International Journal of
Mathematical Modelling & Computations
Vol. 08, No. 01, Winter 2018, 29- 38

An Optimal G^2-Hermite Interpolation by Rational Cubic Bézier Curves

D. Sbibih[a,*] and B. Belkhatir[b]

[a,b]*LANO Laboratoty, University Mohammed First, Faculty of Sciences, Morocco.*

Abstract. In this paper, we study a geometric G^2 Hermite interpolation by planar rational cubic Bézier curves. Two data points, two tangent vectors and two signed curvatures interpolated per each rational segment. We give the necessary and the sufficient intrinsic geometric conditions for two C^2 parametric curves to be connected with G^2 continuity. Locally, the free parameters within a rational cubic Bézier curve should be determined by minimizing a maximum error. We finish by proving and justifying the efficiently of the approaching method with some comparative numerical and graphical examples.

Received: 26 February 2018, Revised: 17 July 2018, Accepted: 06 October 2018.

Keywords: Hermite interpolation; Rational curve; G^2 continuity; Geometric conditions; Optimization.

Index to information contained in this paper

1. Introduction

Geometric interpolation by parametric curves based on the interpolation data such as points, first derivatives and signed curvatures, is an important problem in Computer Aided Geometric Design (CAGD). A lot of work have been given in the literature to deal with the problem of G^1 or G^2 Hermite interpolation. In [2], G^2 Hermite interpolation scheme which provides a sixth-order approximation to smooth curves with non-vanishing curvatures while the approximation can be at most fourth-order accurate near the point with zero curvature was proposed. However the above-mentioned method can be applied only if the distance between the knots is small and even in this case the interpolant need not be unique. The

*Corresponding author. Email: sbibih@yahoo.fr

http://ijm2c.iauctb.ac.ir

problem of quintic G^2 Hermite interpolation by minimizing the approximate strain energy was presented in [7]. However, the aforementioned method use a lowest degree polynomial curves interpolation. In [10], the author propose method based on geometric Hermite interpolation by a family of intrinsically defined planar curves using interpolation of curvature values with Bézier-curves.

In this paper, we propose a method of interpolation and approximation planar curves based on the G^2 Hermite interpolation by rational cubic Bézier curves and on the minimization of the maximum errors formulated in infinite norm using the magnitude tangent vectors. To show the efficiency of our technique, we give a comparison of examples related to the research domain in CAD-Applications: approximation of logarithmic spirals and blending curves.

The present paper is organized as follows. First, we recall some basic facts about the G^2 continuity between tow curves on class C^2 and some properties of plane rational cubic Bézier curves. In Section 3, we propose a solution of the G^2 Hermite interpolation problem. We also describe the algorithm allowing to construct the G^2 interpolant curves. In Section 4, as an application of this work, we give some comparative numerical and graphical examples. Finally, on Section 5, we present our conclusions.

2. Preliminary results

2.1 *Rational Bézier curve*

The parametric form of planar Rational Bézier curves is given by

$$C(t) = \frac{\sum_{i=0}^{3} B_3^i(t) w_i P_i}{\sum_{i=0}^{3} B_3^i(t) w_i} = \frac{A(t)}{B(t)}, \quad t \in [0,1] \tag{1}$$

where $P_i \in \mathbb{R}^2$ are control points, $0 < w_i \in \mathbb{R}$ are weights, and $B_n^i(t) = \binom{n}{i} t^i (1-t)^{n-i}$ are the Bernstein polynomials of degree n, $0 \leqslant i \leqslant n$. It follows that

$$C(0) = P_0 \quad and \quad C(1) = P_3. \tag{2}$$

The first derivative $C'(t)$ of the curve $C(t)$ is defined by

$$C'(t) = \frac{A'(t) - B'(t)C(t)}{B(t)} \tag{3}$$

where

$$A'(t) = 3\sum_{i=0}^{2} B_2^i(t)(w_{i+1}P_{i+1} - w_i P_i) \quad and \quad B'(t) = 3\sum_{i=0}^{2} B_2^i(t)(w_{i+1} - w_i).$$

Then,

$$C'(0) = \frac{3w_1}{w_0}(P_1 - P_0) \quad and \quad C'(1) = \frac{3w_2}{w_3}(P_3 - P_2). \tag{4}$$

