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Dynamics of a Delayed Epidemic Model with Beddington-DeAngelis Incidence Rate and a Constant Infectious Period

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Abstract. In this paper, an SIR epidemic model with an infectious period and a non-linear Beddington-DeAngelis type incidence rate function is considered. The dynamics of this model depend on the reproduction number R_0. Accurately, if $R_0 < 1$, we show the global asymptotic stability of the disease-free equilibrium by analyzing the corresponding characteristic equation and using comparison arguments. In contrast, if $R_0 > 1$, we see that the disease-free equilibrium is unstable and the endemic equilibrium is permanent and locally asymptotically stable and we give sufficient conditions for the global asymptotic stability of the endemic equilibrium.

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1. Introduction

Dynamical behaviors of SIR epidemic model describing the spread of infectious diseases in a population where the population is subdivided into three classes of individuals: susceptible, infective and recovered, are studied by many authors (see, for example, [1, 12, 13, 17] and the references therein).
In the natural world, there are many diseases which the infected population recover

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and become susceptible or removed population by itself after they are infected by some certain time such as tuberculoses, influenza, etc. (see, [4, 9]). In [8, 16, 17], the authors assume that, when a susceptible individual is infected, there is a time $\tau > 0$, during which the infectious individual develop, and only after that time the infected individual becomes the removed individual. The time τ is called infectious period.

Usually the rate of infection in most epidemic models is assumed to be bilinear in the susceptible S and the infective I. However, the actual incidence rate is probably not linear over the entire range of S and I. Thus, it is reasonable to assume that the infection rate of epidemic model is given by the Beddington-DeAnglis function, $\frac{\beta SI}{1+\alpha_1 I+\alpha_2 S}$, where $\alpha_1, \alpha_2 \geqslant 0$ are constants. The incidence function $\frac{\beta SI}{1+\alpha_1 I+\alpha_2 S}$ was introduced by Beddington (see, [1]) and DeAngelis et al. (see, [6]). When $\alpha_1 > 0$, $\alpha_2 = 0$, the Beddington-DeAngelis incidence function is simplified to saturated incidence function introduced by Capasso and Serio (see, [3]).

In [15] R. Xu and Y. Du considered the following delayed SIR epidemic model with saturation incidence and a constant infectious period

$$\begin{aligned}
\dot{S}(t) &= \Lambda - \mu_1 S(t) - \frac{\beta S(t)I(t)}{1+\alpha_1 I(t)},\\
\dot{I}(t) &= \frac{\beta S(t)I(t)}{1+\alpha_1 I(t)} - \frac{\beta e^{-\mu_2\tau}S(t-\tau)I(t-\tau)}{1+\alpha_1 I(t-\tau)} - \mu_2 I(t), \qquad (1)\\
\dot{R}(t) &= \frac{\beta e^{-\mu_2\tau}S(t-\tau)I(t-\tau)}{1+\alpha_1 I(t-\tau)} - \mu_3 R(t),
\end{aligned}$$

where $S(t)$ is the number of susceptible individuals, $I(t)$ is the number of infectious individuals, $R(t)$ is the number of recovered individuals, at time t. In (1) the parameters Λ, α_1, β, τ, μ_1, μ_2 and μ_3 are non-negative constants in which Λ is the constant recruitment rate into the population, β is the average number of adequate contacts of an infectious individuals per unit time, τ is a time delay representing the infection period of the disease; μ_1, $\mu_3 > 0$ are the natural death rates of the susceptible and the removed populations, respectively; $\mu_2 > 0$ represents the rate of natural death and the disease-induced death of the infectious.

In the current paper, motivated by the works of R. Xu, Y. Du [15] and B. Dubey et al. [7], we study the dynamics of the following SIR epidemic model with Beddington-DeAngelis incidence rate and a constant infectious period:

$$\begin{aligned}
\dot{S}(t) &= \Lambda - \mu_1 S(t) - \frac{\beta S(t)I(t)}{1+\alpha_1 I(t)+\alpha_2 S(t)},\\
\dot{I}(t) &= \frac{\beta S(t)I(t)}{1+\alpha_1 I(t)+\alpha_2 S(t)} - \frac{\beta e^{-\mu_2\tau}S(t-\tau)I(t-\tau)}{1+\alpha_1 I(t-\tau)+\alpha_2 S(t-\tau)} - \mu_2 I(t), \qquad (2)\\
\dot{R}(t) &= \frac{\beta e^{-\mu_2\tau}S(t-\tau)I(t-\tau)}{1+\alpha_1 I(t-\tau)+\alpha_2 S(t-\tau)} - \mu_3 R(t),
\end{aligned}$$

here the variables $S(t)$, $I(t)$, $R(t)$ and the parameters Λ, α_1, β, τ, μ_1, μ_2 and μ_3 have the same meanings as in system (1); α_2 is a nonnegative constant.

2. Preliminary results

The initial conditions of delay differential equations (2) are given as

$$\begin{aligned} S(\theta) = \phi_1(\theta),\ I(\theta) = \phi_3(\theta),\ R(\theta) = \phi_4(\theta), \\ \phi_i(\theta) \geqslant 0,\ \theta \in [-\tau, 0],\ i = 1, 2, 3, \end{aligned} \tag{3}$$

where $\phi = (\phi_1, \phi_2, \phi_3) \in C^+\left([-\tau, 0], \Re^3_+\right)$, $C^+\left([-\tau, 0], \Re^3_+\right)$ denotes the nonnegative cone of the Banach space $C\left([-\tau, 0], \Re^3\right)$ of continuous functions mapping the interval $[-\tau, 0]$ into $\Re^3$.
The existence and the uniqueness of the solution $(S(t), I(t), R(t))$ of the system (2) with initial conditions (3) follows from the well known theorem in [12].
It is easy to show that all solutions of system (2) with initial conditions (3) are non-negative on $[0, +\infty)$. Furthermore, if $\phi_1(0), \phi_2(0), \phi_3(0) > 0$, then the solutions are positive on $[0, +\infty)$.

We define the basic reproduction number R_0 of the model (2) by

$$R_0 = \frac{\beta\Lambda}{\mu_2\mu_1 + \mu_2\Lambda\alpha_2}(1 - e^{-\mu_2\tau}).$$

It is the number of newly infectives infected by an infective individual during the whole infection period when all of the individuals in the population are initially susceptible. This quantity determines the thresholds for disease transmissions.
The system (2) always has a disease-free equilibrium $E_1(\frac{\Lambda}{\mu_1}, 0, 0)$. Moreover, if $R_0 > 1$, then system (2) has further the endemic equilibrium $E_2(S_2, I_2, R_2)$, where

$$\begin{aligned} S_2 &= \frac{\mu_2 + \Lambda\alpha_1(1 - e^{-\mu_2\tau})}{(\beta + \mu_1\alpha_1)(1 - e^{-\mu_2\tau}) - \mu_2\alpha_2}, \\ I_2 &= \frac{1}{\mu_2}\left[\frac{\beta\Lambda(1 - e^{-\mu_2\tau}) - \Lambda\mu_2\alpha_2 - \mu_1\mu_2}{(\beta + \mu_1\alpha_1)(1 - e^{-\mu_2\tau}) - \mu_2\alpha_2}\right](1 - e^{-\mu_2\tau}), \\ R_2 &= \frac{1}{\mu_3}\left(\frac{\beta\Lambda(1 - e^{-\mu_2\tau}) - \Lambda\mu_2\alpha_2 - \mu_1\mu_2}{(\beta + \mu_1\alpha_1)(1 - e^{-\mu_2\tau}) - \mu_2\alpha_2}\right)e^{-\mu_2\tau}. \end{aligned}$$

3. Permanence of disease

In the epidemic models, permanence is an important property because it implies that the disease continues to exist for any initial conditions. In the following we show that $R_0 > 1$ is a necessary and sufficient condition for system (2) to be permanent.

