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On a Modification of the Chebyshev Collocation Method for Solving Fractional Diffusion Equation

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Abstract. In this article a modification of the Chebyshev collocation method is applied to the solution of space fractional differential equations.The fractional derivative is considered in the Caputo sense.The finite difference scheme and Chebyshev collocation method are used .The numerical results obtained by this way have been compared with other methods.The results show the reliability and efficiency of the proposed method.

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Index to information contained in this paper

1. Introduction

The fractional partial differential equations (FPDEs) arise in numerous problems of engineering, physics, mathematics, and chemistry, biology, viscoelasticity [1,2,3,4]. Most fractional differential equations do not have exact analytical solutions, thus many authors are seeking ways to numerically solve these problems [5,6].

Recently, some different numerical methods to solve fractional differential equations have been given such as variational iteration method [7], homotopy perturbation method[8], adomian decomposition method [9], homotopy analysis method [10],

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and collocation method [11]. In [12] Fix developed a least square finite element solution of a fractional-order two-point boundary value problems. Darzi and et al. proposed Sumudu transform method for solving fractional differential equations and fractional diffusion-wave equation as well [13]. Neamaty and et al.[31] used wavelet operational method for solving fractional partial differential equations. Liu and et al.[14,28] suggested method of lines to transform the space fractional Fokker-Planck equation into a system of ordinary differential equations.The space fractional diffusion equations are solved numerically by many authors.Khader proposed Chebyshev collocation method to discretize space fractional diffusion equations to obtain a linear system of ordinary differential equations and he solved the resulting system by finite difference method [15]. Dehghan [16] applied Tau approach to solve space fractional diffusion equations.

2. Basic Ideas and Definitions

Definition 2.1. The Caputo fractional derivative operator ${}_0D_x^\alpha$ of order α is defined in the following form:

$${}_0D_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0,$$

where $m-1 < \alpha \leqslant m,\ m \in N,\ x > 0.$

Caputo fractional derivative operator is a linear operation and for the Caputos derivative we have [18]:

$$D^\alpha c = 0, \tag{1}$$

$$D^\alpha x^n = \begin{cases} 0, & n \in N_0 \ \ and \ \ n < \lceil \alpha \rceil, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & n \in N_0 \ \ and \ \ n \geqslant \lceil \alpha \rceil, \end{cases} \tag{2}$$

where c is a constant and $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α and $N_0 = \{1, 2, ...\}$. For $\alpha \in N_0$, the Caputo differential operator coincides with the usual differential of integer order ([17,18,26]).

In this article, we propose a modified approach to obtain the solution of space fractional diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = d(x,t)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + s(x,t), \quad a < x < b, \ \ 0 \leqslant t \leqslant T, \ \ 1 < \alpha \leqslant 2, \tag{3}$$

with initial condition

$$u(x,0) = u^0(x), \quad a < x < b, \tag{4}$$

and boundary conditions

$$u(a,t) = u(b,t) = 0, \tag{5}$$

where the function s(x,t) is a source term. Also we solve numerically the fractional Riccati differential equation

$$D^{\alpha}u(t) + u^2(t) - 1 = 0, \;\; t > 0, 0 < \alpha \leqslant 1,$$

with the initial condition $u(0) = u^0$. Note that for $\alpha = 2$, Eq.(3) is the classical diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = d(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + s(x,t). \tag{6}$$

We use the Chebyshev collocation method to discretize (3) and to get a linear system of ordinary differential equations [15] and use the finite difference method (FDM) [20-22] to solve the resulting system, and obtain the coefficients in the approximate solution. By determining the coefficients ,the associated approximate polynomial will be as:

$$u_m(x,t_n) = \sum_{i=0}^{m} u_i(t_n)T_i^*(x) = \acute{a}_{o,n}T_0^*(x) + \acute{a}_{1,n}T_1^*(x) + \acute{a}_{2,n}T_2^*(x) + ... + \acute{a}_{m,n}T_m^*(x) \tag{7}$$