The signed curvature of $C(t)$ is defined by

$$k(t) = \frac{det(C'(t), C''(t))}{\|C'(t)\|^3},$$

if $C(t)$ is regular, i.e. $\|C'(t)\| \neq 0$. The curvature is positive if the center of the osculating circle is on the left when a curve is traversed in the direction of increasing parameter; otherwise, it is negative.
It follows that

$$k(0) = \frac{2w_0 w_2}{3w_1^2} \frac{det(P_1 - P_0, P_2 - P_1)}{\|P_1 - P_0\|^3} \tag{5}$$

and

$$k(1) = \frac{2w_1 w_3}{3w_2^2} \frac{det(P_2 - P_1, P_3 - P_2)}{\|P_3 - P_2\|^3}. \tag{6}$$

2.2 G^2 *continuity of planar parametric curves*

Definition 2.1 *(See [1]) Let $C_1(t)$ and $C_2(t)$ $(0 \leqslant t \leqslant 1)$ be two parametric curves of class C^2 in $\mathbb{R}^2$ such that $C_1(0) = C_2(1)$ and $C_1'(0) \neq 0$. Then, $C_1(t)$ and $C_2(t)$ are G^2 continuously connected at $C_2(1) = C_1(0)$ if and only if there exist real numbers α and β such that*

$$\begin{cases} C_2'(1) = \alpha C_1'(0), \\ C_2''(1) = \alpha^2 C_1''(0) + \beta C_1'(0). \end{cases} \tag{7}$$

Theorem 2.2 *Let $C_1(t)$ and $C_2(t)$ $(0 \leqslant t \leqslant 1)$ be two parametric curves of class C^2 in $\mathbb{R}^2$ such that $C_1(0) = C_2(1) = Q$ and $C_1'(0) \neq 0$. C_1 and C_2 meet with G^2 continuity at the point Q if and only if they have a common unit tangent and curvature.*

Proof Assume that C_1 and C_2 meet with G^2 continuity at the point Q. According to Definition 2.1, there exist two reals α and β such that $C_2'(1) = \alpha C_1'(0), \alpha > 0$, and

$$C_2''(1) = \alpha^2 C_1''(0) + \beta C_1'(0).$$

Thus, the curvature of the curve C_2 at the point Q is given by

$$\frac{det(C_2'(1), C_2''(1))}{\|C_2'(1)\|^3} = \frac{det(\alpha C_1'(0)), \alpha^2 C_1''(0) + \beta C_1'(0))}{\|\alpha C_1'(0)\|^3},$$

then

$$\frac{det(C_2'(1), C_2''(1))}{\|C_2'(1)\|^3} = \frac{det(C_1'(0), C_1''(0))}{\|C_1'(0)\|^3}.$$

This prove the necessary condition.

For the sufficient condition, if

$$\frac{det(C_1'(0), C_1''(0))}{\|C_1'(0)\|^3} = \frac{det(C_2'(1), C_2''(1))}{\|C_2'(1)\|^3}$$

and $C_2'(1) = \alpha C_1'(0)$ such that $\alpha > 0$,
then

$$det(C_1'(0), \alpha^2 C_1''(0) - C_2''(1)) = 0,$$

thus, there exists $\beta \in \mathbb{R}$ such that

$$C_2''(1) = \alpha^2 C_1''(0) + \beta C_1'(0).$$

■

3. G^2 Hermite interpolation by rational cubic Bézier curves

3.1 *Osculatory interpolation*

Let Q_1 and Q_2 be two points in $\mathbb{R}^2$, v_1, v_2 be two unit tangent vectors, α, β be tow non negatives real numbers and k_1, k_2 be two signed non-null curvatures . We seek a parametric rational cubic Bézier curve $C_{(\alpha,\beta)}$ which interpolates the above data, i.e

$$C_{(\alpha,\beta)}(0) = Q_1, \quad C'_{(\alpha,\beta)}(0) \in \mathbb{R}^{+*}v_1, \quad k(0) = k_1. \tag{8}$$

$$C_{(\alpha,\beta)}(1) = Q_2, \quad C'_{(\alpha,\beta)}(1) \in \mathbb{R}^{+*}v_2, \quad k(1) = k_2, \tag{9}$$

where k is the curvature function. To do this, we choose the parameters of the curve $C_{(\alpha,\beta)}$ as follows:

$$w_0 = 1,\ w_1 = 1,\ P_0 = Q_1,\ P_3 = Q_2,\ P_1 = P_0 + \frac{\alpha}{3}v_1 \quad \text{and} \quad P_2 = P_3 - \frac{\beta}{3}v_2. \tag{10}$$

