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The Tau-Collocation Method for Solving Nonlinear Integro-Differential Equations and Application of a Population Model

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Abstract. This paper present a computational technique that called Tau-collocation method for the developed solution of nonlinear integro-differential equations which involves a population model. To do this, the nonlinear integro-differential equations are transformed into a system of linear algebraic equations in matrix form without interpolation of non polynomial terms of equations. Then, using collocation points, we solve this system and obtain the unknown coefficients. To illustrate the ability and reliability of the method some nonlinear integro-differential equations and population models are presented. The results reveal that the method is very effective and simple.

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Index to information contained in this paper

1. Introduction

Large varieties of physical, chemical and biological phenomena have been modeled by nonlinear equations, like ordinary or partial differential equations, integral and

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integro- differential equations. Recently a lot of attention has been devoted by researchers to formulate the physical phenomena contain integro-differential equations, these equations arises in many fields like fluid dynamics, biological models and chemical kinetics.
Some papers have been devoted to find analytical and numerical solution by Adomian decomposition method [30], Homotopy analysis method[1], Homotopy perturbation method [3], Haar wavelets [27], lagrange functions and Lagrange interpolation [25], [26], Taylor polynomials [19], Chebyshev polynomials [7], sine-cosine wavelets [13], hybrid Legendre and Block Pulse functions [20] and so on.

The idea of the Tau-collocation method for ordinary differential equations with some supplementary conditions is first given by Liu [15] in 1986. In [16] this method is applied successfully to find the numerical solution of eigenvalue problems and in [5] and [31] investigated the Tau-Collocation method in details. As same as idea the Tau method, a perturbation term is added to right hand side of the integro-differential equations to form Tau method. The formulation of the Tau-collocation uses a set of collocation point, the zeros of Chebyshev or Legender polynomial, for solving integro-differential equations. Recently, Allahviranloo et al. provided an efficient numerical approach for multi-order fractional differential equations based on the Tau-Collocation method [2]. Moreover the approximate solution for the nonlinear Volterra-Fredholm-Hammerstein integral equations is obtained by using the Tau-Collocation method in [9].

The structure of this paper is organized as follows: In Section 2, the detailed theorem and formulation of the Tau-collocation method for linear Volterra integro-differential equations is brought. In Section 3, the same is done for linear Fredholm integro-differential equations. In Section 4, we present a method for converting nonlinear equations to linear equations. In Section 5, the Tau-Collocation error estimator function is obtained. Some numerical results are given to clarify the method in Section 6. At the last section, we will have a conclusion of our study.

2. Population model

A mathematical formulation of populations only started in the eighteenth century. In 1767, Leonhard Euler produced the first mathematical model of a human population. This was closely followed, in 1772 by Johann-Heinrich Lambert, using data on human mortality in London (1753-1758), who gave a mathematical formulation of the law of mortality. England has a rich source of data on populations, derived from a steady collection over centuries. In 1798, Thomas Malthus formulated the rate of growth of population size using a first-order differential equation.

Historically, the modern theory of the population dynamic started with Alfred James Lotka (1880-1949), an American Biophysicist. He worked in particular, on the stability of the age composition of a population. In 1907, he published an article [17] which contained two fundamental equations in the study of populations. The most important paper [28], which shall be considered as a rigorous first formulation on the subject of Population Mathematics was that published in 1911 by Lotka and F.R. Sharpe [23]. This equation tracks the rate of female births in the case of a stable population. The basic Lotka one-sex deterministic population model is demonstrated by

$$u(x) = g(x) + \int_0^x K(x-t)u(t)\mathrm{d}t \tag{1}$$

where $K(t-x)$ is net maternity function of females class age x at time t, $g(t)$ is contribution of birth due to female already present at time t and $u(t)$ is the number of female births. In this paper, we consider a more general structure of the equation (1). Let us consider the m-th order nonlinear integro-differential equation in form

$$\sum_{k=0}^{m} p_k(x)\frac{\mathrm{d}^k}{\mathrm{d}x^k}(u(x)) + \lambda_1 \int_0^1 K_1(x,t)G_1(t,u(t))dt + \lambda_2 \int_0^x K_2(x,t)G_2(t,u(t))dt = g(x)$$