$$= a_{0,n} + a_{1,n}x + a_{2,n}x^2 + ... + a_{m,n}x^m,$$

which is of degree m,and $T_i^*(x)$ are shifted Chebyshev polynomials. Now let us change one of the coefficients say $a_{i,n}$, as $\delta a_{i,n}$ and determine δ in such a way that the absolute value of the difference between the exact and approximate solutions be less than or equal to a given $\varepsilon > 0$,. Now the new obtained approximate polynomial is appeared as:

$$u_m(x,t_n) = \sum_{i=0}^{m} u_i(t_n)T_i^*(x) = a_{0,n}+a_{1,n}x+a_{2,n}x^2+...+\delta a_{i,n}x^i+...+a_{m,n}x^m, \tag{8}$$

which approximates the exact solution much better. Such a fact will be demonstrated in solved examples later, by comparing to the other methods.

3. A Review of the Chebyshev Polynomials

The well known Chebyshev polynomials are defined on the interval [-1, 1] as: [29];
$T_0(z) = 1$,
$T_1(z) = z$,
$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z), \;\; n = 1, 2, ...$.

The analytic form of the Chebyshev polynomials $T_n(z)$ of degree n is given by the following:

$$T_n(z) = n\sum_{i=0}^{[\frac{n}{2}]}(-1)^i 2^{n-2i-1}\frac{(n-i-1)!}{(i)!(n-2i)!}z^{n-2i}, \tag{9}$$

where $[\frac{n}{2}]$ denotes the integer part of n/2. The orthogonality condition is

$$\int_{-1}^{1}\frac{T_i(z)T_j(z)}{\sqrt{1-z^2}}dz = \begin{cases}\pi, \ for\ i=j=0,\\ \frac{\pi}{2}, \ for\ i=j\neq 0,\\ 0, \ for\ i\neq j.\end{cases}$$

In order to use these polynomials on the interval $x \in [0,1]$, we define the so called shifted Chebyshev polynomials by introducing the change of variable z=2x-1.We denote $T_n(2x-1)$ by $T_n^*(x)$, defined as:

$$T_n^*(x) = n\sum_{k=0}^{n}(-1)^{n-k}\frac{2^{2k}(n+k-1)!}{(2k)!(n-k)!}x^k, \ \ n=2,3,\dots, \tag{10}$$

where $T_0^*(x)=1$ and $T_1^*(x)=2x-1$.
A function y(x), which is squared integrable in [0, 1], may be expressed in terms of shifted Chebyshev polynomials as:

$$y(x) = \sum_{i=0}^{\infty} c_i T_i^*(x),$$

where

$$c_0 = \frac{1}{\pi}\int_0^1 \frac{y(t)T_0^*(x)}{\sqrt{x-x^2}}dx, \ c_i = \frac{2}{\pi}\int_0^1 \frac{y(t)T_i^*(x)}{\sqrt{x-x^2}}dx, \ \ i=1,2,\dots. \tag{11}$$

Theorem 3.1. [30] Let y(x) be approximated by shifted Chebyshev polynomials as:

$$y_m(x) = \sum_{i=0}^{m} c_i T_i^*(x), \tag{12}$$

and $\alpha > 0$, then

$$D^\alpha(y_m(x)) = \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i w_{i,k}^{(\alpha)} x^{k-\alpha}, \tag{13}$$

where $w_{i,k}^{(\alpha)}$ is given by:

$$w_{i,k}^{(\alpha)} = (-1)^{i-k} \frac{2^{2k} i(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1-\alpha)}. \tag{14}$$

4. Error Analysis

This section is concerned with the studying of the convergence analysis and getting an upper bound for the error of the proposed formula.
Theorem 4.1. (Chebyshev truncation theorem) [29].
The error in approximating y(x) by the sum of its first m terms is bounded by the sum of the absolute values of all the coefficients. If

$$y_m(x) = \sum_{k=0}^{m} c_k T_k(x), \tag{15}$$

then

$$E_T(m) \equiv |y(x) - y_m(x)| \leqslant \sum_{k=m+1}^{\infty} |c_k|, \tag{16}$$

for all y(x), all m, and all $x \in [-1, 1]$.