Definition 3.1 [12]System (2) is said to be permanent if there exists positive constants y_i and k_i, $i = 1, 2, 3$, such that

$$y_1 \leqslant \liminf_{t\to+\infty} S(t) \leqslant \limsup_{t\to+\infty} S(t) \leqslant k_1, ,$$

$$y_2 \leqslant \liminf_{t\to+\infty} I(t) \leqslant \limsup_{t\to+\infty} I(t) \leqslant k_2,$$

$$y_3 \leqslant \liminf_{t\to+\infty} R(t) \leqslant \limsup_{t\to+\infty} R(t) \leqslant k_3,$$

for any positive solution $(S(t), I(t), R(t))$ of system (2) with initial conditions (3). Here y_i and k_i, $i = 1, 2, 3$, are independent of initial conditions (3).

Theorem 3.1 If $R_0 > 1$, then system (2) is permanent.

Proof Let $(S(t), I(t), R(t))$ be any positive solution of system (2) with initial conditions (3).
Noting $N(t) = S(t) + I(t) + R(t)$. Since

$$\dot{N}(t) = \Lambda - \mu_1 S(t) - \mu_2 I(t) - \mu_3 R(t) \leqslant \Lambda - \mu_1 N(t),$$

then

$$\limsup_{t\to+\infty} N(t) \leqslant \frac{\Lambda}{\mu_1} := k_1 := k_2 := k_3, \tag{4}$$

accordingly

$$\limsup_{t\to+\infty} I(t) \leqslant \frac{\Lambda}{\mu_1}.$$

Therefor, for $\epsilon > 0$ sufficiently small, there is a $M_1 > 0$ such that if $t > M_1$, then $I(t) \leqslant \frac{\Lambda}{\mu_1} + \epsilon$.
We derive from the first equation of system (2) that, for $t > M_1$,

$$\dot{S}(t) > \Lambda - (\mu_1 + \frac{\beta(\frac{\Lambda}{\mu_1} + \epsilon)}{1 + \alpha_1(\frac{\Lambda}{\mu_1} + \epsilon)})S(t).$$

By comparison, we have

$$\liminf_{t\to+\infty} S(t) \geqslant \frac{\Lambda[1 + \alpha_1(\frac{\Lambda}{\mu_1} + \epsilon)]}{\mu_1 + (\mu_1\alpha_1 + \beta)(\frac{\Lambda}{\mu_1} + \epsilon)}.$$

Since this inequality hold true for arbitrary $\epsilon > 0$ sufficiently small, we deduce that

$$\liminf_{t\to+\infty} S(t) \geqslant \frac{\Lambda(1 + \alpha_1\frac{\Lambda}{\mu_1})}{\mu_1 + (\mu_1\alpha_1 + \beta)\frac{\Lambda}{\mu_1}} := y_1. \tag{5}$$

Therefor, for $\epsilon > 0$ sufficiently small, there is a $M_2 > M_1$ such that if $t > M_2$, then $S(t) \geqslant y_1 - \epsilon$.
Denote $I^* = \frac{\mu_1\alpha_1[\beta\Lambda(1-e^{-\mu_2\tau}) - \mu_1\mu_2 - \mu_2\alpha_2\Lambda]}{\mu_2(\beta+\mu_1\alpha_1)^2}$. Clearly, if $R_0 > 1$, then $I^* > 0$ and

$$\frac{\beta\Lambda(1 - e^{-\mu_2\tau})}{\mu_2[\mu_1 + \alpha_2\Lambda + I^*(\beta + \mu_1\alpha_1)]} = \frac{\frac{(\beta+\mu_1\alpha_1)^2}{\mu_1\alpha_1} I^* + \mu_1 + \alpha_2\Lambda}{\mu_1 + \alpha_2\Lambda + I^*(\beta + \mu_1\alpha_1)} > 1.$$

Thus, we can choose a positive constant $\lambda > \tau$ such that

$$p := \frac{\beta\Lambda(1 - e^{-\mu_2\tau})}{\mu_2[\mu_1 + \alpha_2\Lambda + I^*(\beta + \mu_1\alpha_1)]}[1 - e^{-(\mu_1 + \frac{\beta I^*}{1+\alpha_1 I^*})\lambda}] > 1. \tag{6}$$

We claim that for any $t_0 > 0$, there exists a time $t_1 \geqslant t_0$ such that $I(t_1) > I^*$. Suppose that the claim is not valid. Then there is a $t_0 > 0$ such that

$$I(t) \leqslant I^*, \quad t \geqslant t_0. \tag{7}$$

It follows from the first equation of system (2) and (7)that for $t \geqslant t_0$,

$$\dot{S}(t) \geqslant \Lambda - (\mu_1 + \frac{\beta I^*}{1 + \alpha_1 I^*})S(t).$$

Therefor, for $t \geqslant t_0 + \lambda$, we obtain

$$\begin{aligned} S(t) &\geqslant S(t_0)e^{-(\mu_1 + \frac{\beta I^*}{1+\alpha_1 I^*})(t-t_0)} + \Lambda \int_{t_0}^{t} e^{-(\mu_1 + \frac{\beta I^*}{1+\alpha_1 I^*})(t-\theta)} d\theta \\ &> \frac{\Lambda(1 + \alpha_1 I^*)}{\beta I^* + \mu_1(1 + \alpha_1 I^*)}[1 - e^{-(\mu_1 + \frac{\beta I^*}{1+\alpha_1 I^*})(t-t_0)}] \\ &\geqslant \frac{\Lambda(1 + \alpha_1 I^*)}{\beta I^* + \mu_1(1 + \alpha_1 I^*)}[1 - e^{-(\mu_1 + \frac{\beta I^*}{1+\alpha_1 I^*})\lambda}] \\ &:= S^*. \end{aligned} \tag{8}$$

Clearly,

$$S^* \leqslant \frac{\Lambda(1 + \alpha_1 I^*)}{\beta I^* + \mu_1(1 + \alpha_1 I^*)} := m. \tag{9}$$

We define

$$V(t) := I(t) - \beta e^{-\mu_2 \tau} \int_{t-\tau}^{t} \frac{S(\theta)I(\theta)}{1 + \alpha_1 I(\theta) + \alpha_2 S(\theta)} d\theta. \tag{10}$$

The derivative of V along the solutions of (2) is given by

$$\frac{dV(t)}{dt} = \mu_2[\frac{\beta(1 - e^{-\mu_2 \tau})S(t)}{\mu_2(1 + \alpha_1 I(t) + \alpha_2 S(t))} - 1]I(t).$$

By (7), (8) and (9), we have

$$\begin{aligned} \frac{dV(t)}{dt} &> \mu_2[\frac{\beta(1 - e^{-\mu_2 \tau})S^*}{\mu_2(1 + \alpha_1 I(t) + \alpha_2 S^*)} - 1]I(t) \\ &\geqslant \mu_2[\frac{\beta(1 - e^{-\mu_2 \tau})S^*}{\mu_2(1 + \alpha_1 I(t) + \alpha_2 m)} - 1]I(t) \\ &= \mu_2(p - 1)I(t), \quad t \geqslant t_0 + \lambda. \end{aligned} \tag{11}$$

We set

$$I_l = \min_{u \in [-\tau, 0]} I(t_0 + \lambda + \tau + u) > 0.$$

We will show that $I(t) \geqslant I_l$ for all $t \geqslant t_0 + \lambda$. Suppose the contrary. Then there is a $t_1 > t_0 + \lambda + \tau$ such that $I(t_1) < I_l$. Denote $t_l = \inf\{t | I(t) < I_l, t > t_0 + \lambda + \tau\}$.