with the given supplementary conditions

$$(\ell_j, u) = \gamma_j, \qquad j = 1, 2, ..., m-1 \tag{2}$$

where ℓ_j is a set of linear evaluation functional acting on $u(x)$ and its derivatives and $(\gamma_j)_{j=1}^{m-1}$ are constant. $(\ell_j, u) = \gamma_j$ stands for the initial, boundary or mixed conditions of the nonlinear integro-differential equations under consideration. In Eq.(2), λ_1 and λ_2 are suitable constant, $p_k(x), g(x), K_1(x,t)$ and $K_2(x,t)$ are the given continuous functions on the interval $0 \leqslant x, t \leqslant 1$ and $u(x)$ is a unknown function. Moreover $G_1(t, u(t))$ and $G_2(t, u(t))$ are given continuous functions which are nonlinear with respect to $u(t)$ and t.

3. Tau-collocation method

In this section, first we recall the definitions of Tau-Collocation method. Our aim is to approximate the solution $u(t)$ by the truncated series

$$u(x) = \sum_{i=0}^{\infty} a_i L_i(x) = \mathbf{a}\underline{\mathbf{L}}\mathbf{X}$$

to be an series expansion of the exact solution of Eq.(2), where $L_i(x),\ i = 0, 1, ...$ are Legendre polynomial, $\underline{\mathbf{L}}$ is a non-singular lower triangular matrix and

$$\mathbf{X} = \Big(1, x, x^2, x^3, \dots\Big)^t \quad , \qquad \mathbf{a} = \Big(a_0, a_1, a_2, \dots\Big) \quad , \qquad \underline{\mathbf{L}} = \begin{pmatrix} l_{11} & 0 & 0 & \cdots \\ l_{21} & l_{22} & 0 & \cdots \\ l_{31} & l_{32} & l_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We look for an approximate solution $u_N(x)$ of the problem (2) in the form

$$u_N(x) = \sum_{i=0}^{N} a_i L_i(x) = \mathbf{a}_n\ \underline{\mathbf{L}}\mathbf{X} \tag{3}$$

where $\mathbf{a}_n = \Big(a_0, a_1, ..., a_N, 0, \cdots\Big)$ is a vector of unknown coefficients and $\underline{\mathbf{L}}\mathbf{X} = \Big(L_0(t), L_1(t), L_3(t), ...\Big)^t$.

To find the approximate polynomial solution $u_N(x)$ for (2), as same as the idea of the Tau Method [22], a perturbation term $H_N(x)$ is added to the right hand side of equation (2). Also by substituting $u_N(x)$ from (3) in (2) we have

$$\begin{cases} \sum_{k=0}^{m} p_k(x)\frac{\mathrm{d}^k}{\mathrm{d}x^k}(u(x)) + \lambda_1 \int_0^1 K_1(x,t)G_1(t,u(t))dt + \lambda_2 \int_0^x K_2(x,t)G_2(t,u(t))dt = g(x) + H_N(x), \\ (\ell_j, u(x)) = \gamma_j \qquad j = 1, ..., \vartheta., \end{cases} \tag{4}$$

The perturbation term $H_N(x)$ usually chosen as

$$H_N(x) = g(x, \tau_0, \tau_1, ..., \tau_{\varphi-1})V_{N-\vartheta+1}(x).$$

Where $g(t, \tau_0, \tau_1, ..., \tau_{\varphi-1})$ is a function of t with some free parameters τ_i, $i = 1, 2, ..., \varphi - 1$ and $V_{N-\vartheta+1}(x)$ is an orthogonal polynomial with degree $N - \vartheta + 1$. Usually, the orthogonal polynomial is chosen in the shifted Chebyshev or Legendre polynomial [16]. Note that the free parameters τ_i, $i = 1, 2, ..., \varphi - 1$, inside function $g(x, \tau_0, \tau_1, ..., \tau_{\varphi-1})$ balance the over determined system of linear algebraic equations. The format of perturbation term $H_N(x)$ is selected for producing exactly $N - \vartheta + 1$ zeros of the orthogonal polynomial $V_{N-\vartheta+1}(x)$. It is not necessary to consider the detail of the function $g(t, \tau_0, \tau_1, ..., \tau_{\varphi-1})$, i.e. the number of free parameter φ throughout the computation because the zeros of the orthogonal polynomial $V_{N-\vartheta+1}(x)$ are used for collocation during the formulation process of the Tau-Collocation method for the nonlinear integro-differential equations [19]. The operational approach of Tau-Collocation method requires that the nonlinear integro-differential Eq.(4) be expressed in the matrix representation of the problem.