Theorem 4.2. [30] The error $|E_T(m)| = |D^\alpha y(x) - D^\alpha y_m(x)|$ in approximating$D^\alpha y(x)$ by $D^\alpha y_m(x)$ is bounded as:

$$|E_T(m)| \leqslant |\sum_{i=m+1}^{\infty} c_i (\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \theta_{i,j,k})|, \tag{17}$$

where
$\theta_{i,j,k} = \frac{(-1)^{i-k} 2i(i+k-1)!\Gamma(k-\alpha+\frac{1}{2})}{h_j \Gamma(k+\frac{1}{2})(i-k)!\Gamma(k-\alpha-j+1)\Gamma(k+j-\alpha+1)}, \quad j = 1, 2, \ldots .$

5. The Process of Solving the Space Fractional Diffusion Equation

In order to use Chebyshev collocation method in Eq.(3), we approximate u(x) as:

$$u_m(x,t) = \sum_{i=0}^{m} u_i(t) T_i^*(x). \tag{18}$$

From Eqs. (3), (12) and Theorem 3.1 we have:

$$\sum_{i=0}^{m} \frac{du_i(t)}{dt} T_i^*(x) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} u_i(t) w_{i,k}^{(\alpha)} x^{k-\alpha} + s(x,t). \tag{19}$$

Collocating, Eq. (19) at $(m+1-\lceil\alpha\rceil)$ points x_p yields:

$$\sum_{i=0}^{m} \frac{du_i(t)}{dt} T_i^*(x_p) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} u_i(t) w_{i,k}^{(\alpha)} x_p^{k-\alpha} + s(x_p,t), \quad P = 0,1,...,m-\lceil\alpha\rceil. \tag{20}$$

Now we use of roots of shifted Chebyshev Polynomials $T^*_{m+1-\lceil\alpha\rceil}(x)$ as suitable collocation points.
By substituting Eqs.(13) and (18) in the boundary conditions (5) we get

$$\sum_{i=0}^{m} (-1)^i u_i(t) = 0, \qquad \sum_{i=0}^{m} u_i(t) = 0. \tag{21}$$

If so, $\lceil\alpha\rceil$ equations obtained from (21), along with m+1-$\lceil\alpha\rceil$ equations obtained from (20) give (m+1) ordinary differential equations which may be solved by using FDM, to get the m unknown u_i, i=0,1,...,m, in various time levels t_n. by determining the unknowns $u_i(t_n)$, the approximate m degree polynomials as obtained as follows:

$$u_m(x,t_n) = \sum_{i=0}^{m} u_i(t_n) T_i^*(x) = \acute{a}_{o,n} T_0^*(x) + \acute{a}_{1,n} T_1^*(x) + \acute{a}_{2,n} T_2^*(x) + ... + \acute{a}_{m,n} T_m^*(x)$$

$$= a_{0,n} + a_{1,n} x^1 + a_{2,n} x^2 + ... + a_{m,n} x^m,$$

i=0,1,...,N, $\triangle t = \frac{T}{N}$, $0 \leqslant t_i \leqslant T$, $t_i = i\triangle t$,

in which T is the final time and $u_i^n = u_i(t_n)$. Now we change one of a coefficients say $a_{i,n}$, as $\delta a_{i,n}$, and obtain δ by using $|u_{ex} - u_{approx}| \leqslant \varepsilon$ where, u_{ex} and u_{approx} are respectively the exact and approximate solutions.The new obtained approximate polynomial is appeared as:

$$u_m(x,t_n) = \sum_{i=0}^{m} u_i(t_n) T_i^*(x) = a_{0,n} + a_{1,n} x + a_{2,n} x^2 + ... + \delta a_{i,n} x^i + ... + a_{m,n} x^m.$$