As in [3], the parameters w_1 and w_2 are calculated as follows

$$w_1 = \frac{2}{3}(\frac{d_1^2 d_0}{k_1^2 k_0})^{\frac{1}{3}} \quad and \quad w_2 = \frac{2}{3}(\frac{d_0^2 d_1}{k_0^2 k_1})^{\frac{1}{3}}, \tag{11}$$

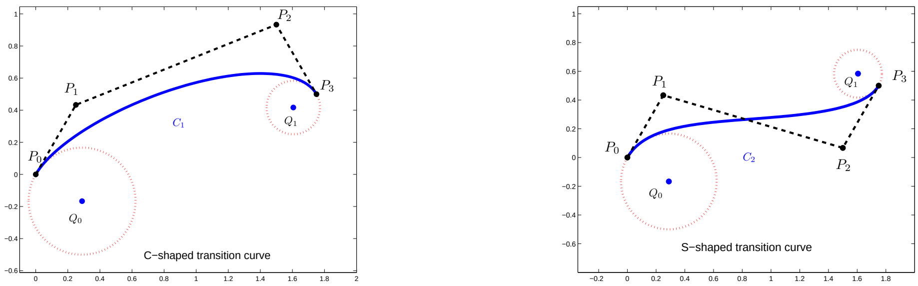
where

$$d_0 = \frac{det(P_1 - P_0, P_2 - P_1)}{\|P_1 - P_0\|^3} \quad and \quad d_1 = \frac{det(P_2 - P_1, P_3 - P_2)}{\|P_3 - P_2\|^3}. \tag{12}$$

Remark 1 The data of the interpolation problem must be homogeneous, i.e.
$d_i k_i > 0$ for $i = 0, 1$.

Algorithm 1 (Construction of the interpolant)

(1) Input data $Q_1, Q_2, v_1, v_2, k_1, k_2, \alpha, \beta$.
(2) Compute the rational Bézier control points P_0, P_1, P_2 and P_3 from Equation (10).
(3) Compute d_0 and d_1 values from Equation (12).



(a) (b)

Figure 1. Two examples of the construction of the interpolant. (a) C-sharp forms. (b) S-sharp forms.

(4) if $(d_i k_i < 0)$ then $k_i \leftarrow (-k_i)$ for $i = 0, 1$.
(5) Compute w_1 and w_2 values from Equation (11).
(6) Return the curve C

To illustrate the result given in the above Algorithm, we plot two curves C_1 and C_2 (see Figure 1) with different forms.

3.2 *Interpolation Problem*

The problem is as follow. Suppose that a sequence of data points

$$P_i \in \mathbb{R}^2, i = 0, 1, \ldots, n \quad P_i \neq P_{i+1},$$

associated respectively with a sequence of data unit tangents v_i and curvatures k_i are given. Our goal is to find a rational cubic G^2 spline curve

$$C(t) : [0, 1] \to \mathbb{R}^2$$

composed of rational cubic Bézier curve between two adjacent points P_{i-1} and P_i (see Figure 2).

$$C_i(t) : [0, 1] \to \mathbb{R}^2,$$

such that

- $C_i(0) = P_{i-1}, \quad C_i(1) = P_i;$
- $C_i'(0) = \alpha_{i-1} v_{i-1}, \quad C_i'(1) = \beta_{i-1} v_i;$
- $k(0) = k_{i-1}, \quad k(1) = k_i,$

where $\alpha_{i-1} > 0$ and $\beta_{i-1} > 0$, for $i = 1, 2, \ldots, n$. The rational curve C_i is the interpolant constructed using Algorithm *1*. From Theorem 2.1, it is easy to prove the following result.

