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Dynamics of Food Chain Model: Role of Alternative Resource for Top Predator

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Abstract. In this paper, effect of alternative resource for top predator in food chain model with holling type III functional response is seen . Proposed model is demonstrated in respect of analytical as well numerical results. Bifurcation study with the variation of alternative resource and half saturation constants are done numerically. Simulation results shows that suitable alternative resource has the capability to prevent top predator extinction.

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1. Introduction

Food chain model described the interaction of the units in natural ecological system. In this world there is no species exists in isolation from other species. Each species is a part of a community; they must be interacting with many other species. The interaction of species may be competition for food, dependency for food etc. Prey dependent prey predator models were studied by many researchers, [1–5, 8]. M.F.Elettreby [3], proposed a new multi team prey predator model, where they considered prey teams help each other. They studied the local as well as global

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stability and persistence of the model. Kar and Matsudan [7], studied harvesting efforts on the prey predator system, and obtained exactly one stable limit cycle that occurs in the system when the positive equilibrium is unstable. Mada Sanjaya WS et. all [12], studied three species food chain with mixed functional response. They analyzed dynamical behavior of system at equilibrium point and found that solution posses Hopf bifurcation.

Mada Sanjaya WS et. all [11], discussed food chain model with Holling type III functional response for middle predator and top predator and found that the solution posses bifurcation . O.P. Misra and Raveendra Babu A. [9], in the paper studied the food chain system considering interference of top predator in a polluted environment, they assumed that the presence of top predator reduces the predatory ability of intermediate predator, and finally shows that the predator rate of intermediate predator is a bifurcation parameter and Hopf-bifurcation occurs at some critical value of this parameter.

T.K.Kar and Bapan Ghosh [6], in their paper developed two species prey predator model in which the predator is partally influenced by alternative prey, and the effects of harvesting efforts on both specie is seen. O.P.Mishra et. all [10], studied a model for three species system with time lag considering two competing species and a predator specie which is partially coupled with an alternative prey. They shows that all the species increase and decrease with respect to constant alternative food resource .

In all of the above studied no mention for the behavior of alternative resource in food chain system with holling type III functional response, therefore in this paper a three species food chain model, with holling type III functional response for middle predator and supper predator. Middle predator is harvested by unnatural activity and top predator's growth effected by the presence of alternative resource which may be an alternative prey for top predator.

2. Mathematical Model

In this chapter we analyse the food chain model composed of a prey, middle predator and top predator of densities x, y and z. respectively. Before introduce the model, we would like to describe a brief sketch of the model which may indicate the biological behavior of the system.

- (H_1). Here all the participated species is assumed to arise from the coupling of three interacting species. x (prey) for y (middle predator) and y for z (top predator), where prime prey x is not interacting with super predator but y(middle predator) interacting with both x and z. y behavior as predator for prey and secondary prey for top predator (z). This is an interesting practical assumption from both mathematical and biological.
- (H_2). We assume that the growth of both prey (x for y and y for z) in absence of predator is logistic in nature with carrying capacity $K > 0$ and $L > 0$ respectively.
- (H_3). Here we considered Holling type III functional response for species $(x; y)$ and also for $(y; z)$
- (H_4). In this model, y (middle predator or secondary prey) is harvested and z (top predator) required besides y as food resource, an alternative food resource A also.

In view of the above, we have developed three species food chain model with the above assumptions.

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{a_1 x^2 y}{x^2 + a} \tag{1}$$

$$\frac{dy}{dt} = sy\left(1 - \frac{y}{L}\right) + \frac{a_2 x^2 y}{x^2 + a} - \frac{b_1 y^2 z}{y^2 + b} - Hy \tag{2}$$

$$\frac{dz}{dt} = \frac{b_2 y^2 z}{y^2 + b} + (1 - A)\, z - \eta z^2 - d_3 z \tag{3}$$

Where $x(0) > 0$, $y(0) > 0$ and $z(0) > 0$.

Here, r is intrinsic growth rate of prey, K is carrying capacity of prey population, a_1 capturing rate of predator for prey population, a_2 Conversion rate of prey, a Half saturation constant, s be the intrinsic growth rate of predator, L carrying capacity of environment for predator population, b_1 capturing rate of top predator for middle predator population, b half saturation constant, H harvesting effort, b_2 conversion rate of middle predator, η depletion rate coefficient of top predator due to self competition and d_3 is natural death rate of top predator. Here all the parameters to be positive constants.

3. Boundenees of the System

In this section, we need to establish that the boundenees of the dependent variables involving in the system of equations (1-3). The region of attraction for the solution of the model is shown by the following theorem.