Then we have $I(t_l) = I_l$ and $I(t) \geqslant I_l$ for $t_0 + \lambda \leqslant t \leqslant t_l$. It follows from the second equation of (2) that

$$I(t) = \beta \int_{t-\tau}^{t} \frac{S(\theta)I(\theta)}{1 + \alpha_1 I(\theta) + \alpha_2 S(\theta)} e^{-\mu_2(t-\theta)} d\theta, \quad t \geqslant \tau.$$

accordingly

$$\begin{aligned} I(t_l) &> \frac{\beta S^* I_l}{1 + \alpha_1 I_l + \alpha_2 S^*} \int_{t_l-\tau}^{t_l} e^{-\mu_2(t_l-\theta)} d\theta \\ &\geqslant \frac{\beta S^* I_l}{1 + \alpha_1 I_l + \alpha_2 m} \int_{t_l-\tau}^{t_l} e^{-\mu_2(t_l-\theta)} d\theta \\ &= \frac{\beta S^* I_l (1 - e^{-\mu_2 \tau})}{\mu_2(1 + \alpha_1 I_l + \alpha_2 m)}. \end{aligned} \tag{12}$$

Since $I_l \leqslant I^*$, then

$$\frac{\beta S^*(1 - e^{-\mu_2 \tau})}{\mu_2(1 + \alpha_1 I_l + \alpha_2 m)} \geqslant \frac{\beta S^*(1 - e^{-\mu_2 \tau})}{\mu_2(1 + \alpha_1 I^* + \alpha_2 m)} = p > 1.$$

As a consequence, (12) leads to $I(t_l) > I_l$. This is a contradiction. Thus, $I(t) \geqslant I_l$ for all $t \geqslant t_0 + \lambda$.
We deduce from (11) and (6) that

$$\frac{dV(t)}{dt} > \mu_2(p-1)I_l > 0,$$

which implies that as $t \to +\infty$, $V(t) \to +\infty$. By using (4) and (10), we obtain

$$\limsup_{t \to +\infty} V(t) \leqslant \frac{\Lambda}{\mu_1} + \frac{\Lambda^2 \beta e^{-\mu_2 \tau} \tau}{\mu_2^2 + \alpha_1 \Lambda \mu_1 + \alpha_2 \Lambda \mu_1},$$

this contradicts $\lim_{t \to +\infty} V(t) = +\infty$. Hence, the claim is proved.
By the claim, we are left to consider two cases. First, $I(t) \geqslant I^*$ for all t large enough. Second, I(t) oscillates about I^* for t large enough. For the second case, let $t^* > 0$ sufficiently large such that $S(t) \geqslant y_1 - \epsilon$ for $\epsilon > 0$ being sufficiently small and $\sigma > 0$ satisfy

$$I^* = I(t^*) = I(t^* + \sigma) \quad and \quad I(t) < I^* \quad for \quad t^* < t < t^* + \sigma.$$

We will restrict study on the interval $[t^*, t^* + \sigma]$. Since $I(t)$ is uniformly continuous, then there is a η ($0 < \eta < \tau$, and η is independent of the choice of t^*) such that $I(t) > \frac{I^*}{2}$ for $t^* < t < t^* + \eta$.
If $\sigma \leqslant \eta$, then $I(t) > \frac{I^*}{2}$ for all $t \in [t^*, t^* + \sigma]$.
Let us consider the case where $\eta < \sigma \leqslant \tau$. For $t^* + \eta < t \leqslant t^* + \sigma$, we obtain

$$\begin{aligned} I(t) &> \beta(y_1 - \epsilon) \int_{t-\tau}^{t} \frac{I(\theta)}{1 + \alpha_1 I(\theta) + \alpha_2(y_1 - \epsilon)} e^{-\mu_2(t-\theta)} d\theta \\ &\geqslant \beta(y_1 - \epsilon) \int_{t^*}^{t^*+\eta} \frac{I(\theta)}{1 + \alpha_1 I(\theta) + \alpha_2(y_1 - \epsilon)} e^{-\mu_2 \tau} d\theta \end{aligned}$$

$$> \frac{\beta(y_1-\epsilon)I^* e^{-\mu_2\tau}\eta}{2(1+\alpha_1\frac{I^*}{2}+\alpha_2(y_1-\epsilon))} := I_0. \tag{13}$$

We set

$$I_1 = \min\{\frac{I^*}{2}, I_0\}. \tag{14}$$

Clearly, $I(t) \geqslant I_1$ for $t \in [t^*, t^*+\sigma]$.
If $\sigma > \tau$, with same reason, $I(t) \geqslant I_1$ for $t \in [t^*, t^*+\tau]$. For $t \in [t^*+\tau, t^*+\frac{3\tau}{2}]$, we obtain

$$\begin{aligned} I(t) &\geqslant \beta(y_1-\epsilon)\int_{t^*+\frac{\tau}{2}}^{t^*+\tau} \frac{I_1}{1+\alpha_1 I_1+\alpha_2(y_1-\epsilon)} e^{-\mu_2(t-\theta)}d\theta \\ &\geqslant \frac{\beta(y_1-\epsilon)I_1 e^{-\mu_2\tau}\tau}{2(1+\alpha_1 I_1+\alpha_2(y_1-\epsilon))} := I_2. \end{aligned}$$

For $t \in (t^*+\frac{3\tau}{2}, t^*+2\tau]$, we obtain

$$\begin{aligned} I(t) &\geqslant \beta(y_1-\epsilon)\int_{t^*+\tau}^{t^*+\frac{3\tau}{2}} \frac{I_2}{1+\alpha_1 I_2+\alpha_2(y_1-\epsilon)} e^{-\mu_2(t-\theta)}d\theta \\ &\geqslant \frac{\beta(y_1-\epsilon)I_2 e^{-\mu_2\tau}\tau}{2(1+\alpha_1 I_2+\alpha_2(y_1-\epsilon))} := I_3. \end{aligned}$$

We set $N = \lceil\frac{\lambda}{\tau}\rceil + 1$ ($\lceil\frac{\lambda}{\tau}\rceil$ is the minimum integer being greater than or equal to $\frac{\lambda}{\tau}$).
Define

$$I_k = \frac{\beta(y_1-\epsilon)I_{k-1}e^{-\mu_2\tau}\tau}{2(1+\alpha_1 I_{k-1}+\alpha_2(y_1-\epsilon))}, \quad k = 2, 3, ..., 2N-1. \tag{15}$$

Continuing the process above, we conclude that

$$I(t) \geqslant I_{2n-2}, \quad t \in (t^*+(n-1)\tau, t^*+(n-\frac{1}{2})\tau],$$

$$I(t) \geqslant I_{2n-1}, \quad t \in (t^*+(n-\frac{1}{2})\tau, t^*+n\tau], \quad n = 2, 3, .., N.$$