Theorem 3.1 *For $i = 1, 2, \ldots, n-1$, the both curves C_i and C_{i+1} are connected with G^2 continuity at the point P_i. Consequently, C is G^2 continuous.*

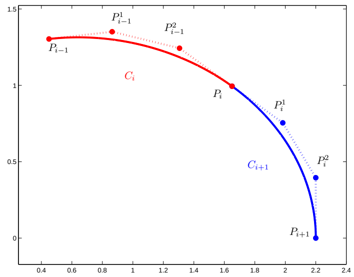
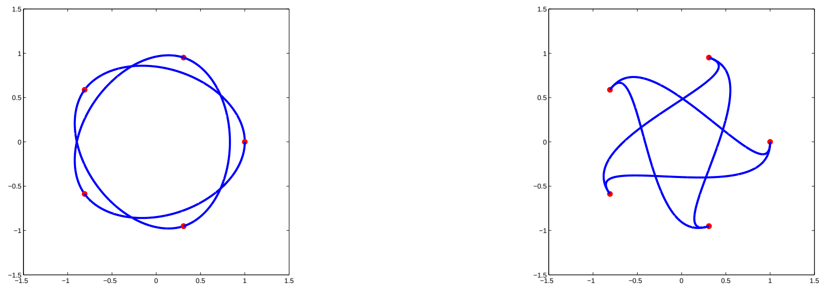


Figure 2. Construction of two curves C_i and C_{i+1} with their control points, connected with G^2 continuity at the point P_i.



(a) (b)

Figure 3. (a) Interpolation of C^1G^2 pentagon curves. (b) Interpolation with the same data as (a) except for $\alpha_i = -1$ and $\beta_i = 1$.

To illustrate the result given in the above Theorem, we use the following data given in [2](see Figure 3).

$$
\begin{aligned}
P_i &= (cos(4i\pi/5), sin(4i\pi/5)),\\
v_i &= (-sin(4i\pi/5), cos(4i\pi/5)),\\
k_i &= 2, \quad i = 0, 1, \ldots, 5,
\end{aligned}
$$

4. Applications

In this section, some examples are given to illustrate the effectiveness of the proposed method of planar G^2 Hermite interpolation. Comparison with other existing methods are also presented.

4.1 *Approximation*

Let $C(t)$ $(0 \leqslant t \leqslant 1)$ be a planar parametric curves of class C^2. By using Algorithm *1*, we approximate locally the original curve C by a piecewise G^2 interpolating curve $\tilde{C}$ with a good optimization, i.e. we look locally for two positive parameters $\hat{\alpha}$ and $\hat{\beta}$ (see (10)) so that the error functional is minimal

$$e(\alpha, \beta) = \|C - \tilde{C}(\alpha, \beta)\|_\infty = max(\|C_x - \tilde{C}_x(\alpha, \beta)\|_\infty, \|C_y - \tilde{C}_y(\alpha, \beta)\|_\infty), \quad (13)$$

where

$$\|C_x - \tilde{C_x}(\alpha,\beta)\|_\infty = \sup_{t\in[0,1]} \|C_x(t) - \tilde{C_x}(\alpha,\beta)(t)\|_\infty \quad (14)$$

and

$$\|C_y - \tilde{C_y}(\alpha,\beta)\|_\infty = \sup_{t\in[0,1]} \|C_y(t) - \tilde{C_y}(\alpha,\beta)(t)\|_\infty. \quad (15)$$

To better characterize the performance of this approximation method, we compute the relative error between the total length of the original curve and that of the reconstructed curve, i.e.

$$e_L = \frac{|L_{org} - L_{rec}|}{L_{org}}.100\% \quad (16)$$

where

$$L_{rec} = \int_0^1 \|C'(u)\|du \quad and \quad L_{org} = \int_0^1 \|\tilde{C}'(u)\|du.$$

To evaluate the tow integrals L_{org} and L_{rec}, we use the classical method Composite Simpson's rule.

4.1.1 Numerical and graphical example

We present some examples showing the efficiency of our method and we also compare it with those introduced in [9, 10].
Example 1(logarithmic spiral) A logarithmic spiral C in $\mathbb{R}^2$ can be defined by Cartesian coordinates as

$$C(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r_0 e^{\theta t} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} \in \mathbb{R}^2, r_0 \in \mathbb{R}^+, \theta \in \mathbb{R}. \quad (17)$$

The spiral curve is rotating around the pole O (origin) of the spiral.
Referring to the article [9], an original logarithmic spiral is defined by

$$C(t) = 0,1\exp(0,72\pi t) \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}, \quad 0 \leqslant t \leqslant 1.$$