3.1 *Matrix representation of ordinary differential part*

THEOREM 3.1 *Let $u_N(x) = \sum_{i=0}^{N} a_i L_i(x) = \underline{\boldsymbol{a}}\ \underline{\boldsymbol{L}}\boldsymbol{X}$. The i^{th} derivative of $u_N(x)$ with respect to t can be written as*

$$\frac{d^k}{dx^k} u_N(x) = \underline{\boldsymbol{a}}\ \underline{\boldsymbol{L}}\ \ddot{\Pi}_k(x) \tag{5}$$

where $\ddot{\Pi}_k(x)$ is a column vector and

$$\text{the } \ r-th \text{ element of } \ \ddot{\Pi}_k(x) = \begin{cases} 0 & , \quad for\ r = 1, ..., k; \\ \frac{(r-1)!}{(r-1-k)!}x^{r-1-k} & , for\ r = k+1, ..., N+1. \end{cases}$$

Proof .It is proved directly by induction. ■

3.2 *Matrix representation for the integration term*

Let the nonlinear analytic function $G(t, u(t))$ defined on $[0, x] \times \mathbb{R}$, be approximated as:

$$G(x, u_N(x)) = \sum_{m=0}^{k} \gamma_m(x) u_N^m(x) \tag{6}$$

For use of the Tau-Collocation method, $u_N^m(x)$ must be written as the product of a matrix and a vector. The following result is concerned with approximation of the nonlinear function.

LEMMA 3.2 *(See [9]) Let $u_N(x) = \sum_{i=0}^{N} a_i L_i(x) = \underline{\boldsymbol{a}}\ \underline{\boldsymbol{L}}\boldsymbol{X}$ be the orthogonal series expansion of $u_N(x)$, where $\boldsymbol{X} = \Big(1, x, x^2, \cdots\Big)^t$, $\underline{\boldsymbol{a}} = \Big(a_0, a_1, a_2, \cdots, a_N, 0, 0, ...\Big)$ be infinite vectors and $\underline{\boldsymbol{L}}$ is a non-singular lower triangular matrix. Then*

$$\underline{\boldsymbol{a}}\ \underline{\boldsymbol{L}}\boldsymbol{X}\ \underline{\boldsymbol{a}}\ \underline{\boldsymbol{L}}\boldsymbol{X} = \underline{\boldsymbol{a}}\boldsymbol{B}(x)\underline{\boldsymbol{L}}\boldsymbol{X} \tag{7}$$

where $\boldsymbol{B}$ is an diametrical matrix as:

$$\boldsymbol{B}_{ij}(x) = \begin{cases} \sum_{i=0}^{N} a_i L_i(x), \text{ if } i=j, & i, j=1, 2, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

and for any positive integer m, the relation

$$u_N^m(x) = \underline{\boldsymbol{a}}(\boldsymbol{B}(x))^{m-1}\underline{\boldsymbol{L}}\boldsymbol{X} \tag{8}$$

is valid.

Now, we present the matrix representation of the integration term for a class of integro-differential equations. Also, we show that $\int_0^x K(x,t)u_N^m(t)$ can be represented as the product of a matrix and vector.