Denote

$$y_2 = \min_{1\leqslant k\leqslant 2N-1} I_k,$$

with I_k ($1 \leqslant k \leqslant 2N-1$) are defined in (13), (14) and (15).
Clearly, $I(t) \geqslant y_2$ for all $t \in [t^*, t^*+N\tau]$. Since $\frac{\lambda}{\tau} \leqslant N$, then $I(t) \geqslant y_2$ for all $t \in [t^*, t^*+\lambda]$.
If $\sigma \leqslant \lambda$, then $I(t) \geqslant y_2$ for $t \in (t^*, t^*+\sigma]$.
We will show that if $\sigma > \lambda$, then $I(t) \geqslant y_2$ for $t \in (t^*+\lambda, t^*+\sigma]$. Suppose the contrary, then there is a $\delta > 0$ such that $I(t) \geqslant y_2$ for $t^* \leqslant t \leqslant t^*+\delta+\lambda \leqslant t^*+\sigma$

and $I(t^*+\delta+\lambda) = y_2$. Noting that $I(t) \leqslant I^*$ for $t \in [t^*, t^*+\sigma]$, then $S(t) > S^*$ for $t \geqslant t^*+\lambda$. Hence, for $t = t^*+\delta+\lambda$,

$$\begin{aligned} I(t) &> \frac{\beta S^* y_2}{1+\alpha_1 y_2+\alpha_2 S^*}\int_{t-\tau}^{t} e^{-\mu_2(t-\theta)}d\theta \\ &\geqslant \frac{\beta S^* y_2(1-e^{-\mu_2\tau})}{\mu_2(1+\alpha_1 y_2+\alpha_2 m)}. \end{aligned} \tag{16}$$

By using $I^* \geqslant y_2$ and (6), we conclude that

$$1 < p = \frac{\beta S^*(1-e^{-\mu_2\tau})}{\mu_2(1+\alpha_1 I^*+\alpha_2 m)} \leqslant \frac{\beta S^*(1-e^{-\mu_2\tau})}{\mu_2(1+\alpha_1 y_2+\alpha_2 m)}. \tag{17}$$

It follows from (16) and (17) that $I(t = t^*+\delta+\lambda) > y_2$, which contradicts $I(t = t^*+\delta+\lambda) = y_2$. Consequently, $I(t) \geqslant y_2$ for $t \in (t^*+\lambda, t^*+\sigma]$. Since this kind of interval $[t^*, t^*+\sigma]$ is chosen in an arbitrary way (we only need t^* to be large), we conclude that $I(t) \geqslant y_2$ for all large t in the second case. therefor, $\liminf_{t\to+\infty} I(t) \geqslant y_2$.
Thus, for $\epsilon > 0$ sufficiently small, there is a $M_3 > M_2$ such that if $t > M_3$, then $I(t) \geqslant y_2 - \epsilon$. We derive from the third equation of (2) that, for $t > M_3$,

$$\dot{R}(t) \geqslant \frac{\beta e^{-\mu_2\tau}(y_1-\epsilon)(y_2-\epsilon)}{1+\alpha_1(y_2-\epsilon)+\alpha_2(y_1-\epsilon)} - \mu_3 R(t),$$

by comparison it follows that

$$\liminf_{t\to+\infty} R(t) \geqslant \frac{\beta e^{-\mu_2\tau}(y_1-\epsilon)(y_2-\epsilon)}{\mu_3(1+\alpha_1(y_2-\epsilon)+\alpha_2(y_1-\epsilon))}.$$

Since this inequality hold true for arbitrary $\epsilon > 0$ sufficiently small, we deduce that

$$\liminf_{t\to+\infty} R(t) \geqslant \frac{\beta e^{-\mu_2\tau} y_1 y_2}{\mu_3(1+\alpha_1 y_2+\alpha_2 y_1)} := y_3.$$

This completes the proof. ■

4. Global asymptotic stability of the disease-free equilibrium state

In this section, we establish the global asymptotic stability of the disease-free equilibrium E_1 of system (2).

Lemma 4.1 Let $\mu, \tau \in \mathbb{R}$. If $\mu, \tau > 0$, then

$$(1+\frac{\gamma}{\mu})^2 + \frac{w^2}{\mu^2} \geqslant \frac{1-2\cos(w\tau)e^{-(\gamma+\mu)\tau}+e^{-2(\gamma+\mu)\tau}}{(1-e^{-\mu\tau})^2},$$

for all $\gamma, w \geqslant 0$.

Proof Define the function

$$H(w) := (1+\frac{\gamma}{\mu})^2 + \frac{w^2}{\mu^2} - \frac{1-2\cos(w\tau)e^{-(\gamma+\mu)\tau}+e^{-2(\gamma+\mu)\tau}}{(1-e^{-\mu\tau})^2}.$$

The derivative of H is given by

$$\dot{H}(w) = \frac{2w}{\mu_2^2} - \frac{2\tau \sin(w\tau) e^{-(\gamma+\mu_2)\tau}}{(1-e^{-\mu_2\tau})^2}.$$

The derivative of $\dot{H}$ is given by

$$\ddot{H}(w) = \frac{2}{\mu_2^2} - \frac{2\tau^2 \cos(w\tau) e^{-(\gamma+\mu_2)\tau}}{(1-e^{-\mu_2\tau})^2} \geqslant \frac{2}{\mu_2^2} - \frac{2\tau^2 e^{-\mu_2\tau}}{(1-e^{-\mu_2\tau})^2}.$$

Since $\dot{H}(0) = 0$ and $\ddot{H}(w) \geqslant \frac{2}{\mu_2^2} - \frac{2\tau^2 e^{-\mu_2\tau}}{(1-e^{-\mu_2\tau})^2}$, then $\frac{2}{\mu_2^2} - \frac{2\tau^2 e^{-\mu_2\tau}}{(1-e^{-\mu_2\tau})^2} \geqslant 0$ and $H(0) = (1+\frac{\gamma}{\mu_2})^2 - \frac{(1-e^{-(\gamma+\mu_2)\tau})^2}{(1-e^{-\mu_2\tau})^2} \geqslant 0$ are sufficient conditions to ensure that $H(w) \geqslant 0$. Define de function

$$R(\gamma) := 1 + \frac{\gamma}{\mu_2} - \frac{1-e^{-(\gamma+\mu_2)\tau}}{1-e^{-\mu_2\tau}}.$$

It is clear that $R(0) = 0$ and the derivative of R is given by

$$\dot{R}(\gamma) = \frac{1}{\mu_2} - \frac{\tau e^{-(\gamma+\mu_2)\tau}}{1-e^{-\mu_2\tau}} \geqslant \frac{1}{\mu_2} - \frac{\tau e^{-\mu_2\tau}}{1-e^{-\mu_2\tau}}.$$

It is not difficult to show that $\psi_1(u) = 1 - \frac{u^2 e^{-u}}{(1-e^{-u})^2} > 0$, $\psi_2(u) = 1 - \frac{u e^{-u}}{1-e^{-u}} > 0$ for all $u > 0$. ■

Theorem 4.1 If $R_0 < 1$, then the disease-free equilibrium E_1 of system (2) is locally asymptotically stable; if $R_0 > 1$, then E_1 is unstable.

