Eqs.(22) and (23), we can write

$$\epsilon_N^2 = ||\sum_{i=N+1}^{\infty}\sum_{j=1}^{n}\beta ij\hat{\psi}_{ij}(t)||^2,\ t \in [0, t_{k+1}]$$

and

$$\epsilon_{N-1}^2 = ||\sum_{i=N}^{\infty}\sum_{j=1}^{n}\beta ij\hat{\psi}_{ij}(t)||^2 = \sum_{i=N}^{\infty}(\sum_{j=1}^{n}\beta ij)^2,\ t \in [0, t_{k+1}].$$

Clearly,$\epsilon_N \leq \epsilon_{N-1}$, and consequently $\{\epsilon_N\}_{N=1}^{\infty}$ are decreasing in the sense of$||.||_{W_2^2}$. By Theorem 6, we know that

$$\sum_{i=N}^{\infty}\sum_{j=1}^{n}\beta ij\hat{\psi}_{ij}(t),\ t \in [0, t_{k+1}]$$

is convergent. Thus,$\epsilon_N^2 \to 0$ or $\epsilon_N \to 0$. So, the proof of the theorem is complete. ■

6. Numerical examples

In this section, we solve the following one example appearing in Ref.[26], by using the method discussed above. All experiments were performed in MATHEMATICA 8.0. For solving fuzzy impulsive fractional differential equations using the following algorithm.

Algorithm: The approximate and exact solution $y_N(t,r)$ and $y(t,r)$ for fuzzy impulsive fractional differential equations (9)-(11), we do the following main steps:
Step 1. Fixed $t, y \in [a,b]$,
if $y \leq t$ set $R_t^2(y) = \frac{1}{6}(y-a)(2a^2 - y^2 + 3t(2+y) - a(6+3t+y))$
Else set $R_t^2(y) = \frac{1}{6}(t-a)(2a^2 - t^2 + 3y(2+t) - a(6+3y+t))$
For $i = 1, 2, \ldots, n$,$h = 1, 2, \ldots, m$ and $j = 1, 2$,do the following:
Set $t_i = \frac{i-1}{n-1}$,
Set $r_h = \frac{h-1}{m-1}$,
Set

$$\psi_i(t_i) = L^{-1}R_t^2(y)|_{y=t_i}, \quad t \in [0, t_{k+1}].$$

Output: The orthogonal function system $\psi_i(t_i)$.
Step 2.

$$B_{ik} = \langle\psi_i, \hat{\psi}_i\rangle = \psi_i(a)\hat{\psi}_i(a) + \psi_i'(a)\hat{\psi}_i'(a) + \int_a^b \psi_i''(t)\hat{\psi}_i''(t)\mathrm{d}t, \quad t \in (0, t_{k+1}].$$

$$\beta_{ii} = 1/\Big(\sqrt{[(\psi_i(a))^2 + (\psi_i'(a))^2 + \int_a^b(\psi_i''(t))^2\mathrm{d}t - \sum_{k=1}^{i-1}B_{ik}^2]}\Big), \quad t \in (0, t_{k+1}].$$

$$\beta_{ij} = \beta_{ii} * (-\sum_{k=j}^{i-1} B_{ik} * \beta_{kj})\ (i = 1, 2, \ldots),\ (j = 1, 2, \ldots, i-1),\ (k = 1, 2, \ldots, i-1).$$

Output: The orthogonalization coefficients β_{ik}
Step 3. Set $\hat{\psi}_i(t) = \sum_{k=1}^{i} \beta_{ik}\psi_k(t), \qquad (\beta_{ii} > 0, \quad i = 1, 2, \ldots).$
Output: The orthogonal function system $\hat{\psi}_i(t)$
Step 4. Set $y_0(t_1) = y(t_1)$
Step 5. Set $n = 1$
Step 6. Set

$$c_n = \sum_{k=1}^{n} \beta_{nk} y_{k-1}(t_k), \quad t \in (0, t_{k+1}].$$

Step 7. Set

$$y_n(t) = \sum_{i=1}^{n} c_i \hat{\psi}_i(t), \quad \in (0, t_{k+1}].$$

Step 8. Set

$$\underline{y_n(t,r)} = \sum_{i=1}^{n} \underline{c_i \hat{\psi}_i}, \quad t \in (0, t_{k+1}].$$

And

$$\overline{y_n(t,r)} = \sum_{i=1}^{n} \underline{c_i \hat{\psi}_i}, \quad t \in (0, t_{k+1}].$$

if $y(t,r)$ is [(i)-gH]-differentiability then:

$$y(t,r) = \Big[\sum_{i=1}^{n} \underline{c_i \hat{\psi}_i}, \sum_{i=1}^{n} \overline{c_i} \overline{\hat{\psi}_i}\Big].$$