Theorem 3.3 *(See [9])Let $(x_\ell)_{\ell=0}^N$ be the set of the $(N+1)$ Gauss or Gauss-Radau, or Gauss-Lobatto points of the shifted Legendre polynomials in [0, 1] and $(\omega_\kappa)_{\kappa=0}^N$ be the corresponding quadrature weights. Assume that the approximated solution $u_N(x)$ is given by Eq.(3) and K is a bivariate given continuous function, then*

$$\int_0^{x_\ell} K(x_\ell, t)u_N^m(t)dt = \underline{\boldsymbol{a}}\Theta^{m-1}(x_\ell) \tag{9}$$

where

$$\Theta_{lj}^{m-1}(x_\ell) = x_\ell\Big(\sum_{\kappa=0}^{N} \widetilde{K}_l(x_\ell, s_\kappa x_\ell)(\sum_{i=0}^{N} a_i L_i(s_\kappa x_\ell))^{m-1}\omega_\kappa\Big) \quad j=1, \ l=1,2,..., \ m=1,2,...,k.$$

Remark 1 Clearly, for $\int_0^1 K(x,t)u_N^m(t)$, we don't need to use the change of variable, and by Gauss quadrature formulas, we obtain

$$\int_0^1 K(x,t)u_N^m(t) = \underline{\mathbf{a}}\ddot{\Theta}^{m-1}(x_\ell) \quad , \quad m=1,2,...,k \quad , \quad \ell=0,1,...,N.$$

where

$$\ddot{\Theta}_{lj}^{m-1}(x_\ell) = \sum_{\kappa=0}^{N}\Big(\breve{K}_l(x_\ell, s_\kappa)(\sum_{i=0}^{N} a_i L_i(s_\kappa))^{m-1}\omega_\kappa\Big) \quad j=1, \quad l=1,2,....$$

3.3 Matrix representation for supplementary conditions

We introduce the vectors $\underline{\ell} = (\ell_1, ..., \ell_\vartheta, 0, ...)$ and $\underline{\gamma} = (\gamma_1, ..., \gamma_{m-1}, 0, ...)$ and the matrix $\widehat{B} = [\widehat{b}_{ij}]$ such that $\widehat{b}_{ij} = \ell_j(L_i(x))_{i=0}^N$ for $j=1,...,\vartheta$.
We have $u_N(x) = \sum_{i=0}^N a_i L_i(x)$, then the supplementary conditions take the form:

$$\begin{aligned}\underline{\ell}u_N(x) &= (\ell_1 u_N(x), ..., \ell_\vartheta u_N(x), 0, ...) = (\ell_1\sum_{i=0}^{N} a_i L_i(x), ..., \ell_\vartheta\sum_{i=0}^{N} a_i L_i(x), 0, ...) \\ &= \underline{\mathbf{a}}(\ell_1(L_i(x))_{i=0}^N, ..., \ell_\nu(L_i(x))_{i=0}^N, 0, ...) = \underline{\mathbf{a}}(\widehat{b}_1, ..., \widehat{b}_\vartheta, 0, ...).\end{aligned}$$

We conclude that

$$\underline{\ell}u_N = \underline{\mathbf{a}}\widehat{B} = \underline{\gamma}. \tag{10}$$

3.4 *Tau-Collocation approximation of the Population equation*

Here we apply the previous results for constructing the Tau-Collocation approximate solution of the Population equation

$$\begin{cases} \sum_{k=0}^{m} p_k(x)\frac{\mathrm{d}^k}{\mathrm{d}x^k}(u(x)) + \lambda_1 \int_0^1 K_1(x,t)G_1(t,u(t))dt + \lambda_2 \int_0^x K_2(x,t)G_2(t,u(t))dt = g(x) + H_N(x), \\ (\ell_j, u(x)) = \gamma_j \qquad j = 1, ..., \vartheta., \end{cases} \tag{11}$$

Without loss of generality, we can assume that in Eq.(11), the nonlinear analytic function may be expanded as

$$G_1(t,u(t)) = \sum_{m_1=0}^{k} \gamma_{m_1}(t)u^{m_1}(t) \ , \qquad G_2(t,u(t)) = \sum_{m_2=0}^{k} \mu_{m_2}(t)u^{m_2}(t) \qquad k \leqslant n$$

By considering $\widehat{K}_{m_1}(x,t) = K_1(t,x)\gamma_{m_1}(t)$ and $\overline{K}_{m_2}(x,t) = K_2(t,x)\mu_{m_2}(t)$, Eq.(11), is as follows:

$$\sum_{k=0}^{m} p_k(x)\frac{\mathrm{d}^k}{\mathrm{d}x^k}(u_N(x)) + \lambda_1 \sum_{m_1=0}^{k} \int_0^x \widehat{K}_{m_1}(x,t)u_N^{m_1}(t)dt + \lambda_2 \sum_{m_2=0}^{k} \int_0^1 \overline{K}_{m_2}(x,t)u_N^{m_2}(t)dt = g(x)$$