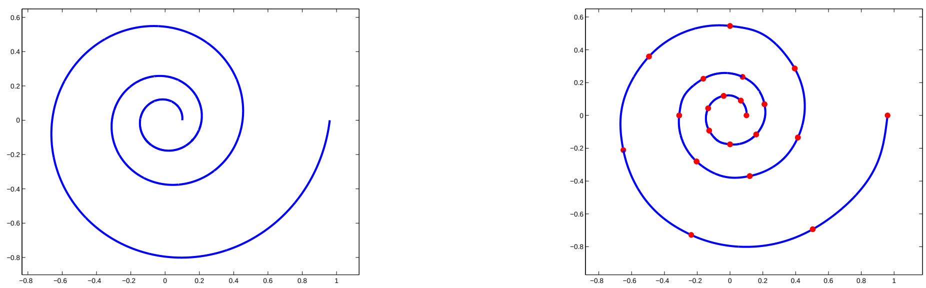
We approximate the spiral by constructing 20 rational B$\acute{e}$zier arcs that interpolate the boundary data $C(i * 0,05), i = 0, 1, \cdots, 20$ of the original spiral. Figure 4 illustrates the original spiral and the interpolating rational arc that consists of 20 arcs. The errors of the local interpolants are given in Table 1. The relative error associated with the curve C is

$$e_L = 3.6E^{-5}.100\%$$

By comparing the proposed rational cubic B$\acute{e}$zier curve with the method cited in [9] via Example 1, it is clear that our method produces better results, i.e. the approximative curve is G^2 continuous and has maximum approximation error equal to $2.7E^{-2}$ (see Figure 4 and Table 2). On the other hand, the constructed curve by the method in question can interpolate selected points, tangents at the points

Table 1. Optimal Approximation errors for 20 rational arc Bézier curves of the curve C.

Arc(i)	$(\hat{\alpha}, \hat{\beta})$	Error	Arc(i)	$(\hat{\alpha}, \hat{\beta})$	Error
Arc(0)	(0.0320 , 0.0360)	$2.8E^{-4}$	Arc(10)	(0.1000, 0.1130)	$2.0E^{-4}$
Arc(1)	(0.0388 , 0.0487)	$1.4E^{-3}$	Arc(11)	0.1200 , 0.1510)	$4.4E^{-3}$
Arc(2)	(0.0427 , 0.0500)	$8.1E^{-4}$	Arc(12)	(0.1323 , 0.1556)	$2.4E^{-4}$
Arc(3)	(0.0461 , 0.0521)	$2.0E^{-4}$	Arc(13)	(0.1430 , 0.1610)	$6.7E^{-4}$
Arc(4)	(0.0640 , 0.0790)	$6.4E^{-4}$	Arc(14)	(0.2000 , 0.2500)	$1.9E^{-2}$
Arc(5)	(0.0578 , 0.0650)	$2.8E^{-4}$	Arc(15)	(0.2000 , 0.2000)	$1.2E^{-2}$
Arc(6)	(0.0848 , 0.0676)	$2.3E^{-3}$	Arc(16)	(0.2100 ,0.2628)	$7.1E^{-3}$
Arc(7)	(0.0748 , 0.0876)	$1.2E^{-3}$	Arc(17)	(0.2000 , 0.3000)	$1.5E^{-2}$
Arc(8)	(0.0810 , 0.0920)	$5.2E^{-4}$	Arc(18)	(0.2900 , 0.2900)	$2.5E^{-3}$
Arc(9)	(0.1151 , 0.1411)	$8.3E^{-3}$	Arc(19)	(0.4001 , 0.4194)	$2.7E^{-2}$



(a) (b)

Figure 4. (a) The origin spiral C; (b) The approximate spiral $\tilde{C}$.

with G^1 continuity. It is noticed that the maximum approximation error is $5.1E^{-2}$.
Example 2 Let C be the original curve, introduced in [10], and defined by

$$C(t) = \begin{cases} x(t) = 0.1\cos(2t) + \cos(t) + \cos(3t) + 0.1\cos(4t), \\ y(t) = 0.6\sin(t) + \sin(3t), \end{cases} t \in [0, 1]. \quad (18)$$