And if $y(t,r)$ is [(ii)-gH]-differentiability then:

$$y(t,r) = \Big[\sum_{i=1}^{n} \overline{c_i} \overline{\hat{\psi}_i}, \sum_{j=1}^{n} \underline{c_i \hat{\psi}_i}\Big].$$

Example 6.1 Let us consider the fuzzy impulsive fractional differential equation,

$$
\begin{aligned}
&{}_c D^{\frac{1}{\alpha}} y(t) = \tfrac{1|y(t)|}{10(1+|y(t)|)}, \quad t \in J := [0,1], \quad t \neq \tfrac{1}{2}, \ m-1 < \tfrac{1}{\alpha} < m, \quad m \in \mathbb{N}\ , \\
&\Delta y|_{t=\frac{1}{2}} = \tfrac{|y(\frac{1^-}{2})|}{3+|y(\frac{1^-}{2})|}, \\
&y(0) = [\tilde{0}, \tilde{0}].
\end{aligned}
$$

Set

$$I_k(t) = [\tfrac{(3r-1)t}{t+3}, \tfrac{(3-r)t}{t+3}],$$

$$f(t, y_{ii.gH}(t)) = \Big[\tfrac{(r-1)t}{10(1+t)}, \tfrac{(1-r)t}{10(1+t)}\Big].$$

Table 1. Numerical results of Example 1 for $\underline{y_{i.gH}}(t)$ and $\underline{y_{ii.gH}}(t)$.

r/t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.55	-0.00540	-0.00480	-0 .00420	-0.00360	-0.0030	-0.00243	-0.00180	-0.00120	-0.00060	0
0.60	-0.00662	-0.00589	-0.00515	-0.00441	-0.00368	-0.00294	-0.00221	-0.00147	-0.00074	0
0.65	-0.00802	-0.00713	-0.00748	-0.00535	-0.00445	-0.00365	-0.00267	-0.00178	-0.00089	0
0.70	-0.00961	-0.00854	-0 .00888	-0.00761	-0.00534	-0.00427	-0.00320	-0.00213	-0.00107	0
0.75	- 0.01142	-0.01015	-0.01047	-0.00897	-0.00634	-0.00507	-0.00380	-0.00254	-0.00127	0
0.80	-0.01346	-0.01196	-0.01225	-0.01012	-0.00747	-0.00598	-0.00448	-0.00350	-0.00149	0
0.85	-0.01576	-0.01401	-0.01426	-0.01050	-0.00875	-0.00700	-0.00525	-0.00407	-0.00175	0
0.90	- 0.01834	-0.01630	-0.01426	-0.01212	-0.01018	-0.00814	-0.00610	-0.22024	-0.00203	0
0.95	-0.02122	-0.01886	-0.01650	- 0.01414	-0.01178	-0.00942	-0.00706	-0.00471	-0.00235	0
1	-0.02443	- 0.02171	-0 .01899	-0.01628	-0.01365	-0.01084	-0.00813	-0.00542	-0.00271	0

Table 2. Numerical results of Example 1 for $\overline{y_{i.gH}}(t)$ and $\overline{y_{ii.gH}}(t)$.

r/t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.55	0.00539	0.00479	0 .00419	0.00359	0.00288	0.00239	0.00179	0.00119	0.00059	0
0.60	0.00660	0.00587	0.00514	0.00440	0.00367	0.00293	0.002204	0.00147	0.00073	0
0.65	0.00800	0.00711	0.00622	0.00533	0.00444	0.00355	0.00266	0.00342	0.00089	0
0.70	0.00958	0.00852	0 .00745	0.00639	0.00533	0.00426	0.00319	0.00178	0.00107	0
0.75	0.01138	0.01012	0 .00885	0.00759	0.00633 0	0.00506	0.00379	0.00253	0.00126	0
0.80	0.01341	0.01192	0 .01043	0.00895	0.00746 0	0.00596	0.00447	0.00298	0.00149	0
0.85	0.01570	0.01396	0 .01221	0.01047	0.00087	0.00698	0.00524	0.00349	0.00174	0
0.905	0.01826	0.01623	0 .01421	0.01218	0.01015	0.00812	0.00609	0.00406	0.00203	0
0.95	0.02112	0.01878	0 .01644	0.01409	0.01174	0.00940	0.00703	0.00470	0.00235	0
1	0.02431	0.02162	0 .01892	0.01622	0.01352	0.01822	0.00811	0.00541	0.00270	0

Table 3. Numerical results of Example 2 for $\underline{y_{i.gH}}(t)$ and $\underline{y_{ii.gH}}(t)$.