Following Theorem 3.1, Theorem 3.3, Remark *1* and Eq.(4) in collocation point $(x_\ell)_{\ell=0}^N$, Eq.(12) is as follows

$$\underline{\mathbf{a}}\Big(\sum_{k=0}^{m} p_k(x_\ell)\ \underline{\mathbf{L}}\ \ddot{\Pi}_k(x_\ell) + \lambda_1 \sum_{m_1=0}^{k} \Theta^{m_1-1}(x_\ell) + \lambda_2 \sum_{m_2=0}^{k} \ddot{\Theta}^{m_2-1}(x_\ell)\Big) = g(x_\ell) (12)$$

We denote

$$\xi_n(x_\ell) = \sum_{k=0}^{m} p_k(x_\ell)\ \underline{\mathbf{L}}\ \ddot{\Pi}_k(x_\ell) + \lambda_1 \sum_{m_1=0}^{k} \Theta^{m_1-1}(x_\ell) + \lambda_2 \sum_{m_2=0}^{k} \ddot{\Theta}^{m_2-1}(x_\ell)$$

Then the formula (12) can be written in compact form:

$$\underline{\mathbf{a}}\xi_n(x_\ell) = f(x_\ell) \tag{13}$$

Therefore, to find the coefficient $\underline{\mathbf{a}} = (a_0, a_1, ..., a_n, 0, ...)$, by Combining the system (10) and (13) we have the following nonlinear system of algebraic equations:

$$\underline{\mathbf{a}}\mathbb{G}_N = \mathbb{R}_N \tag{14}$$

$$\mathbb{G}_N = [\widehat{B}|\ddot{\Pi}] \qquad , \qquad \mathbb{R} = (\underline{\gamma}, F_{N-\vartheta+1}).$$

Solving this linear system of algebraic equations by any method provides $textbf\underline{a} = (a_0, a_1, ..., a_N, 0, \cdots)$.

4. Numerical examples

In this section, first we present an example to illustrate the procedure of the Tau-Collocation method. Moreover, we evaluate the numerical solution of the problem (2) to show the efficiency of the present method in comparison with other methods by solving other examples.The computations associated with the examples are performed using Matlab.

Example 4.1 (see[32]) Consider following integro-differential equation

$$u''(x) + xu'(x) - xu(x) = 8x^2 - \frac{25}{3}x + 7x^3 - 3x^4 + 4 + \int_{-1}^{1} xu(t)dt \tag{15}$$

with conditions

$$u(0) = 7 \quad , \quad u'(0) = -4$$

By applying the technique described in preceding section with $N = 3$ and by using Chebyshev zeros for collocation point ($x_1 = -0.7071$, $x_2 = 0.7071$), we approximate solution as

$$\underline{\Gamma}(x) = \begin{pmatrix} -x \\ -x^2 + x \\ -x^3 + 2x^2 + 2 \\ -x^4 + 3x^3 + 6x \end{pmatrix} \quad , \quad \ddot{\Pi}(x) = \begin{pmatrix} 2x \\ 0 \\ \frac{2x}{3} \\ 0 \end{pmatrix} \quad , \quad \xi_3(x) = \begin{pmatrix} -x - 2x \\ -x^2 + x \\ -x^3 + 2x^2 + 2 - \frac{2x}{3} \\ -x^4 + 3x^3 + 6x \end{pmatrix}$$

$$\widehat{\Pi} = [\xi_3(x_0)|\xi_3(x_1)] = \begin{pmatrix} 2.1213 & -2.1213 \\ -1.2071 & 0.2071 \\ 5.5286 & 6.4714 \\ -18.5919 & 19.5919 \end{pmatrix}, \underline{F}_2 = \begin{pmatrix} 2.9718 & 5.4761 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -3 \end{pmatrix}$$