6. Numerical Results

Example 6.1. In this section, we consider space fractional diffusion equation (3) with $\alpha = 1.8$, of the form:

$$\frac{\partial u(x,t)}{\partial t} = d(x,t)\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + s(x,t),$$

where, $0 < x < 1$, with the diffusion coefficient: d(x,t)= $\Gamma(1.2)x^{1.8}$, and the source function: s(x,t)=$3x^2(2x-1)e^{-t}$. The initial and boundary conditions are respectively as:
u(x,0)=$x^2(1-x)$,
u(0,t)=u(1,t)=0.
The exact solution of this problem is u(x,t)= $x^2(1-x)e^{-t}$.

We apply the present method with m=3, and approximate the solution as follows:

$$u_3(x,t) = \sum_{i=0}^{3} u_i(t)T_i^*(x). \tag{22}$$

Using Eq. (20) we have:

$$\sum_{i=0}^{3} \frac{du_i(t)}{dt} T_i^*(x_p) = d(x_p,t)\sum_{i=2}^{3}\sum_{k=2}^{i} u_i(t)w_{i,k}^{(1.8)} x_p^{k-1.8} + s(x_p,t), \;\; p = 0,1, \tag{23}$$

note that the $x_p^{,}s$ are roots of shifted Chebyshev polynomial $T_2^*(x)$, i.e.

$$x_0 = 0.146447, \quad x_1 = 0.887298.$$

By using Eqs. (21) and (23), the following system of ordinary differential equations is obtained :

$$\dot{u}_0(t) + k_1\dot{u}_0(t) + k_2\dot{u}_0(t) = R_1u_2(t) + R_2u_3(t) + s_0(t), \tag{24}$$

$$\dot{u}_0(t) + k_{11}\dot{u}_0(t) + k_{22}\dot{u}_0(t) = R_{11}u_2(t) + R_{22}u_3(t) + s_1(t), \tag{25}$$

$$u_0(t) - u_1(t) + u_2(t) - u_3(t) = 0, \tag{26}$$

$$u_0(t) + u_1(t) + u_2(t) + u_3(t) = 0, \tag{27}$$

where:
$k_1 = T_1^*(x_0), k_2 = T_3^*(x_0), k_{11} = T_1^*(x_1), k_{22} = T_3^*(x_1)$,

$R_1 = d(x_0,t)w_{2,2}^{(1.8)}x_0^{0.2}, R_2 = d(x_0,t)[w_{3,2}^{(1.8)}x_0^{0.2} + w_{3,3}^{(1.8)}x_0^{1.2}]$,

$R_{11} = d(x_1,t)w_{2,2}^{(1.8)}x_1^{0.2}, R_{22} = d(x_1,t)[w_{3,2}^{(1.8)}x_1^{0.2} + w_{3,3}^{(1.8)}x_1^{1.2}]$.

Now, the system (24)-(27) is solved, by using finite difference method (FDM). In this example, we consider $t_i = i\Delta t, 0 \leqslant t_i \leqslant T, \Delta t = \tau = \frac{T}{N}$, for i=0,1,...,N, and $u_i^n = u_i(t_n), s_i^n{=}s_i(t_n)$. If so, the system (24)-(27), is discretized in time and takes the following form:

$$\frac{u_0^n - u_0^{n-1}}{\triangle t} + k_1\frac{u_1^n - u_1^{n-1}}{\triangle t} + k_2\frac{u_3^n - u_3^{n-1}}{\triangle t} = R_1u_2^n + R_2u_3^n + S_0^n, \tag{28}$$

$$\frac{u_0^n - u_0^{n-1}}{\triangle t} + k_{11}\frac{u_1^n - u_1^{n-1}}{\triangle t} + k_{22}\frac{u_3^n - u_3^{n-1}}{\triangle t} = R_{11}u_2^n + R_{22}u_3^n + S_1^n, \tag{29}$$