Proof The characteristic equation of the linearized system of system (2) near the disease-free equilibrium E_1 takes the form

$$(\lambda+\mu_1)(\lambda+\mu_3)(\lambda+\mu_2 - \frac{\beta\Lambda}{\mu_1+\Lambda\alpha_2} + \frac{\beta\Lambda}{\mu_1+\Lambda\alpha_2} e^{-(\mu_2+\lambda)\tau}) = 0. \quad (18)$$

Clearly, (18) always has two negative roots $\lambda_1 = -\mu_1$, $\lambda_2 = -\mu_3$. The other roots of (18) are determined by the following equation

$$\lambda + \mu_2 - \frac{\beta\Lambda}{\mu_1+\Lambda\alpha_2} + \frac{\beta\Lambda}{\mu_1+\Lambda\alpha_2} e^{-(\mu_2+\lambda)\tau} = 0. \quad (19)$$

Let $\lambda = \gamma + iw$. Thanks to the property of symmetry we can assume that $w \geqslant 0$. Separating real and imaginary parts of (19), it follows that

$$\begin{aligned} \mu_2 + \gamma &= \frac{\beta\Lambda}{\mu_1+\Lambda\alpha_2}[1 - e^{-(\mu_2+\gamma)\tau}\cos(w\tau)], \\ w &= \frac{\beta\Lambda}{\mu_1+\Lambda\alpha_2} e^{-(\mu_2+\gamma)\tau}\sin(w\tau). \end{aligned} \quad (20)$$

Squaring and adding the two equations of (20) and using $\frac{\beta\Lambda}{\mu_1+\Lambda\alpha_2} = \frac{\mu_2 R_0}{1-e^{-\mu_2\tau}}$, we obtain that

$$\frac{1}{R_0^2}[(1+\frac{\gamma}{\mu_2})^2+\frac{w^2}{\mu_2^2}]=\frac{1-2\cos{(w\tau)}e^{-(\gamma+\mu)\tau}+e^{-2(\gamma+\mu)\tau}}{(1-e^{-\mu\tau})^2}.$$

We assume for contradiction that there exists $\gamma \geqslant 0$ satisfy the two equations of (20). It follows from Lemma 4.1. that if $R_0 < 1$, then

$$\frac{1}{R_0^2}[(1+\frac{\gamma}{\mu_2})^2+\frac{w^2}{\mu_2^2}] > (1+\frac{\gamma}{\mu_2})^2+\frac{w^2}{\mu_2} \geqslant \frac{1-2\cos{(w\tau)}e^{-(\gamma+\mu)\tau}+e^{-2(\gamma+\mu)\tau}}{(1-e^{-\mu\tau})^2},$$

which is a contradiction. Consequently, the disease-free equilibrium E_1 is locally asymptotically stable when $R_0 < 1$.
For λ real, denote

$$g(\lambda)=\lambda+\mu_2-\frac{\beta\Lambda}{\mu_1+\Lambda\alpha_2}+\frac{\beta\Lambda}{\mu_1+\Lambda\alpha_2}e^{-(\mu_2+\lambda)\tau}.$$

If $R_0 > 1$, clearly,

$$g(0)=\mu_2(1-R_0)<0, \quad \lim_{\lambda\to+\infty} g(\lambda)=+\infty.$$

Consequently, (19) admits at least one positive root. Hence, if $R_0 > 1$, E_1 is unstable. ■

Lemma 4.2 [10] Let $(f_n)_{n\in\mathbb{N}}$ be a measurable sequence of non-negative uniformly bounded functions. Then

$$\int \liminf f_n \leqslant \liminf \int f_n \leqslant \limsup \int f_n \leqslant \int \limsup f_n.$$

Theorem 4.2 If $R_0 < 1$, then the disease-free equilibrium E_1 of system (2) is globally asymptotically stable; if $R_0 > 1$, then E_1 is unstable.

Proof In Theorem 4.1, we have given the local asymptotic stability of E_1. We now prove that E_1 is globally attractive.
Let $(S(t), I(t), R(t))$ be any non-negative solution of (2) with initial conditions (3). Clearly,

$$\limsup_{t\to+\infty} S(t) \leqslant \frac{\Lambda}{\mu_1}. \tag{21}$$

Therefore, for $\epsilon > 0$ sufficiently small, there is a $M_1 > 0$ such that if $t > M_1$, $S(t) \leqslant \frac{\Lambda}{\mu_1}+\epsilon$.
It follows from the second equation of system (2) that

$$\begin{aligned}\limsup_{t\to+\infty} I(t) &= \limsup_{t\to+\infty}\int_0^\tau \frac{\beta I(t-\theta)S(t-\theta)}{1+\alpha_1 I(t-\theta)+\alpha_2 S(t-\theta)}e^{-\mu_2\theta}d\theta\\ &\leqslant \int_0^\tau \frac{\limsup_{t\to+\infty}\beta I(t-\theta)\limsup_{t\to+\infty}S(t-\theta)}{1+\limsup_{t\to+\infty}\alpha_1 I(t-\theta)+\limsup_{t\to+\infty}\alpha_2 S(t-\theta)}e^{-\mu_2\theta}d\theta\\ &\leqslant \frac{\Lambda}{\mu_1\mu_2}(1-e^{-\mu_2\tau})\frac{\limsup_{t\to+\infty}\beta I(t)}{1+\limsup_{t\to+\infty}\alpha_1 I(t)+\alpha_2\frac{\Lambda}{\mu_1}},\end{aligned}$$

$$\leqslant \frac{\Lambda}{\mu_1\mu_2}(1-e^{-\mu_2\tau})\frac{\limsup_{t\to+\infty}\beta I(t)}{1+\alpha_2\frac{\Lambda}{\mu_1}},$$
$$= \limsup_{t\to+\infty} I(t)R_0,$$

hence,

$$\limsup_{t\to+\infty} I(t)(R_0-1)\geqslant 0. \tag{22}$$

Since $R_0<1$, then (22) ensures that

$$\limsup_{t\to+\infty} I(t)=0.$$

Thus, for $\epsilon>0$ sufficiently small, there is a $M_2>M_1$ such that if $t>M_2$, $I(t)\leqslant\epsilon$. We derive from the third equation of system (2) that, for $t>M_2+\tau$,

$$\dot{R}(t)\leqslant \frac{\beta e^{-\mu_2\tau}(\frac{\Lambda}{\mu_1}+\epsilon)\epsilon}{1+\alpha_1\epsilon+\alpha_2(\frac{\Lambda}{\mu_1}+\epsilon)}-\mu_3R(t).$$

By using a comparison argument, we obtain

$$\limsup_{t\to+\infty} R(t)\leqslant \frac{\beta e^{-\mu_2\tau}(\frac{\Lambda}{\mu_1}+\epsilon)\epsilon}{\mu_3(1+\alpha_1\epsilon+\alpha_2(\frac{\Lambda}{\mu_1}+\epsilon))}.$$

Since this inequality holds true for arbitrary $\epsilon>0$ sufficiently small, we conclude that

$$\lim_{t\to+\infty} R(t)=0.$$

It follows from the first equation of system (2) that, for $t>M_2$,

$$\dot{S}(t)\geqslant \Lambda-\mu_1S(t)-\frac{\beta\epsilon}{1+\alpha_1\epsilon}S(t).$$

By using a comparison argument, we obtain

$$\liminf_{t\to+\infty} S(t)\geqslant \frac{\Lambda(1+\alpha_1\epsilon)}{\beta\epsilon+\mu_1(1+\alpha_1\epsilon)}.$$

Since this inequality holds true for arbitrary $\epsilon>0$ sufficiently small, we conclude that

$$\liminf_{t\to+\infty} S(t)\geqslant \frac{\Lambda}{\mu_1}.$$

We deduce from this inequality and (21) that

$$\lim_{t\to+\infty} S(t)=\frac{\Lambda}{\mu_1}.$$

This completes the proof. ■

5. Global asymptotic stability of the endemic equilibrium state

In this section, we establish the local asymptotic stability of the endemic equilibrium E_2 of system (2) and we give the sufficient conditions for the global asymptotic stability of the endemic equilibrium E_2.