Then we have

$$A = \begin{pmatrix} 1 & 0 & 2.1213 & -2.1213 \\ 0 & 1 & -1.2071 & 0.2071 \\ -1 & 0 & 5.5286 & 6.4714 \\ 0 & -3 & -18.5919 & 19.5919 \end{pmatrix} \quad , \quad R = \begin{pmatrix} 7.0000 & -4.0000 & 2.9718 & 5.4761 \end{pmatrix}$$

By solving system $(a_0, a_1, a_2, a_3)A = R$, we get $a_0 = 7.0000$, $a_1 = -4.0000$, $a_2 = 2.0000$, $a_3 = 3.0000$ and the Tau-collocation approximation is $u_3(x) = 3x^3 + 2x^2 - 4x + 7$, which is the exact solution.

Example 4.2 Let us consider the Fredholm integro-differential equation

$$x^2u'(x) + e^x u(x) + \int_{-1}^{1} e^{(x+1)t}u(t)dt = (x^2 + e^x)e^x + \frac{2sinh(x+2)}{(x+2)} \qquad -1 \leqslant x \leqslant 1$$

with $u(-1) = \frac{1}{e}$.The exact solution of this problem is $u(x) = e^x$.In Table 1, we compare the absolute errors obtained by the present method and the Tau method [11].

Example 4.3 Let us consider the following nonlinear integro-differential equation

$$u(x) = x^3 + \frac{cos(1) - 1}{3} + \int_{0}^{1} t^2 sin(u(t))dt \qquad 0 \leqslant x \leqslant 1$$

Table 1. Numerical results for Example 4.2.

x	N=10		N=15	
	Tau-collocation	Tau	Tau-collocation	Tau
-1.00	3.13353×10^{-10}	9.57838×10^{-4}	5.55111×10^{-15}	3.11849×10^{-6}
-0.75	1.33032×10^{-10}	8.90142×10^{-5}	4.32986×10^{-15}	2.03418×10^{-7}
-0.25	2.55736×10^{-11}	1.18903×10^{-5}	7.77156×10^{-16}	1.42467×10^{-7}
0.25	1.84971×10^{-9}	1.09536×10^{-4}	6.66133×10^{-16}	1.42467×10^{-7}
0.75	1.85911×10^{-10}	3.07668×10^{-4}	4.44089×10^{-16}	6.42180×10^{-7}
1.00	2.52126×10^{-10}	9.57838×10^{-4}	1.77635×10^{-15}	3.11849×10^{-6}

The exact solution of this problem is $u(x) = x^3$. The absolute errors for mentioned method is compared by the Tau method [24] in Table 2.

Table 2. Numerical results for Example 4.3.

x	N=10		N=15	
	Tau-collocation	Tau	Tau-collocation	Tau
0.00	3.40475×10^{-7}	3.4×10^{-3}	3.83574×10^{-8}	1.3×10^{-4}
0.20	3.40475×10^{-7}	3.4×10^{-3}	3.83574×10^{-8}	1.3×10^{-4}
0.40	3.40475×10^{-7}	3.4×10^{-3}	3.83574×10^{-8}	1.3×10^{-4}
0.80	3.40475×10^{-7}	3.4×10^{-3}	3.83574×10^{-8}	1.3×10^{-4}
1.00	3.40475×10^{-7}	3.4×10^{-3}	3.83574×10^{-8}	1.3×10^{-4}

In rest of the examples, we are going to compare Tau-collocation method with recently research such as Bernsteins approximation, meshless method and sinc-collocation method. It is shown that the Tau-collocation method yields comparable or better results.

Example 4.4 (see[18]) Consider the linear Volterra integral equation

$$u(x) = cos(x) - e^x sin(x) + \int_0^x e^x u(t)dt \qquad , \qquad 0 \leqslant x \leqslant 1 \tag{16}$$

where the exact solution is $u(x) = \cos(x)$. The computational results have been reported in Table 3.
The maximum absolute error of Bernsteins approximation in [18] obtained $\|E_{10}\| = \max_{0\leqslant x\leqslant 1} |E_{10}(x)| = 3.340973 \times 10^{-10}$.