$$u_o^n - u_1^n + u_2^n - u_3^n = 0, \tag{30}$$

$$u_o^n + u_1^n + u_2^n + u_3^n = 0. \tag{31}$$

The above system (28)-(31) can be written in the following matrix form:

$$\begin{pmatrix} 1 & k_1 & -\tau R_1 & k_2 - \tau R_2 \\ 1 & k_{11} & -\tau R_{11} & k_{22} - \tau R_{22} \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}^n = \begin{pmatrix} 1 & k_1 & 0 & k_2 \\ 1 & k_{11} & 0 & k_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}^{n-1} + \tau \begin{pmatrix} s_0 \\ s_1 \\ 0 \\ 0 \end{pmatrix}^n, \tag{32}$$

or,

$$AU^n = BU^{n-1} + \tau S^n, \;\; or, \;\; U^n = A^{-1}(BU^{n-1}) + A^{-1}(\tau S^n), \tag{33}$$

where:

$U^n = (u_0^n, u_1^n, u_2^n, u_3^n)$ and $S^n = (s_0^n, s_1^n, 0, 0)$.

Note that for n=1, the initial solution $U^0 = \{\frac{1}{16}, \frac{1}{32}, -\frac{1}{16}, -\frac{1}{32}\}$ is obtained from the initial condition of the problem, u(x,0) and using Eq.(11). We show the obtained numerical results by means of the proposed method with the initial solution U^0, in Tables 1 and 2 .In Table 1, the absolute error, between the exact solution u_{ex} and the approximate solution u_{approx} at m=3 and time step $\tau = 0.0025$, with the final time T=2 is given. Note that, the new approximating polynomial is considered as:

$$u_3(x,t_n) = \sum_{i=0}^{3} u_i(t_n)T_i^*(x) = a_{0,n} + \delta a_{1,n}x + a_{2,n}x^2 + a_{3,n}x^3, \tag{34}$$

where in (34), δ can be considered as: $\delta a_{i,n}x^i$, i=0,1,2,3. as well By using the inequality:

$$|u_{approx(\delta)} - u_{ex}| = |(a_{0,n} + \delta a_{1,n}x + a_{2,n}x^2 + a_{3,n}x^3) - (x^2(1-x)e^{-t})| \leqslant 10^{-10},$$

the values of δ are calculated for various points ,time and ε. In Table 2, by means of the modified method, values δ and $|u_{approx(\delta)} - u_{ex}|$, at m=3 and time step $\tau = 0.0025$, with the final time T=2, and $\varepsilon = 10^{-10}$ are given. note that $u_3(x, 2)$ considered as:

$$u_3(x, 2) = \sum_{i=0}^{3} u_i(t_{800})T_i^*(x) = a_{0,800} + a_{1,800}x + a_{2,800}x^2 + a_{3,800}x^3.$$

It is notable that by considering $\Delta t = 0.0025$, we will has 800 ($\frac{T}{\Delta t} = \frac{2}{0.0025} = 800$) level time for the solution of the system of equations (33),i.e. we get 800 approximate solutions $u(x, t_n)$, $0 < x < 1$.
In the above example all 800 values of $u^n = (u_0^n, u_1^n, u_2^n, u_3^n)$ are calculated by utilizing mathematica.

Table 1: Comparsion of absolute errors for u(x,2)at m=3from example 6.1.