Theorem 5.1 If $R_0 > 1$, then the endemic equilibrium E_2 of system (2) is locally asymptotically stable.

Proof The characteristic equation of the linearized system of system (2) near the endemic equilibrium E_2 takes the form

$$(\lambda + \mu_3)[\lambda^2 + p_1(\tau)\lambda + p_0(\tau) + (q_1(\tau)\lambda + q_0(\tau))e^{-\lambda\tau}] = 0, \tag{23}$$

where

$$\begin{aligned} p_1(\tau) &= \mu_1 + \mu_2 + l_1 - l_2, & p_0(\tau) &= \mu_2(\mu_1 + l_1) - \mu_1 l_2, \\ q_1(\tau) &= l_2 e^{-\mu_2\tau}, & q_0(\tau) &= \mu_1 l_2 e^{-\mu_2\tau}, \\ l_1 &= \frac{\beta I_2(1+\alpha_1 I_2)}{(1+\alpha_1 I_2 + \alpha_2 S_2)^2}, & l_2 &= \frac{\beta S_2(1+\alpha_2 S_2)}{(1+\alpha_1 I_2 + \alpha_2 S_2)^2}. \end{aligned}$$

Clearly, (23) always has a negative root $\lambda = -\mu_3$. Other roots of (23) are determined by the following equation

$$\lambda^2 + p_1(\tau)\lambda + p_0(\tau) + (q_1(\tau)\lambda + q_0(\tau))e^{-\lambda\tau} = 0. \tag{24}$$

Let $\lambda = \gamma + iw$. Thanks to the property of symmetry we can assume that $w \geqslant 0$. Separating real and imaginary parts of (24), it follows that

$$\begin{aligned} (\gamma+\mu_1)(\gamma+\mu_2-l_2) - w^2 + (\gamma+\mu_2)l_1 = -l_2[(\gamma+\mu_1)\cos(w\tau) + w \\ \times \sin(w\tau)]e^{-(\gamma+\mu_2)\tau}, \end{aligned} \tag{25}$$

$$w[\gamma+\mu_2-l_2+\gamma+\mu_1+l_1] = -l_2[w\cos(w\tau) - (\gamma+\mu_1)\sin(w\tau)]e^{-(\gamma+\mu_2)\tau}. \tag{26}$$

Multiplying (25) by w and (26) by $\gamma+\mu_1$ and subtracting the products, we obtain that

$$\frac{w}{(\gamma+\mu_1)^2+w^2}(\mu_2-\mu_1)l_1 - w = -l_2\sin(w\tau)e^{-(\gamma+\mu_2)\tau}. \tag{27}$$

Multiplying (25) by $\gamma+\mu_1$ and (26) by w and adding the products, we obtain that

$$\gamma+\mu_2 + \frac{(\gamma+\mu_1)(\gamma+\mu_2)+w^2}{(\gamma+\mu_1)^2+w^2} l_1 = l_2[1 - \cos(w\tau)e^{-(\gamma+\mu_2)\tau}]. \tag{28}$$

We set

$$T_1 = \frac{[\gamma+\mu_1)(\gamma+\mu_2)+w^2]^2}{[(\gamma+\mu_1)^2+w^2]^2} l_1^2 + \frac{w^2}{[(\gamma+\mu_1)^2+w^2]^2}(\mu_2-\mu_1)^2 l_1^2,$$

$$T_2 = \frac{2\mu_2(\gamma+\mu_1)(\gamma+\mu_2)}{(\gamma+\mu_1)^2+w^2}l_1 + \frac{2w^2\mu_1}{(\gamma+\mu_1)^2+w^2}l_1 + \frac{2\gamma[(\gamma+\mu_1)(\gamma+\mu_2)+w^2]}{(\gamma+\mu_1)^2+w^2}l_1.$$

Squaring and adding the two equations (27) and (28), we get that

$$(\gamma+\mu_2)^2 + w^2 + T_1 + T_2 = l_2^2[1 - 2\cos(w\tau)e^{-(\gamma+\mu_2)\tau} + e^{-2(\gamma+\mu_2)\tau}]. \qquad (29)$$

We assume for contradiction that there exists $\gamma \geqslant 0$ satisfy equations (25) and (26). Since $T_1 + T_2 > 0$, then it suffices to prove that $(\gamma+\mu_2)^2 + w^2 \geqslant l_2^2[1 - 2\cos(w\tau)e^{-(\gamma+\mu_2)\tau} + e^{-2(\gamma+\mu_2)\tau}]$.

Define the functions

$$G(w) := (1+\frac{\gamma}{\mu_2})^2 + (\frac{w}{\mu_2})^2,$$

$$Z(w) := \frac{1-2\cos(w\tau)e^{-(\gamma+\mu_2)\tau} + e^{-2(\gamma+\mu_2)\tau}}{(1-e^{-\mu_2\tau})^2}.$$

It is clear that if $R_0 > 1$, then $\frac{\mu_2}{l_2} = (1-e^{-\mu_2\tau})\frac{\beta+\alpha_1(\mu_1+\Lambda\alpha_2)R_0}{\beta+\alpha_1(\mu_1+\Lambda\alpha_2)} \geqslant 1-e^{-\mu_2\tau}$. It follows from Lemma 4.1. that $G(w) \geqslant Z(w)$ for all $w \geqslant 0$, which is a contradiction. ■

Theorem 5.2 Suppose that $\alpha_1\mu_1 \geqslant \beta$. If $R_0 > 1$, then the endemic equilibrium E_2 of system (2) is globally asymptotically stable.

Proof In Theorem 5.1, we have given the local asymptotic stability of E_2. We now prove that E_2 is globally attractive.
Let $(S(t), I(t), R(t))$ be any positive solution of (2) with initial conditions (3). Denote

$$S_s = \limsup_{t\to+\infty} S(t), \quad I_s = \limsup_{t\to+\infty} I(t), \quad R_s = \limsup_{t\to+\infty} R(t),$$

$$S_i = \liminf_{t\to+\infty} S(t), \quad I_i = \liminf_{t\to+\infty} I(t), \quad R_i = \liminf_{t\to+\infty} R(t).$$

In the following we prove that $S_s = S_i = S_2$, $I_s = I_i = I_2$, $R_s = R_i = R_2$.
Clearly,

$$\limsup_{t\to+\infty} S(t) \leqslant \frac{\Lambda}{\mu_1} := X_1^S.$$

Consequently, for $\epsilon > 0$ sufficiently small there is a $M_1 > 0$ such that if $t \geqslant M_1$,, $S(t) \leqslant X_1^S + \epsilon$.
We derive from the second equation of system (2) that

$$I(t) = \int_0^\tau \frac{\beta I(t-\theta)S(t-\theta)}{1+\alpha_1 I(t-\theta) + \alpha_2 S(t-\theta)} e^{-\mu_2\theta} d\theta. \qquad (30)$$

