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Galerkin Method for the Numerical Solution of the Advection-Diffusion Equation by Using Exponential B-Splines

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Abstract. In this paper, the exponential B-spline functions are used for the numerical solution of the advection-diffusion equation. Two numerical examples related to pure advection in a finitely long channel and the distribution of an initial Gaussian pulse are employed to illustrate the accuracy and the efficiency of the method. Obtained results are compared with some early studies.

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1. Introduction

One known problem of our age is the environmental pollution. This problem is increasingly reduce the quality of our water. Scientists are benefiting from the solution of the advection-diffusion equation (ADE) in modelling this problem. Since both advection and diffusion terms exist in the ADE, it also arises very frequently in transferring mass, heat, energy, velocity and vorticity in engineering and chemistry. Thus, the heat transfer in a draining film, dispersion of tracers in porous media,

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the intrusion of salt water into fresh water aquifers, the spread of pollutants in rivers and streams, the dispersion of dissolved material in estuaries and coastal sea, contaminant dispersion in shallow lakes, the absorption of chemicals into beds, the spread of solute in a liquid flowing through a tube, long-range transport of pollutants in the atmosphere, forced cooling by fluids of solid material such as windings in turbo generators, thermal pollution in river systems and flow in porous media, etc. are modeled by ADE [16].

It is well known that the solution of the advection-diffusion boundary value problem displays sharp boundary layers. To cope with the sharp solutions, some of the spline based methods for the numerical solution of ADE are suggested such as the quasi-Lagrangian cubic spline method [19, 20], the characteristic methods integrated with splines [25, 28], the cubic B-spline Galerkin method [10], the quadratic B-spline subdomain collocation method [11], the spline approximation with the help of upwind collocation nodes [9], the exponential spline interpolation in characteristic based scheme [29] , the cubic spline interpolation for the advection component and the Crank-Nicolson scheme for the diffusion component [1, 2], the meshless method based on thin-plate spline radial basis functions [3], the least-square B-spline finite element method [6], the standard finite difference method [26, 27], the cubic B-spline collocation method [12–14], the quadratic/cubic B-spline Taylor-Galerkin methods [5], the cubic/quadratic B-spline least-squares finite element techniques [7, 8, 15], the cubic B-spline differential quadrature method [16], the quadratic Galerkin finite elements method [4].

The exponential B-spline basis functions are used to establish the numerical methods. Thus the exponential B-spline based collocation method are constructed to solve the differential equations. Numerical solution of the singular perturbation problem is solved with a variant of exponential B-spline collocation method in the work [23], the cardinal exponential B-splines is used for solving the singularly perturbed problems [21], the exponential B-spline collocation method is built up for finding the numerical solutions of the self-adjoint singularly perturbed boundary value problems in the work [22], the numerical solutions of the convection-diffusion equation is obtained by using the exponential B-spline collocation method [18]. As far as we search, no study exists solving the advection-diffusion problems using the exponential B-spline Galerkin method. Thus advection-diffusion equation is fully integrated with combination of the exponential B-spline Galerkin method (EB-SGM) for space discretization and Crank-Nicolson method for time discretization.

The study is organized as follows. In section 2, exponential B-splines are introduced and their some basic relations are given. In section 3, the application of the numerical method to the ADE is given. The efficiency and the accuracy of the present method are investigated by using two numerical experiments related to pure advection in an infinitely long channel and the distribution of an initial Gaussian pulse.

2. Exponential B-splines and finite element solution

The mathematical model describing the transport and diffusion processes is the one dimensional ADE

$$\frac{\partial u}{\partial t}+\xi\frac{\partial u}{\partial x}-\lambda\frac{\partial^2 u}{\partial x^2}=0, \tag{1}$$

where the function $u(x,t)$ represents the concentration at position x and time t with uniform flow velocity ξ and constant diffusion coefficient λ. The initial condition

Table 1. Error norm at $t = 5$, $\xi = 0.8\ m/s$, $\lambda = 0.005\ m^2/s$, $\Delta t = 0.0125$.