Table 3. Numerical results for Example 4.4.

x	N=10	N=15
	Tau-collocation	Tau-collocation
0.00	0	0
0.20	4.36839×10^{-12}	1.11022×10^{-16}
0.40	9.58733×10^{-12}	2.22044×10^{-16}
0.60	1.04368×10^{-11}	4.44089×10^{-16}
0.80	5.15421×10^{-12}	9.99200×10^{-16}
1.00	1.06668×10^{-11}	3.10862×10^{-15}

5. Application of the population model

In this section, we will apply the Tau-Collocation method for the population model.

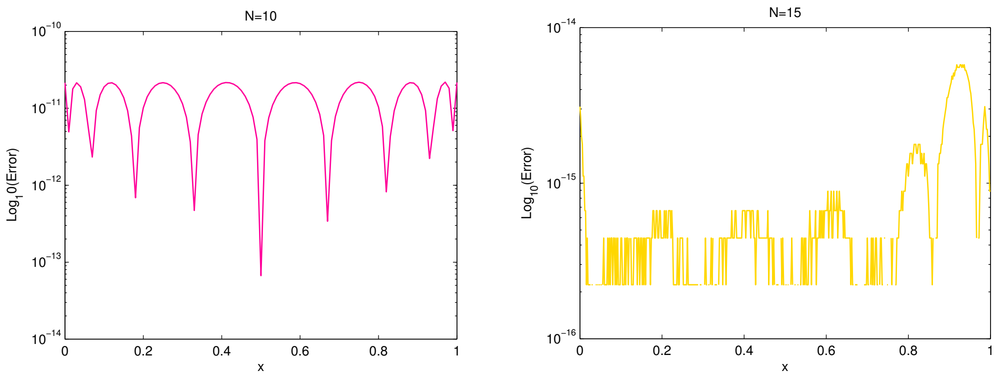

Figure 1. Graph of the Tau-Collocation approximation error of Example 5.1 in Legendre-Gauss point.

Example 5.1 Let us consider the number of female births for $g(x) = e^x$, $K(x,t) = x - t$ and $x \in [0,1]$. Then, Eq.(1) converts to the problem

$$u(x) = e^x + \int_0^x (x-t)u(t)\mathtt{d}t. \tag{17}$$

The exact solution of this problem is $u(x) = \frac{1}{2}\Big[e^x + \cos(x) + \sin(x)\Big]$.

It is seen from Fig. 1 and Table 4 that the accuracy of the solutions increases as N is increased.

Table 4. Numerical results for Example 5.1.

x	N=10 Tau-collocation	N=15 Tau-collocation
0.00	2.14435×10^{-11}	3.10862×10^{-15}
0.20	1.01314×10^{-11}	4.44089×10^{-16}
0.40	2.09956×10^{-11}	6.66133×10^{-16}
0.60	2.09248×10^{-11}	6.66133×10^{-16}
0.80	1.03508×10^{-11}	1.33226×10^{-15}
1.00	2.16515×10^{-11}	8.88178×10^{-16}

Example 5.2 (From [11],[12])Consider the following population model

$$u'(x) + u(x) - \int_0^x x(1+2x)e^{t(x-t)}u(t)dt = 1 + 2x \qquad , \qquad 0 \leqslant x \leqslant 1 \tag{18}$$

with $u(0) = 1$, whose exact solution is given by $u(x) = e^{x^2}$.

The computational results for various N, with Legendre bases and in Legendre–Gauss point have been reported in Table 5.

Example 5.3 (see[8]) Consider the following nonlinear population model

$$u'(x) = 1 - \frac{x}{2} + \frac{xe^{-x^2}}{2} + \int_0^x txe^{-u^2(t)}dt \qquad\qquad 0 \leqslant x \leqslant 1$$

with $u(0) = 0$. The exact solution of this problem is $u(x) = x$.
The maximum absolute error of meshless method in [8] obtained $\|E_{129}\| = \max_{0\leqslant x\leqslant 1}|E_{129}(x)| = 1.26 \times 10^{-6}$.