It follows from (31) and Lemma (4.2) that

$$\limsup_{t\to+\infty} I(t) = \limsup_{t\to+\infty} \int_0^\tau \frac{\beta I(t-\theta)S(t-\theta)}{1+\alpha_1 I(t-\theta) + \alpha_2 S(t-\theta)} e^{-\mu_2\theta} d\theta$$

$$\leqslant \int_0^\tau \frac{\limsup_{t\to+\infty} \beta I(t-\theta) \limsup_{t\to+\infty} S(t-\theta)}{1+\limsup_{t\to+\infty}\alpha_1 I(t-\theta)+\limsup_{t\to+\infty}\alpha_2 S(t-\theta)} e^{-\mu_2\theta} d\theta$$
$$\leqslant \frac{\beta X_1^S}{\mu_2}(1-e^{-\mu_2\tau})\frac{\limsup_{t\to+\infty} I(t)}{1+\limsup_{t\to+\infty}\alpha_1 I(t)+\alpha_2 X_1^S} - \mu_3 R(t),$$

which leads to

$$\limsup_{t\to+\infty} I(t) \leqslant \frac{(\beta(1-e^{-\mu_2\tau})-\mu_2\alpha_2)X_1^S-\mu_2}{\mu_2\alpha_1} := X_1^I.$$

Consequently, for $\epsilon > 0$ sufficiently small there is $M_2 > M_1$ such that if $t > M_2$, $I(t) \leqslant X_1^I + \epsilon$.
We derive from the third equation of system (2) that, for $t > M_2 + \tau$,

$$\dot{R}(t) \leqslant \frac{\beta e^{-\mu_2\tau}(X_1^S+\epsilon)(X_1^I+\epsilon)}{1+\alpha_1(X_1^I+\epsilon)+\alpha_2(X_1^S+\epsilon)},$$

which leads to

$$\limsup_{t\to+\infty} R(t) \leqslant \frac{\beta e^{-\mu_2\tau}(X_1^S+\epsilon)(X_1^I+\epsilon)}{\mu_3(1+\alpha_1(X_1^I+\epsilon)+\alpha_2(X_1^S+\epsilon))}.$$

Since this inequality holds true for arbitrary $\epsilon > 0$ sufficiently small, we conclude that

$$\limsup_{t\to+\infty} R(t) \leqslant \frac{\beta e^{-\mu_2\tau} X_1^S X_1^I}{\mu_3(1+\alpha_1 X_1^I+\alpha_2 X_1^S)} := X_1^R.$$

Consequently, for $\epsilon > 0$ sufficiently small there is $M_3 > M_2+\tau$ such that if $t > M_3$, $R(t) \leqslant X_1^R + \epsilon$.
We derive from the first equation of system (2) that, for $t > M_3$,

$$\dot{S}(t) \geqslant \Lambda - \mu_1 S(t) - \frac{\beta(X_1^S+\epsilon)(X_1^I+\epsilon)}{1+\alpha_1(X_1^I+\epsilon)+\alpha_2(X_1^S+\epsilon)},$$

which yields

$$\liminf_{t\to+\infty} S(t) \geqslant \frac{1}{\mu_1}[\Lambda - \frac{\beta(X_1^S+\epsilon)(X_1^I+\epsilon)}{1+\alpha_1(X_1^I+\epsilon)+\alpha_2(X_1^S+\epsilon)}].$$

Since this inequality holds true for arbitrary $\epsilon > 0$ sufficiently small, we conclude that

$$\liminf_{t\to+\infty} S(t) \geqslant \frac{1}{\mu_1}(\Lambda - \frac{\beta X_1^S X_1^I}{1+\alpha_1 X_1^I+\alpha_2 X_1^S}) := Y_1^S.$$

Consequently, for $\epsilon > 0$ sufficiently small there is $M_4 > M_3$ such that if $t > M_4$, $S(t) \geqslant Y_1^S - \epsilon$.
We derive from the second equation of system (2) that

$$\liminf_{t\to+\infty} I(t) = \liminf_{t\to+\infty} \int_0^\tau \frac{\beta I(t-\theta)S(t-\theta)}{1+\alpha_1 I(t-\theta)+\alpha_2 S(t-\theta)} e^{-\mu_2\theta} d\theta$$

$$\geqslant \int_0^{\tau} \frac{\liminf_{t\to+\infty} \beta I(t-\theta) \liminf_{t\to+\infty} S(t-\theta)}{1+\liminf_{t\to+\infty} \alpha_1 I(t-\theta)+\liminf_{t\to+\infty} \alpha_2 S(t-\theta)} e^{-\mu_2\theta} d\theta$$
$$\geqslant \frac{\beta Y_1^S}{\mu_2}(1-e^{-\mu_2\tau}) \frac{\liminf_{t\to+\infty} \beta I(t)}{1+\liminf_{t\to+\infty} \alpha_1 I(t)+\alpha_2 Y_1^S}. \tag{31}$$

It follows from section 3. that $\liminf_{t\to+\infty} I(t) > 0$, we conclude from (31) that

$$\liminf_{t\to+\infty} I(t) \geqslant \frac{(\beta(1-e^{-\mu_2\tau})-\mu_2\alpha_2)Y_1^S-\mu_2}{\mu_2\alpha_1} := Y_1^I.$$

Consequently, for $\epsilon > 0$ sufficiently small there is $M_5 > M_4$ such that if $t > M_5$, $I(t) \geqslant Y_1^I - \epsilon$.
We derive from the third equation of system (2) that, for $t > M_5 + \tau$,

$$\dot{R}(t) \geqslant \frac{\beta e^{-\mu_2\tau}(Y_1^S-\epsilon)(Y_1^I-\epsilon)}{1+\alpha_1(Y_1^I-\epsilon)+\alpha_2(Y_1^S-\epsilon)} - \mu_3 R(t),$$

which leads to

$$\liminf_{t\to+\infty} R(t) \geqslant \frac{\beta e^{-\mu_2\tau}(Y_1^S-\epsilon)(Y_1^I-\epsilon)}{\mu_3(1+\alpha_1(Y_1^I-\epsilon)+\alpha_2(Y_1^S-\epsilon))}.$$

Since this inequality holds true for arbitrary $\epsilon > 0$ sufficiently small, we conclude that

$$\liminf_{t\to+\infty} R(t) \geqslant \frac{\beta e^{-\mu_2\tau} Y_1^S Y_1^I}{\mu_3(1+\alpha_1 Y_1^I+\alpha_2 Y_1^S)} := Y_1^R.$$

Consequently, for $\epsilon > 0$ sufficiently small there is $M_6 > M_5 + \tau$ such that if $t > M_6$, $R(t) \geqslant Y_1^R - \epsilon$.
Another time, we derive from the first equation of system (2) that, for $t > M_6$,

$$\dot{S}(t) \leqslant \Lambda - \mu_1 - \frac{\beta(Y_1^S-\epsilon)(Y_1^I-\epsilon)}{1+\alpha_1(Y_1^I-\epsilon)+\alpha_2(Y_1^S-\epsilon)},$$

which leads to

$$\limsup_{t\to+\infty} S(t) \leqslant \frac{1}{\mu_1}[\Lambda - \frac{\beta(Y_1^S-\epsilon)(Y_1^I-\epsilon)}{1+\alpha_1(Y_1^I-\epsilon)+\alpha_2(Y_1^S-\epsilon)}].$$