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$\phi_i(x)$	0	$\frac{s-ph}{2(phc-s)}$	1	$\frac{s-ph}{2(phc-s)}$	0
$\phi_i'(x)$	0	$\frac{p(c-1)}{2(phc-s)}$	0	$\frac{p(1-c)}{2(phc-s)}$	0
$\phi_i''(x)$	0	$\frac{p^2 s}{2(phc-s)}$	$\frac{-p^2 s}{phc-s}$	$\frac{p^2 s}{2(phc-s)}$	0

of Eq. (1) is

$$u(x,0) = u_0(x), \quad 0 \leqslant x \leqslant L \tag{2}$$

and the boundary conditions are

$$u(0,t) = f_0(t), \quad u(L,t) = f_L(t) \quad \text{or} \ -\lambda \frac{\partial u}{\partial x}|_L = \phi_L(t) \tag{3}$$

where L is the length of the channel, ϕ_L is the flux at the boundary $x = L$ and u_0, f_0, f_L are imposed functions.

Let us consider a uniform mesh Γ with the knots x_i on $[a,b]$ such that

$$\Gamma : a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$$

where $h = \dfrac{b-a}{N}$ and $x_i = x_0 + ih$.

Let $\phi_i(x)$ be the exponential B-splines at the points of Γ together with knots x_i, $i = -3,-2,-1,N+1,N+2,N+3$ outside the interval $[a,b]$ and having a finite support on the four consecutive intervals $[x_i + kh, x_i + (k+1)h]_{k=-3}^{0}$, $i = 0,...,N+2$. According to McCartin [17], the $\phi_i(x)$ can be defined as

$$\phi_i(x) = \begin{cases} b_2\left[(x_{i-2}-x) - \frac{1}{p}\left(\sinh\left(p\left(x_{i-2}-x\right)\right)\right)\right] & \text{if } x \in [x_{i-2}, x_{i-1}]; \\ a_1 + b_1(x_i - x) + c_1 e^{p(x_i-x)} + d_1 e^{-p(x_i-x)} & \text{if } x \in [x_{i-1}, x_i]; \\ a_1 + b_1(x - x_i) + c_1 e^{p(x-x_i)} + d_1 e^{-p(x-x_i)} & \text{if } x \in [x_i, x_{i+1}]; \\ b_2\left[(x - x_{i+2}) - \frac{1}{p}\left(\sinh\left(p\left(x-x_{i+2}\right)\right)\right)\right] & \text{if } x \in [x_{i+1}, x_{i+2}]; \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

where

$$\begin{aligned} & p = \max_{0 \leqslant i \leqslant N} p_i, \ s = \sinh(ph), \ c = \cosh(ph) \\ & b_2 = \frac{p}{2(phc-s)}, \ a_1 = \frac{phc}{phc-s}, \ b_1 = \frac{p}{2}\left[\frac{c(c-1)+s^2}{(phc-s)(1-c)}\right], \\ & c_1 = \frac{1}{4}\left[\frac{e^{-ph}(1-c) + s\left(e^{-ph}-1\right)}{(phc-s)(1-c)}\right], \ d_1 = \frac{1}{4}\left[\frac{e^{ph}(c-1) + s\left(e^{ph}-1\right)}{(phc-s)(1-c)}\right]. \end{aligned}$$

Each basis function $\phi_i(x)$ is twice continuously differentiable. The values of $\phi_i(x)$, $\phi_i'(x)$ and $\phi_i''(x)$ at the knots x_i's are given in Table 1.

The $\phi_i(x)$, $i = -1,\ldots,N+1$ form a basis for functions defined on the interval $[a,b]$. We seek an approximation $U(x,t)$ to the analytical solution $u(x,t)$ in terms for the exponential B-splines

$$u(x,t) \approx U(x,t) = \sum_{i=-1}^{N+1} \phi_i(x)\,\delta_i(t) \tag{5}$$

where $\delta_i(t)$ are time dependent unknown to be determined from the boundary conditions and Galerkin approach to the equation (1). The approximate solution and their derivatives at the knots can be found from the Eq. (4-5) as

$$\begin{aligned} U_i &= U(x_i,t) = \alpha_1\delta_{i-1} + \delta_i + \alpha_1\delta_{i+1}, \\ U_i' &= U'(x_i,t) = \alpha_2\delta_{i-1} - \alpha_2\delta_{i+1}, \\ U_i'' &= U''(x_i,t) = \alpha_3\delta_{i-1} - 2\alpha_3\delta_i + \alpha_3\delta_{i+1} \end{aligned} \tag{6}$$

where $\alpha_1 = \dfrac{s-ph}{2(phc-s)}, \alpha_2 = \dfrac{p(1-c)}{2(phc-s)}, \alpha_3 = \dfrac{p^2 s}{2(phc-s)}$.