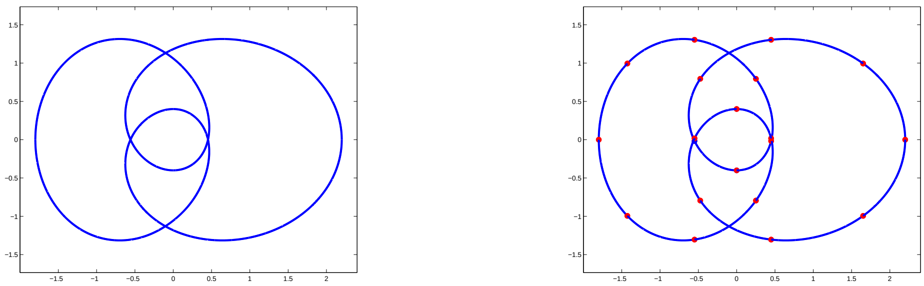
We approximate C by constructing 20 rational Bézier arcs that interpolate the boundary data $C(i*0,05), i = 0, 1, \cdots, 20$ of the original spiral. Figure 5 illustrates the original spiral and the interpolating rational arc that consists of 20 arcs in Table 2. The relative error associated with the curve C is

$$e_L = 2.7E^{-6}.100\%$$

By comparing the proposed rational cubic Bézier curves with the method introduced in [10] via Example 2, it is clear that our method produces better results i.e., the approximative curve is G^2 continuous and has maximum approximation errors equal to $9.3E^{-5}$ (see Figure 5 and Table 3). On the other hand, the constructed curve by the method in question can interpolate selected points, tangents and sometimes curvatures at the points, but approximate the rest parts of the curve. It is noticed that the maximum approximation error is $1.18E^{-4}$.

Table 2. Optimal Approximation errors for 20 rational arc Bézier curves of the curve C.

Arc(i)	$(\hat{\alpha}, \hat{\beta})$	Error	Arc(i)	$(\hat{\alpha}, \hat{\beta})$	Error
Arc(0)	(0.3862 , 0.4279)	$3.4E^{-5}$	Arc(10)	(0.3847 , 0.3453)	$4.9E^{-5}$
Arc(1)	(0.4169 , 0.4277)	$5.1E^{-5}$	Arc(11)	(0.3467 , 0.3268)	$4.0E^{-5}$
Arc(2)	(0.4185 , 0.3242)	$7.9E^{-5}$	Arc(12)	(0.3264 ,0.3289)	$2.4E^{-5}$
Arc(3)	(0.3246 , 0.2672)	$4.9E^{-5}$	Arc(13)	(0.3295 ,0.2477)	$5.1E^{-5}$
Arc(4)	(0.2168 , 0.2143)	$5.0E^{-5}$	Arc(14)	(0.2482 , 0.2153)	$6.7E^{-5}$
Arc(5)	(0.2153 , 0.2481)	$9.3E^{-5}$	Arc(15)	(0.2144 , 0.2668)	$4.9E^{-5}$
Arc(6)	(0.2477 , 0.3290)	$2.4E^{-5}$	Arc(16)	(0.2672 , 0.3246)	$4.9E^{-5}$
Arc(7)	(0.3289 , 0.3264)	$2.4E^{-5}$	Arc(17)	(0.3242 , 0.4185)	$7.9E^{-5}$
Arc(8)	(0.3268 , 0.3467)	$3.3E^{-5}$	Arc(18)	(0.4269 , 0.4770)	$2.3E^{-5}$
Arc(9)	(0.34531, 0.3847)	$3.6E^{-5}$	Arc(19)	(0.4079 , 0.3862	$3.0E^{-5}$



(a) (b)

Figure 5. (a) The origin curve C; (b) The approximate curve $\tilde{C}$.

4.2 *Blending curves with minimal strain energy*

Let $C_1(t)$ and $C_2(t)$ $(0 \leqslant t \leqslant 1)$ be two parametric curves of class C^2. Our aim is to find a blend $C(t), 0 \leqslant t \leqslant 1)$, (see [8]) rational cubic Bézier curve such that

- $C(t)$ and C_1 are connected with G^2 continuity at the point $A = C_1(1)$,
- $C(t)$ and C_2 are connected with G^2 continuity at the point $B = C_2(0)$.