x	present method	Method[12]	Method [20]	Method [13]
0.0	8.67362×10^{-19}	1.70849×10^{-4}	4.483787×10^{-3}	0.00
0.1	8.23560×10^{-5}	2.10940×10^{-5}	4.479660×10^{-3}	2.89×10^{-5}
0.2	1.49747×10^{-4}	1.76609×10^{-4}	4.201329×10^{-3}	1.09×10^{-4}
0.3	2.00921×10^{-4}	3.01420×10^{-4}	3.695172×10^{-3}	2.20×10^{-4}
0.4	2.34628×10^{-4}	4.04138×10^{-4}	3.007566×10^{-3}	3.40×10^{-4}
0.5	2.49617×10^{-4}	4.89044×10^{-4}	2.184889×10^{-3}	4.45×10^{-4}
0.6	2.44636×10^{-4}	4.89044×10^{-4}	1.273510×10^{-3}	5.15×10^{-4}
0.7	2.18435×10^{-4}	5.63305×10^{-4}	0.319831×10^{-3}	5.27×10^{-4}
0.8	1.69763×10^{-4}	6.33367×10^{-4}	0.629793×10^{-3}	4.60×10^{-4}
0.9	9.73680×10^{-5}	7.05677×10^{-4}	1.528978×10^{-3}	2.91×10^{-4}
1.0	2.60209×10^{-18}	8.82821×10^{-4}	2.331347×10^{-3}	0.00

Example 6.2. Consider the fractional Riccati differential equation of the form

$$D^{\alpha}u(t) + u^2(t) - 1 = 0,\ t > 0,\ 0 < \alpha \leqslant 1, \tag{35}$$

with the initial condition

$$u(0) = u^0, \tag{36}$$

and the parameter α, refers to the fractional order of the time derivative.
For $\alpha = 1$; the Eq.(35) is the standard Riccati differential equation

$$\frac{du(t)}{dt} + u^2(t) - 1 = 0.$$

Table 2: Absolute errors for u(x,2) with m=3 and $\varepsilon = 10^{-10}$ from example 6.1.

x	δ	$\lvert u_{approx(\delta)} - u_{ex}\rvert$
0.1	0.0790163028	9.99948×10^{-11}
0.2	0.1626942205	9.99998×10^{-11}
0.3	0.2510358034	9.99976×10^{-11}
0.4	0.3440407717	9.99864×10^{-11}
0.5	0.4417090696	9.99985×10^{-11}
0.6	0.5440406785	9.99750×10^{-11}
0.7	0.6510355900	9.97496×10^{-11}
0.8	0.7626938012	9.99788×10^{-11}
0.9	0.8790153088	9.99866×10^{-11}
1.0	1.0000001118	9.99734×10^{-11}

The exact solution to this equation is

$$u(t) = \frac{e^{2t}-1}{e^{2t}+1}.$$

Now we approximate the function u(t) by using formula (12) and its Caputo derivative $D^{\alpha}u(t)$ by using the presented formula (13) with m=5.Then fractional Riccati differential equation (35) is transformed to the following approximated form

$$\sum_{i=1}^{5}\sum_{k=1}^{i} c_i w_{i,k}^{(\alpha)} t^{k-\alpha} + (\sum_{i=0}^{5} c_i T_i^*(t))^2 - 1 = 0, \tag{37}$$

where $w_{i,k}^{(\alpha)}$ is defined in (14). Also the initial condition (36) is given by :

$$\sum_{i=0}^{5} c_i(T_i^*(0)) = u^0. \tag{38}$$

We now collocate Eq. (37) at $(m+1-\lceil\alpha\rceil)$ points t_p as:

$$\sum_{i=1}^{5}\sum_{k=1}^{i} c_i w_{i,k}^{(\alpha)} t_p^{k-\alpha} + (\sum_{i=0}^{5} c_i T_i^*(t_p))^2 - 1 = 0, \quad p = 0,1,2,3,4. \tag{39}$$

Note that $t_p's$ are roots of shifted Chebyshev polynomial $T_5^*(t)$, i.e.

$$t_0 = 0.5,\ t_1 = 0.206107,\ t_2 = 0.793893,\ t_3 = 0.024471,\ t_4 = 0.975528.$$

By using Eqs.(38) and (39), we obtain a system of non-linear algebraic equations which contains 6 equations for the unknowns $c_i, i = 0,1,...,5$.
By solving the previous system, utilizing the Newton iteration method, we obtain the unknown $c_i, i = 0,1,...,5$, and therefore, the approximate solution is obtained via:

$$u_5(t) = \sum_{i=0}^{5} c_i T_i^*(t).$$

In figure 1, comparison between the exact and approximate solutions using the introduced method with different values of α ($\alpha = 1; 0.9; o.8; o.75; 0.5$); $u_0 = 0$, is presented.