Applying the Galerkin method to the ADE with the exponential B-splines as weight function over the interval $[a,b]$ gives

$$\int_a^b \phi_i(x)\left(u_t + \xi u_x - \lambda u_{xx}\right) dx = 0. \tag{7}$$

The approximate solution U over the element $[x_m, x_{m+1}]$ can be written as

$$U^e = \phi_{m-1}(x)\,\delta_{m-1}(t) + \phi_m(x)\,\delta_m(t) + \phi_{m+1}(x)\,\delta_{m+1}(t) + \phi_{m+2}(x)\,\delta_{m+2}(t) \tag{8}$$

where quantities $\delta_j(t)$, $j = m-1, ..., m+2$ are element parameters and $\phi_j(x)$, $j = m-1, ..., m+2$ are known as the element shape functions.

The contribution of the integral equation (7) over the sample interval $[x_m, x_{m+1}]$ is given by

$$\int_{x_m}^{x_{m+1}} \phi_j(x)\left(u_t + \xi u_x - \lambda u_{xx}\right) dx. \tag{9}$$

Applying the Galerkin discretization scheme by replacing U_t, U_x, U_{xx}, which are derivatives of the approximate solution U^e in Eq. (8), into u_t, u_x, u_{xx}, which are derivatives of the exact solution u, respectively, we obtain a system of equations in the unknown parameters δ_j

$$\sum_{i=m-1}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} \phi_j \phi_i dx \right) \overset{\bullet}{\delta}_i + \xi \left(\int_{x_m}^{x_{m+1}} \phi_j \phi_i' dx \right) \delta_i - \lambda \left(\int_{x_m}^{x_{m+1}} \phi_j \phi_i'' dx \right) \delta_i \right\} \tag{10}$$

where i and j take only the values $m-1$, m, $m+1$, $m+2$ for $m = 0, 1, \ldots, N-1$ and $\bullet$ denotes time derivative.

In the above system of differential equations, when A_{ji}^e, B_{ji}^e and C_{ji}^e are denoted by

$$A_{ji}^e = \int_{x_m}^{x_{m+1}} \phi_j \phi_i dx, \quad B_{ji}^e = \int_{x_m}^{x_{m+1}} \phi_j \phi_i' dx, \quad C_{ji}^e = \int_{x_m}^{x_{m+1}} \phi_j \phi_i'' dx \tag{11}$$

where $\mathbf{A}^e$, $\mathbf{B}^e$ and $\mathbf{C}^e$ are the element matrices of which dimensions are 4×4, the matrix form of the Eq.(10) can be written as

$$\mathbf{A}^e\dot{\boldsymbol{\delta}}^e+\left(\xi\mathbf{B}^e-\lambda\mathbf{C}^e\right)\boldsymbol{\delta}^e \tag{12}$$

where $\boldsymbol{\delta}^e=\left(\delta_{m-1},...,\delta_{m+2}\right)^T$.

Gathering the systems (12) over all elements, we obtain global system

$$\mathbf{A}\dot{\boldsymbol{\delta}}+\left(\xi\mathbf{B}-\lambda\mathbf{C}\right)\boldsymbol{\delta}=0 \tag{13}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are derived from the corresponding element matrices $\mathbf{A}^e$, $\mathbf{B}^e$, $\mathbf{C}^e$ and $\boldsymbol{\delta}=\left(\delta_{-1},...,\delta_{N+1}\right)^T$ contain all elements parameters.