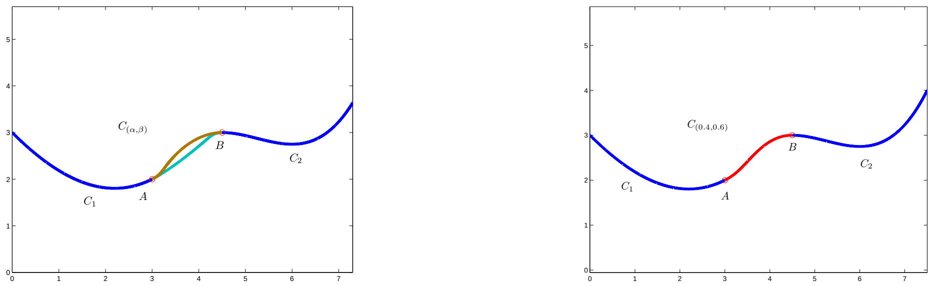
From 3.1, the interpolant $C_{(\alpha,\beta)}(t)$ satisfies the two last conditions.
For fixed data, this problem presents several solutions based on α and β parameters. Therefore, we seek an optimal solution by minimizing the energy operator

$$E(\alpha, \beta) = \int_0^1 (k(t))^2 dt \tag{19}$$

where $k(t)$ is the curvature of $C_{(\alpha,\beta)}$ at point t.

4.2.1 Numerical and graphical example

Given both cubic Bézier curves $C_1(t)$ and $C_2(t)$ with control points: $P_0 = (0, 3)$, $P_1 = (3, 2)$, $P_2 = (2, 1.5)$, $P_3 = (3, 2)$ and $Q_0 = (4.5, 3)$, $Q_1 = (5.5, 3)$, $Q_2 = (6, 5.2)$, $Q_3 = (7.5, 4)$ respectively. From the both curves C_1 and C_2, we first construct the blending curve $C_{(\alpha,\beta)}$ with different values of α and β (see Figure 6 (a)). Then, we construct the blending curves $C_{(0.4,0.6)}$ which presents the minimal strain energy (see Figure 6 (b)).



(a) (b)

Figure 6. (a) Construction of the curves $C_{(\alpha,\beta)}$ with different values of α and β; (b) Construction of the optimal solution curve, ($\alpha = 0.4$ and $\beta = 0.6$).

5. Conclusion

In this paper, we have proposed a method of interpolation and approximation planar curves based on the G^2 Hermite interpolation by rational cubic Bézier curves and on the minimization of the maximum errors formulated in infinite norm using the magnitude tangent vectors. The proposed method can be used to construct interpolating spiral with minimal maximum-errors and blending curves with the minimal strain energy.

References

[1] B. Belkhatir, D. Sbibih and A. Zidna, Construction of flexible blending parametric surface via curves, J. Mathematics and computer in Simulation, **79** (2009) 3599-3608.
[2] C. de Boor, K. Höllig and M. Sabin, High accuracy geometric interpolatio, Computer-Aided Geometric Design, **4** (1987) 269-278.
[3] G. Farin, Curves and Surfaces for CAGD: A Practical Guide, Fifth edition, Elsevier Science, (2002).
[4] R. T. Farouki, M. al-Kandari and T. Sakkalis, Hermite interpolation by rotation-invariant spatial Pythagorean-hodograph curves, Adv. Comput. Math, **17** (2002) 369-383.
[5] B. Jüttler, Hermite interpolation by Pytagorean hodograph curves of degree seven, Math. Comp, **70** (2001) 1089-1111.
[6] H. Lin, Y. Xiong and H. Liao, Semi-structured B-spline for blending two B-spline surfaces, Computers and Mathematics with Application, **68** (2014) 706-718.
[7] L. Lu, Planar quintic G^2 Hermite interpolation with minimum strain energy, Journal of Computational and Applied Mathematics, **274** (2015) 109-117.
[8] D. S. Meek and D. J. Walton, Blending two parametric curves, Computer-Aided Geometric Design, **41** (2009) 423-431.
[9] X. Yang, Geometric Hermite interpolation by logarithmic splines, Computer Aided Geometric Design, **31** (2014) 707-711.
[10] X. Yang and W. Wu, Geometric Hermite by a family of intrinsically defined planar curves, Computer Aided Geometric Design, **77** (2016) 86-97.