The obtained numerical results by of the proposed technique and the exact solution values of α are shown in the Table 3. Also in the Tables 4,5,6 the absolute error between the exact solution u_{ex} and the approximate solution u_{approx} with different values of α ($\alpha = 1; 0.9; o.8; o.75; 0.5$); and δ, by means of the proposed modified method are given.

Table 3: Numerical results with different values of α.and exact solution for Ex.6.2.

x	u_{ex}	$\alpha = 0.5$ $u_{approx(0.5)}$	$\alpha = 0.75$ $u_{approx(0.75)}$	$\alpha = 0.8$ $u_{approx(0.8)}$	$\alpha = 0.9$ $u_{approx(0.9)}$	$\alpha = 1.0$ $u_{approx(1.0)}$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.096668	0.337455	0.184532	0.163128	0.127437	0.099693
0.2	0.197375	0.456273	0.310062	0.284370	0.237669	0.197438
0.3	0.291313	0.497307	0.402443	0.339867	0.334655	0.291345
0.4	0.379949	0.531235	0.476923	0.459384	0.420799	0.379927
0.5	0.462117	0.578471	0.540807	0.527890	0.497371	0.462073
0.6	0.537049	0.629083	0.596122	0.587143	0.564917	0.537033
0.7	0.604368	0.662705	0.642285	0.637269	0.623684	0.604398
0.8	0.664037	0.668449	0.678764	0.678349	0.674033	0.664084
0.9	0.716298	0.664829	0.707743	0.711999	0.716859	0.716312
1.0	0.761594	0.719666	0.736791	0.742953	0.754008	0.761590

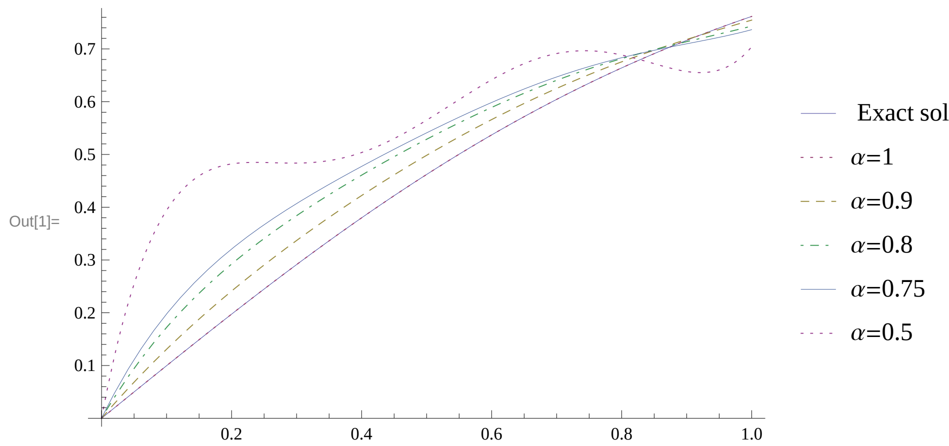

Figure 1. Comparison between the exact and approximate solution with different values of α ($\alpha = 1; 0.9; o.8; o.75; 0.5$)

Table 4: The absolute error with different values of α and δ with $\varepsilon = 10^{-10}$ for Ex.6.2.