The unknown parameters $\boldsymbol{\delta}$ are interpolated between two time levels n and $n+1$ with the Crank-Nicolson method

$$\boldsymbol{\delta}=\frac{\boldsymbol{\delta}^{n+1}+\boldsymbol{\delta}^n}{2},\ \dot{\boldsymbol{\delta}}=\frac{\boldsymbol{\delta}^{n+1}-\boldsymbol{\delta}^n}{\Delta t},$$

we obtain iterative formula for the time parameters $\boldsymbol{\delta}^n$:

$$\left[\mathbf{A}+\frac{\Delta t}{2}\left(\xi\mathbf{B}-\lambda\mathbf{C}\right)\right]\boldsymbol{\delta}^{n+1}=\left[\mathbf{A}-\frac{\Delta t}{2}\left(\xi\mathbf{B}-\lambda\mathbf{C}\right)\right]\boldsymbol{\delta}^{n}. \tag{14}$$

The set of equations consist of $(N+3)$ equations with $(N+3)$ unknown parameters. Before starting the iteration procedure, boundary conditions must be adapted into the system. For this purpose, we delete first and last equations from the system (14) and eliminate the terms δ_{-1}^{n+1} and δ_{N+1}^{n+1} from the remaining system (14) by using boundary conditions in (3), which give the following equations:

$$\begin{aligned}u\left(a,t\right)&=\alpha_1\delta_{-1}^n+\delta_0^n+\alpha_1\delta_1^n=\beta_1,\\ u\left(b,t\right)&=\alpha_1\delta_{N-1}^n+\delta_N^n+\alpha_1\delta_{N+1}^n=\beta_2,\end{aligned}$$

we obtain a septa-diagonal matrix with the dimension $(N+1)\times(N+1)$.

To start evolution of the vector of initial parameters δ^0, it must be determined by using the initial condition (2) and boundary conditions (3):

$$\begin{aligned}u_0'(x_0,0)&=\frac{p\left(1-c\right)}{2\left(phc-s\right)}\delta_{-1}+\frac{p\left(c-1\right)}{2\left(phc-s\right)}\delta_1\\ u\left(x_m,0\right)&=\frac{s-ph}{2\left(phc-s\right)}\delta_{m-1}+\delta_m+\frac{s-ph}{2\left(phc-s\right)}\delta_{m+1},\ m=0,...,N\\ u'\left(x_N,0\right)&=\frac{p\left(1-c\right)}{2\left(phc-s\right)}\delta_{N-1}+\frac{p\left(c-1\right)}{2\left(phc-s\right)}\delta_{N+1}\end{aligned} \tag{15}$$

The solution of matrix equation (15) with the dimensions $(N+1)\times(N+1)$ is obtained by the way of Thomas algorithm. Once δ^0 is determined, we can start the iteration of the system to find the parameters δ^n at time $t^n=n\Delta t$. Thus the approximate solution U (5) can be determined by using these δ values.

To investigate the stability of system of the difference scheme(14), we apply the von Neumann stability analysis. A typical member of the linearized equation corresponding to Eq. (14) is given by

$$\begin{aligned}&\gamma_1\delta_{m-3}^{n+1}+\gamma_2\delta_{m-2}^{n+1}+\gamma_3\delta_{m-1}^{n+1}+\gamma_4\delta_{m}^{n+1}+\gamma_5\delta_{m+1}^{n+1}+\gamma_6\delta_{m+2}^{n+1}+\gamma_7\delta_{m+3}^{n+1}\\&=\gamma_7\delta_{m-3}^{n}+\gamma_6\delta_{m-2}^{n}+\gamma_5\delta_{m-1}^{n}+\gamma_4\delta_{m}^{n}+\gamma_3\delta_{m+1}^{n}+\gamma_2\delta_{m+2}^{n}+\gamma_1\delta_{m+3}^{n}\end{aligned}$$

where the parameters γ_i, $i=1,\dots,7$ are determined from system (14), in which these values are not prefered to document here due to being too long.