Since this inequality holds true for arbitrary $\epsilon > 0$ sufficiently small, we conclude that

$$\limsup_{t\to+\infty} S(t) \leqslant \frac{1}{\mu_1}(\Lambda - \frac{\beta Y_1^S Y_1^I}{1+\alpha_1 Y_1^I+\alpha_2 Y_1^S}) := X_2^S.$$

Consequently, for $\epsilon > 0$ sufficiently small there is $M_7 > M_6$ such that if $t > M_7$, $S(t) \leqslant X_2^S + \epsilon$.
Continuing the process above, we obtain six sequences $(X_n^S)_{n\in\mathbb{N}}$, $(X_n^I)_{n\in\mathbb{N}}$,

$(X_n^R)_{n\in\mathbb{N}}$, $(Y_n^S)_{n\in\mathbb{N}}$, $(Y_n^I)_{n\in\mathbb{N}}$, $(Y_n^R)_{n\in\mathbb{N}}$ such that, for $n \geqslant 2$,

$$\begin{aligned} X_n^S &= \frac{1}{\mu_1}(\Lambda - \frac{\beta Y_{n-1}^S Y_{n-1}^I}{1+\alpha_1 Y_{n-1}^I + \alpha_2 Y_{n-1}^S}), \\ Y_n^S &= \frac{1}{\mu_1}(\Lambda - \frac{\beta X_n^S X_n^I}{1+\alpha_1 X_n^I + \alpha_2 X_n^S}), \\ X_n^I &= \frac{(\beta(1-e^{-\mu_2\tau}) - \mu_2\alpha_2)X_n^S - \mu_2}{\mu_2\alpha_1} \\ Y_n^I &= \frac{(\beta(1-e^{-\mu_2\tau}) - \mu_2\alpha_2)Y_n^S - \mu_2}{\mu_2\alpha_1} \\ X_n^R &= \frac{\beta e^{-\mu_2\tau} X_n^S X_n^I}{\mu_3(1+\alpha_1 X_n^I + \alpha_2 X_n^S)} \\ Y_n^R &= \frac{\beta e^{-\mu_2\tau} Y_n^S Y_n^I}{\mu_3(1+\alpha_1 Y_n^I + \alpha_2 Y_n^S)}. \end{aligned} \tag{32}$$

Clearly,

$$Y_n^S \leqslant S_i \leqslant S_s \leqslant X_n^S, Y_n^I \leqslant I_i \leqslant I_s \leqslant X_n^I, Y_n^R \leqslant R_i \leqslant R_s \leqslant X_n^R. \tag{33}$$

It follows from (32) that

$$\begin{aligned} X_{n+1}^S = \frac{1}{\mu_1}\{(1 - \frac{1}{\mu_1}\frac{\beta(1-e^{-\mu_2\tau}) - \mu_2\alpha_2}{\alpha_1(1-e^{-\mu_2\tau})})(\Lambda + \frac{\mu_2}{\alpha_1(1-e^{-\mu_2\tau})}) \\ + \frac{1}{\mu_1}[\frac{\beta(1-e^{-\mu_2\tau}) - \mu_2\alpha_2}{\alpha_1(1-e^{-\mu_2\tau})}]^2 X_n^S\}. \end{aligned} \tag{34}$$

We derive from (34) that

$$\begin{aligned} X_{n+1}^S - X_n^S = \frac{1}{\mu_1}\{1 - \frac{1}{\mu_1}\frac{\beta(1-e^{-\mu_2\tau}) - \mu_2\alpha_2}{\alpha_1(1-e^{-\mu_2\tau})}\}\{\Lambda + \frac{\mu_2}{\alpha_1(1-e^{-\mu_2\tau})} - \mu_1(1 + \frac{1}{\mu_1} \\ \times \frac{\beta(1-e^{-\mu_2\tau}) - \mu_2\alpha_2}{\alpha_1(1-e^{-\mu_2\tau})})X_n^S\}, \end{aligned}$$

since $X_n^S \geqslant S_2$ and $(1-e^{-\mu_2\tau})(\mu_1\alpha_1 - \beta) \geqslant 0 \geqslant -\mu_2\alpha_2$, then

$$\begin{aligned} X_{n+1}^S - X_n^S &\leqslant \frac{1}{\mu_1}\{1 - \frac{1}{\mu_1}\frac{\beta(1-e^{-\mu_2\tau}) - \mu_2\alpha_2}{\alpha_1(1-e^{-\mu_2\tau})}\}\{\Lambda + \frac{\mu_2}{\alpha_1(1-e^{-\mu_2\tau})} \\ &- \mu_1(1 + \frac{1}{\mu_1}\frac{\beta(1-e^{-\mu_2\tau}) - \mu_2\alpha_2}{\alpha_1(1-e^{-\mu_2\tau})})S_2\} \\ &= 0. \end{aligned}$$

Hence, the sequence $(X_n^S)_{n\in\mathbb{N}}$ is monotonically non-increasing. Thus, $(X_n^S)_{n\in\mathbb{N}}$ is convergent. We derive from (34) that

$$\lim_{n\to+\infty} X_n^S = \frac{\mu_2 + \Lambda\alpha_1(1-e^{-\mu_2\tau})}{(\beta+\mu_1\alpha_1)(1-e^{-\mu_2\tau}) - \mu_2\alpha_2} = S_2. \tag{35}$$

It follows from (32) and (35) that

$$\lim_{n\to+\infty} X_n^S = S_2, \quad \lim_{n\to+\infty} X_n^I = S_2, \quad \lim_{n\to+\infty} X_n^R = R_2,$$
$$\lim_{n\to+\infty} Y_n^S = S_2, \quad \lim_{n\to+\infty} Y_n^I = I_2, \quad \lim_{n\to+\infty} Y_n^R = R_2. \tag{36}$$

We deduce from (33) and (36) that

$$S_s = S_i = S_2, \quad I_s = I_i = I_2, \quad R_s = R_i = R_2.$$

Consequently

$$\lim_{t\to+\infty} S(t) = S_2, \quad \lim_{t\to+\infty} I(t) = I_2, \quad \lim_{t\to+\infty} R(t) = R_2.$$

This completes the proof.

■

6. Discussion

In this work, we have studied an SIR epidemic model with nonlinear incidence rate which has a more general form, and with a constant infectious period. The dynamical behaviors of the model are almost completely determined by the reproduction number R_0. When $R_0 < 1$, the disease-free equilibrium E_1 is globally asymptotically stable, and no other equilibria exist. When $R_0 > 1$, the equilibrium E_1 loses its stability, and an unique endemic equilibrium E_2 appears which is permanent, locally asymptotically stable and if $\alpha_1\mu_1 \geqslant \beta$, the endemic equilibrium is globally asymptotically stable. The global asymptotic stability of both the disease-free and endemic equilibrium was established by analyzing the corresponding characteristic equation and using comparison arguments. The Lemma 4.1. was necessary and important to show the local asymptotic stability of the equilibria. The global attractiveness of the endemic equilibrium E_2 when $\alpha_1\mu_1 \geqslant \beta$ was based on the result $\liminf I(t) > 0$ obtained in the proof of the permanence of the endemic equilibrium E_2. In future work, we would like to study the global asymptotic stability of the endemic equilibrium E_2 without restrictions on the parameter values.

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