r/t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.55	0.013806	- 0.01804	-0.04988	-0.00817	-0.11385	-0.14543	-0.17729	-0.20915	-0.27288	0
0.60	0.027612	-0.005760	-0.03913	-0.07251	- 0.10589	0.139280	-0.17267	-0.20607	-0.23947	0
0.65	0.044673	-0.009418	- 0.02584	-0.06111	-0.09639	-0.13167	-0.16696	-0.20226	-0.23757	0
0.70	0.065519	0.027961	-0.009610	-0.04719	-0.08477	- 0.12237	-0.15998	-0.19760	-0.23523	0
0.75	0.090734	0.050394	0 .010039	-0.03033	-0.07072	- 0.11112	-0.15153	-0.19176	-0.23241	0
0.80	0.120964	0.077291	0 .033597	-0.01012	-0.05386	- 0.09762	-0.14140	-0.18520	-0.22903	0
0.85	0.156914	0.109282	0 .061620	0.013927	-0.03380	-0.08155	-0.12933	-0.17715	-0.22500	0
0.90	0.199357	0.147954	0 .094710	0.042324	-0.01010	- 0.06257	-0.11509	-0.22024	-0.27288	0
0.95	0.249129	0.191353	0 .133522	0.075635	0.01769	- 0.04031	-0.09836	-0.15648	-0.21465	0
1	0.307135	0.242986	0 .178764	0.114469	0.05010	-0.01434	-0.07886	-0.14346	-0.20813	0

Using Eq.(22) and taking $k = 2$ and $\frac{1}{\alpha} = \frac{1}{2}$, the results are shown in Tables 1 and 2 and figures 1 and 2.

Example 6.2 Let us consider the fuzzy impulsive fractional equation,

$$_cD^{\frac{k}{\alpha}}y(t) = \frac{ty^2(t)}{(3+t)(1+y^2(t))},\ t \in J := [0,1],\ t \neq \tfrac{1}{2},\ \ m-1 < \tfrac{1}{\alpha} < m,\ m \in N\ ,$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{|y(\frac{1}{2}^{-})|}{2+|y(\frac{1}{2}^{-})|},$$

$$y(0) = [\tilde{0}, \tilde{0}].$$

Set

$$I_k(t) = \frac{t}{t+2},\ t \in [0, \infty),$$

$$f(t, y(t)) = \left[\frac{t^3(r-1)}{(3+t)(1+t^2)}, \frac{t^3(1-r)}{(3+t)(1+t^2)}\right].$$

Using Eq.(22) and taking $k = 2$ and $\frac{1}{\alpha} = \frac{1}{2}$, the results are shown in Tables 3 and 4 and figures 3 and 4.

Table 4. Numerical results of Example 2 for $\overline{y_{i.gH}}(t)$ and $\overline{y_{ii.gH}}(t)$.

r/t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.55	0.028928	0.025721	0 .022512	0.019302	0.016089	0.012875	0.009659	0.006441	0.003221	0
0.60	0.020779	0.018476	0.016171	0.013865	0.011557	0.009249	0.006938	0.004627	0.002314	0
0.65	0.015366	0.013663	0.011958	0.011025	0.008547	0.006839	0.005131	0.003422	0.001711	0
0.70	0.011649	0.010357	0 .009065	0.007773	0.006479	0.005185	0.003890	0.002594	0.001297	0
0.75	0.009022	0.008022	0 .007022	0.006020	0.005019	0.004016	0.003013	0.002009	0.001005	0
0.80	0.007121	0.006331	0 .005542	0.007452	0.003961	0.003170	0.002378	0.001586	0.000793	0
0.85	0.005713	0.00508	0 .004447	0.003813	0.003178	0.002543	0.001908	0.001273	0.000636	0
0.90	0.004652	0.004137	0 .003621	0.003105	0.002588	0.002071	0.001554	0.001036	0.000518	0
0.95	0.003838	0.003413	0 .002987	0.002561	0.002135	0.001709	0.001282	0.000855	0.000428	0
1	0.003204	0.002849	0 .002494	0.002138	0.001783	0.001427	0.001070	0.000714	0.000357	0

7. Conclusion

In this paper, a new algorithm was presented to solve linear and nonlinear fuzzy impulsive fractional differential equations. This algorithm should be implemented based on the reproducing Hilbert space method . In this algorithm, the fuzzy impulsive fractional differential equations is converted to linear and nonlinear differential equations. There is an important point to make here, the results obtained by the reproducing Hilbert space method are very effective and convenient in linear and nonlinear cases with less computational iterative steps, work, and time. The numerical representations indicate the complete validity, reliability and efficiency of the presented method with a great potential in scientific applications.

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