Substituting the Fourier mode $\delta_m^n = s^n e^{im\varphi}$ into the linearized from of the difference equation becomes

$$s^{n+1} = qs^n.$$

Here, the growth factor is determined as

$$q = \frac{\hat{a} - i\hat{b}}{\hat{a} + i\hat{b}}$$

where

$$\begin{aligned}\hat{a} &= (\gamma_1+\gamma_7)\cos(3\varphi) + (\gamma_2+\gamma_6)\cos(2\varphi) + (\gamma_3+\gamma_5)\cos(\varphi) + (\gamma_4),\\ \hat{b} &= (\gamma_1-\gamma_7)\sin(3\varphi) + (\gamma_2-\gamma_6)\sin(2\varphi) + (\gamma_3-\gamma_5)\sin(\varphi).\end{aligned}$$

Since the magnitude of the growth factor is $|q| = 1$, the difference scheme (14) is unconditionally stable.

3. Test problems

We have carried out two test problems to demonstrate the performance of the given algorithm. Accuracy of the method is measured by the error norm

$$L_\infty = \left\| u^{\text{exact}} - u^{\text{numeric}} \right\|_\infty = \max_{0\leqslant j\leqslant N} \left| u_j^{\text{exact}} - u_j^{\text{numeric}} \right|. \quad (16)$$

In numerical calculations, the determination of p in the exponential B-spline is made by experimentally. The Courant number is defined by the ratio of the flow velocity ξ to the mesh velocity $\frac{h}{\Delta t}$, i.e.,

$$C_r = \xi\frac{\Delta t}{h},$$

and the time and space pointwise rate of convergence for numerical method are computed by the formulas

$$\text{order}=\frac{\log\left|(L_\infty)_{h_i}/(L_\infty)_{h_{i+1}}\right|}{\log|h_i/h_{i+1}|},\text{order}=\frac{\log\left|(L_\infty)_{\Delta t_i}/(L_\infty)_{\Delta t_{i+1}}\right|}{\log|\Delta t_i/\Delta t_{i+1}|},$$

where $(L_\infty)_{h_i}$ and $(L_\infty)_{\Delta t_i}$ are the error norms L_∞ for space step h_i and time step Δt_i, respectively.

Table 2. Peak concentrations at $t = 9600$ s for various C_r and $\Delta t = 50$.

C_r	h	p	EBSGM	[25]	[6]	[10]
0.25	100	6.8E-6	9.992	9.816	9.926	9.986
0.50	50	13.6E-6	9.992	9.836	9.932	9.986
0.75	33.3	2.04E-5	9.992	9.934	9.949	9.993
1.00	25	3.59E-5	9.992	10.000	9.961	9.986
1.50	16.6	4.91E-5	9.992	9.941	9.959	9.994
2.00	12.5	7.18E-5	9.992	10.000	9.961	9.986
3.20	7.8	7.50E-6	9.993	9.988	9.962	9.999
Exact	10					

3.1 *Pure advection in an infinitely long channel*

In the first example, we consider the pure advection that is $\lambda = 0$, in an infinitely long channel is of long constant cross-section, bottom slope and in which constant velocity is $\xi = 0.5$ m/s. The analytical solution is

$$u(x,t) = 10 \exp\left(-\frac{1}{2\rho^2}(x - x_0 - \xi t)^2\right) \tag{17}$$

where $\rho = 264$ m is the standard deviation and the initial distribution is $x_0 = 2$ km away from the start. The initial concentration can be obtained from (17) by taking $t = 0$. At the boundaries the following conditions are taken:

$$u(0,t) = u(L,t) = 0$$

where $L = 9$ km. Since the velocity is 0.5 m/s, the initial distribution is transported 4.8 km after 9600 s. Fig. 1 shows this transportation.