x	u_{ex}	$\alpha = 0.5$ δ	$\|u_{approx(0.5,\delta)} - u_{ex}\|$	$\alpha = 0.75$ δ	$\|u_{approx(0.75,\delta)} - u_{ex}\|$
0.0	0.000000	1.0000000000	0.00000	1.0000000000	0.00000
0.1	0.096668	0.5509367733	6.92721×10^{-11}	0.6118280529	8.67221×10^{-11}
0.2	0.197375	0.7828683978	1.48167×10^{-11}	0.7557098658	9.94548×10^{-11}
0.3	0.291313	0.9022085415	1.57780×10^{-11}	0.8474079058	6.33089×10^{-11}
0.4	0.379949	0.9528339326	5.49787×10^{-11}	0.9035663569	7.07466×10^{-11}
0.5	0.462117	0.9689178977	1.00396×10^{-10}	0.9375555912	7.96814×10^{-11}
0.6	0.537049	0.9734178132	1.80849×10^{-10}	0.9594396045	9.91023×10^{-11}
0.7	0.604368	0.9810632337	2.94385×10^{-10}	0.9759452075	2.91887×10^{-11}
0.8	0.664037	0.9953461638	2.98471×10^{-10}	0.9904367802	5.61777×10^{-11}
0.9	0.716298	1.0100142158	1.51601×10^{-10}	1.0028985149	1.03043×10^{-10}
1.0	0.761594	1.0087667241	3.74314×10^{-10}	1.0099243331	4.65192×10^{-11}

Table 5: The absolute error with different values of α and δ with $\varepsilon = 10^{-10}$ for Ex.6.2.

x	u_{ex}	$\alpha = 0.8$ δ	$\|u_{approx(0.8,\delta)} - u_{ex}\|$	$\alpha = 0.9$ δ	$\|u_{approx(0.9,\delta)} - u_{ex}\|$
0.0	0.000000	1.0000000000	0.00000	1.0000000000	0.00000
0.1	0.096668	0.6530401769	9.52008×10^{-11}	0.7834457254	8.68039×10^{-11}
0.2	0.197375	0.7728985841	6.36302×10^{-11}	0.8482476922	8.63411×10^{-11}
0.3	0.291313	0.8522406578	9.08493×10^{-11}	0.8943227887	7.85992×10^{-11}
0.4	0.379949	0.9033615454	6.00183×10^{-11}	0.9267068960	9.98892×10^{-11}
0.5	0.462117	0.9362341985	5.69637×10^{-11}	0.9496111110	1.07309×10^{-11}
0.6	0.537049	0.9584700450	5.00371×10^{-11}	0.9663644649	3.86520×10^{-11}
0.7	0.604368	0.9752817828	3.36036×10^{-11}	0.9793608133	5.01296×10^{-11}
0.8	0.664037	0.9894528911	4.51679×10^{-11}	0.9900127621	9.40892×10^{-12}
0.9	0.716298	1.0013161863	8.01994×10^{-11}	0.9987223389	7.21614×10^{-11}
1.0	0.761594	1.0087416449	3.28626×10^{-12}	1.0048624114	9.39442×10^{-11}

7. Conclusion

In this paper we proposed a numerical method ,based on the shifted Chebyshev collocation method and finite difference scheme, to find the solution of the space fractional diffusion equations and fractional Riccati differential equation. In this method, the fractional derivatives are described in the Caputo sense. Comparison between our proposed method and other methods , shows that this scheme is superior and evidently the error gets smaller.

Table 6: The absolute error with different values of α and δ with $\varepsilon = 10^{-10}$ for Ex.6.2.

x	u_{ex}	$\alpha = 1.$ δ	$\|u_{approx(1.,\delta)} - u_{ex}\|$
0.0	0.000000	1.0000000000	0.00000
0.1	0.096668	0.9997414696	9.10855×10^{-11}
0.2	0.197375	0.9996883291	8.89971×10^{-11}
0.3	0.291313	0.9998913577	9.38171×10^{-11}
0.4	0.379949	1.0000541869	9.60665×10^{-11}
0.5	0.462117	1.0000876557	9.44814×10^{-11}
0.6	0.537049	1.0000274985	9.00673×10^{-11}
0.7	0.604368	0.9999564718	5.07978×10^{-11}
0.8	0.664037	0.9999405416	4.11708×10^{-11}
0.9	0.716298	0.9999840010	9.90938×10^{-11}
1.0	0.761594	1.0000039826	9.96059×10^{-11}

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