Table 5. Numerical results for Example 5.2.

x	N=10		N=15	
	Tau-collocation	Tau	Tau-collocation	Tau
0.00	0	0	0	0
0.20	1.38457×10^{-7}	$5.72156E \times 10^{-12}$	7.32525×10^{-13}	$1.63300E \times 10^{-16}$
0.40	8.99195×10^{-8}	$2.38451E \times 10^{-8}$	5.76205×10^{-13}	$1.08446E \times 10^{-11}$
0.60	4.20218×10^{-8}	$3.18608E \times 10^{-6}$	1.40332×10^{-12}	$7.28709E \times 10^{-9}$
0.80	1.44747×10^{-7}	$1.04921E \times 10^{-4}$	1.49502×10^{-12}	$7.51118E \times 10^{-7}$
1.00	6.74720×10^{-6}	$1.61516E \times 10^{-3}$	4.33431×10^{-13}	$2.78602E \times 10^{-5}$

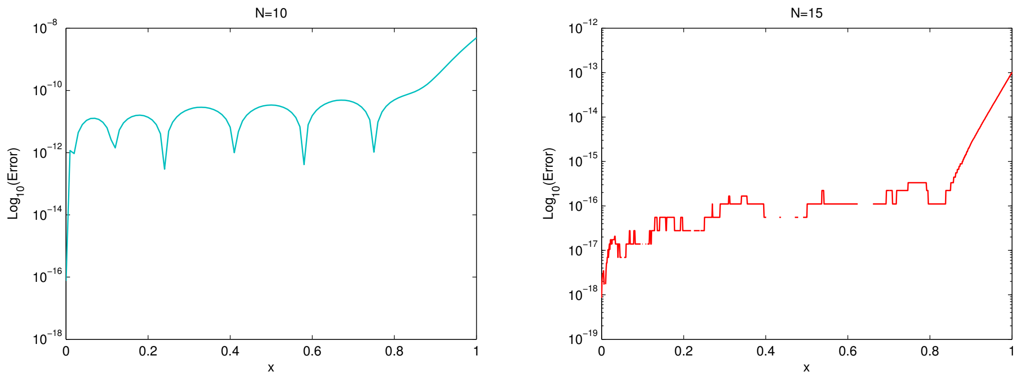

Figure 2. Graph of the Tau-Collocation approximation error of Example 5.3.

Numerical results in Table 6 and Figure 2 show the robustness of the method. We observe that the extended Tau-Collocation approximation is an extremely good approximation to solution of the population and is much better in fact than the other existing methods.

Table 6. Numerical results for Example 5.3.

x	N=10	N=15
	Tau-collocation	Tau-collocation
0.00	0	0
0.20	1.37070×10^{-11}	2.77555×10^{-17}
0.40	6.58101×10^{-12}	5.55111×10^{-17}
0.60	1.54539×10^{-11}	1.11022×10^{-16}
0.80	4.95783×10^{-11}	2.22044×10^{-16}
1.00	4.94000×10^{-9}	1.00808×10^{-13}

Example 5.4 (See [21]) Consider the following population model

$$u(x) = x - x^2 - \frac{x^5}{4} + \frac{2x^6}{5} - \frac{x^7}{6} + \int_0^x xtu^2(t)dt \qquad x \in [0, 1]$$

whose the exact solution is $u(x) = x - x^2$.
The maximum absolute error of sinc-collocation method in [21] obtained $\|E_{10}\| = \max_{0\leqslant x\leqslant 1} |E_{10}(x)| = 2.56270 \times 10^{-6}$.

Table 7. Numerical results for Example 5.4.

x	N=10 Tau-collocation	N=15 Tau-collocation
0.00	1.45831×10^{-13}	7.82477×10^{-14}
0.20	4.00163×10^{-9}	1.35558×10^{-13}
0.40	3.21796×10^{-8}	2.08444×10^{-10}
0.60	1.10285×10^{-7}	2.08143×10^{-8}
0.80	2.67241×10^{-6}	5.31782×10^{-7}
1.00	5.24484×10^{-6}	6.41715×10^{-7}

6. Conclusion

In this paper, we have successfully applied the Tau-Collocation method to solve the population model. Our result have shown a good agreement and high accuracy for the solution of mentioned problems. Moreover any nonlinear integro-differential equation can be solved by the present method. One of the main advantages of the proposed approach is we don't need interpolation polynomial for determining the non-polynomial terms of equation.

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