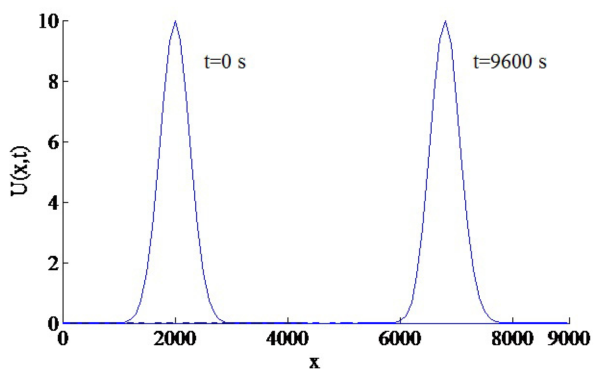

Figure 1. Transportation of the initial distribution with $C_r = 0.25$ and $\Delta t = 50$

The obtained peak concentrations at $t = 9600$ s for various Courant numbers are given in Table 2. As it is seen from the table that during the running of time period, the calculations show that the results of EBSGM are considerably near to the exact value and in general, it leads to more accurate results than the other methods. To see the errors along the whole domain for various Courant numbers, Table 3 is documented. According to this table, the results of EBSGM are mostly more accurate than the those produce by the least square finite element method and extended cubic B-spline collocation method for various Courant numbers. The absolute error distributions of the EBSGM at $t = 9600$ is illustrated in Fig. 2. Maximum error occurred around the peak concentration.

Table 3. Errors at $t = 9600$ s with $\xi = 0.5$ m/s.

C_r	h	Δt	p	EBSGM	[14]	[6]
0.125	200	50	3.30E-6	1.63E-1	1.29E-0	5.18E-1
0.25	100	50	6.80E-6	8.60E-2	3.25E-1	3.76E-1
0.50	50	50	13.6E-6	9.07E-2	1.98E-1	3.73E-1
0.50	10	10	1.53E-4	3.51E-3	7.51E-3	
0.50	1	1	3.04E-4	3.53E-5	7.50E-5	
0.50	0.5	0.5	3.40E-3	1.20E-5	1.88E-5	
0.75	33.3	50	2.04E-5	9.03E-2		3.76E-1
1.00	25	50	3.59E-5	9.02E-2		3.79E-1
1.50	16.6	50	4.91E-5	8.96E-2		3.78E-1
2.00	12.5	50	7.18E-5	9.02E-2		3.79E-1
3.20	7.8	50	7.50E-6	8.90E-2		3.80E-1

Table 4. The time pointwise order of convergence at $t = 9600$ s with $h = 10$ and $p = 0.0000373$.

Δt	L_∞	order
50	0.189757890344378	2.008789561436494
20	0.030117721143528	1.998144047815728
10	0.007539122739676	1.982361076968232
5	0.001907966158771	1.725030542270457
2	0.000392746206108	

The time rate of convergence is calculated by fixing space step as $h = 10$. For the valuation of the time steps given in the Table 4, the rate of convergence is found as 2 approximately. When higher time space is used, we have reached the convergence rate of 2 especially. Same calculation is performed for the space rate of convergence, but the rate of convergence is varied according to selection of depending parameter p.

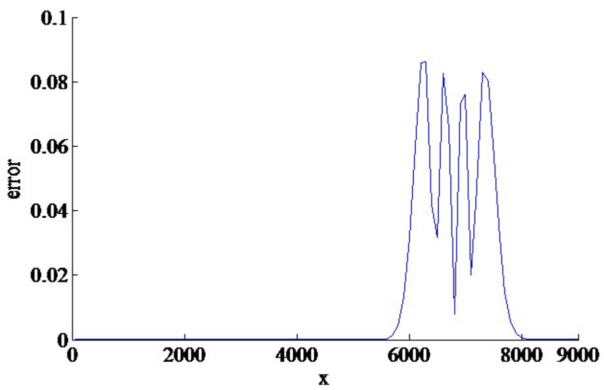

Figure 2. Absolute error distributions at $t = 9600$ with $C_r = 0.25$ and $\Delta t = 50$

3.2 *The distribution of an initial Gaussian pulse*

As a second test problem, we deal with both advection and diffusion. The analytical solution to the one-dimensional ADE of a Gaussian pulse of unit height over the domain $[0, 9]$ is given as

$$u(x,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{(x - x_0 - \xi t)^2}{\lambda\sqrt{4t+1}}\right) \tag{18}$$

where ξ is the velocity, λ is diffusion coefficient and x_0 is the centre of the initial Gaussian pulse [24].

The initial condition is chosen as the analytical value of the Eq. (18) for $t = 0$ and the boundary conditions are chosen as

$$u(0, t) = u(9, t) = 0.$$

The results presented here are computed for time step $\Delta t = 0.0125\ s$. Parameters in the equation are used as $\lambda = 0.005\ m^2/s$ and $\xi = 0.8\ m/s$. Fig. 3 shows the behavior of the numerical and analytical solutions (which are graphed with continuous lines) for various times until the simulation terminating time $t = 5$. Thus, the decay in time of the initial pulse is modeled. So that the effect of the diffusion term has been observed in this test problem. The absolute error distributions of the EBSGM at $t = 5$ is illustrated in Fig. 4.

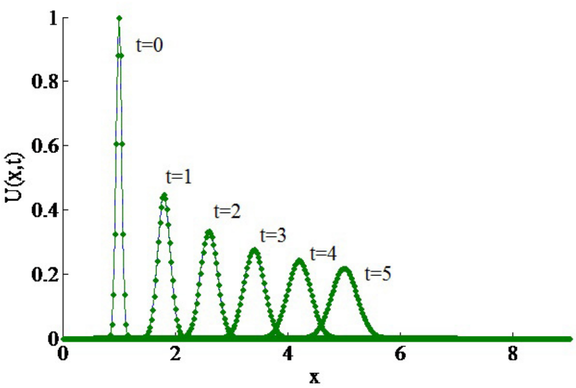

Figure 3. Distribution of an initial Gaussian pulse

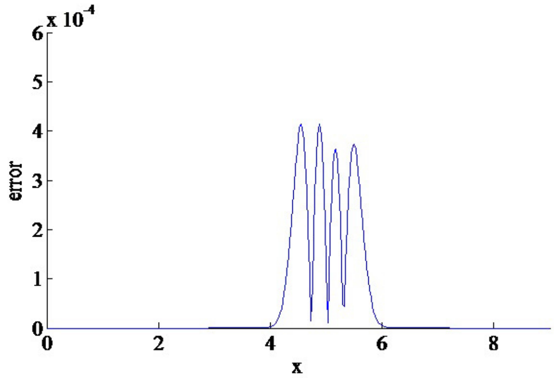

Figure 4. Absolute error distributions at $t = 5$ with $h = 0.025, \Delta t = 0.0125$

For comparison, the advection-diffusion equation is solved for various Courant numbers and computed errors at $t = 5\ s$ are presented in Table 5. As expected time rate of convergence is found as 2 approximately seen in the Table 6 due to using the Crank-Nicolson time integrator, when the space increment is fixed as $h = 0.05$. Once more we couldn't get regular space rate of convergence given in Table 7 for the second test problem.

Table 5. Error norm at $t = 5$, $\xi = 0.8\ m/s$, $\lambda = 0.005\ m^2/s$, $\Delta t = 0.0125$.

C_r	h	EBSGM (p=0.05286)	Method I [16]	Method II [16]
0.05	0.2	0.1326154	0.1253926	0.1361437
0.10	0.1	0.0042464	0.0069553	0.0145554
0.20	0.05	0.0008333	0.0012117	0.0002886
0.40	0.025	0.0004134	0.0003071	0.0000181

Table 6. The time pointwise order of convergence at $t = 9600\ s$ with $h = 0.05$.

Δt	L_∞	order
0.1	0.053345581280198	1.920846239041120
0.05	0.014088544133159	2.048129736530073
0.025	0.003406572459735	2.031342703643572
0.0125	0.00083334060923	2.054630426980018
0.00625	0.00020059363910	

Table 7. The space pointwise order of convergence at $t = 9600\ s$ with $\Delta t = 0.0125$.

h	L_∞	order
0.2	0.132615441852962	4.964862117923087
0.1	0.004246407564140	2.349264655307255
0.05	0.000833340609231	1.011432200464369
0.025	0.000413381574304	

4. Conclusion

In this paper, we have proposed a new algorithm for the numerical solution of the ADE. This algorithm is obtained by employing exponential B-spline functions to the well known Galerkin finite element method. To see achievement of the method is studied two test problems. The resulting numerical solutions for various Courant numbers are compared with the previous studies in Tables 3 and 5. Accordingly, we can say that the proposed method give acceptable results.

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