THEOREM 3.1 *The region of attraction of the system is given as*

$$R = \{(x, y, z) : x \leqslant K, y \leqslant \delta and z \leqslant \epsilon\} \tag{4}$$

where $\delta = \left(s - H + \frac{a_2 K}{K^2 + a}\right)\frac{L}{s}$ *and* $\epsilon = \frac{1}{\eta}\left(\frac{b_2 \delta^2}{\delta^2 + b} - d_3 + (1 - A)\right)$.

Proof **:** *From the equation (1) of the model, we have*

$$\frac{dx}{dt} \leqslant rx\left(1 - \frac{x}{K}\right) \tag{5}$$

then by usual comparison theorem, (Hale 1969), we get

$$\lim_{t\to\infty} \sup x(t) \leqslant K. \tag{6}$$

Now from equatiom (2) of the model, we have

$$\frac{dy}{dt} \leqslant sy\left(1 - \frac{y}{L}\right) + \frac{a_2 K^2 y}{K^2 + a} - Hy$$

Then, by usual comparison theorem, (Hale 1969), we get

$$\lim_{t\to\infty} \sup y(t) \leqslant \left\{ s - H + \frac{a_2 K^2}{K^2 + a} \right\} \frac{L}{s} = \delta(say) \tag{7}$$

From equation (3) of the model, we get

$$\frac{dz}{dt} = \frac{b_2 y^2 z}{y^2 + b} - d_3 z + (1 - A)z - \eta z^2 \tag{8}$$

now using, value of y from equation (7) in above equation, we get

$$\frac{dz}{dt} \leqslant \frac{b_2 \delta^2 z}{\delta^2 + b} - d_3 z + (1 - A)z - \eta z^2 \tag{9}$$

by usual comparison theorem, (Hale 1969), we get the following result

$$\lim_{t\to\infty} \sup z(t) \leqslant \left(\frac{b_2 \delta^2}{\delta^2 + b^2} - d_3 + (1 - A) \right) \frac{1}{\eta} = \epsilon(say) \tag{10}$$

This completes the proof of theorem.

4. Existence of Equilibrium Points

The above system of equations has eight positive equilibrium points, $E_0(0,0,0)$, $E_1(K,0,0)$, $E_2(0,(1-H)L,0)$, $E_3(0,0,z_3)$, $E_4(x_4,y_4,0)$, $E_5(x_5,0,z_5)$, $E_6(0,y_6,z_6)$ and interior equilibrium point $\bar{E}(\bar{x},\bar{y},\bar{z})$.

(i). Equilibrium point $E_0(0,0,0)$ is trivial.

(ii). Equilibrium points $E_1(K,0,0)$, $E_2(0,(1-H)L,0)$ and $E_3(0,0,z_3)$ exists if $H < 1$ and $0 < A + d_3 < 1$ where $z_3 = \dfrac{(1 - A - d_3)}{\eta}$.

(iii). The existence of equilibrium point $E_4(x_4,y_4,0)$ is given by the equations

$$rx\left(1 - \frac{x}{K}\right) - \frac{a_1 x^2 y}{x^2 + a} = 0 \tag{11}$$

$$sy\left(1 - \frac{y}{L}\right) + \frac{a_2 x^2 y}{x^2 + a} - Hy = 0 \tag{12}$$

from the equations (11) and (12) we have,

$$\phi(x) = r\left(1 - \frac{x}{K}\right) - \frac{a_1 x L}{x^2 + a}\left(1 - \frac{H}{s} + \frac{a_2 x^2}{s(x^2 + a)}\right) \tag{13}$$

From (13) at $x = 0$, $\phi(0) = r > 0$ and at $x = K$, $\phi(K) = -\dfrac{a_1KL}{K^2+a}\left(1 - \dfrac{H}{s} + \dfrac{a_2K^2}{s(K^2+a)}\right) < 0$, for $s > H$ then there exits a positive point $x_4 \in (0, K)$ such that $\phi(x_4) = 0$ and $\phi'(x_4) < 0$, iff $a > x_4^2$.

After finding the value of x_4 we obtained the value of y_4 from equation (11) or (12).

(iv). Equilibrium point $E_5(x_5, 0, z_5)$, is obtained
$x_5 = K$ and $z_5 = \frac{(1-A)-d_3}{\eta}$.

(v). Equilibrium point $E_6(0, y_6, z_6)$ is obtained by the equations

$$sy\left(1-\frac{y}{L}\right) - \frac{b_1y^2z}{y^2+b} - Hy = 0 \tag{14}$$

$$\frac{b_2y^2}{y^2+b} + (1-A) - \eta z - d_3 = 0 \tag{15}$$

From (15), we get

$$z = \frac{1}{\eta}\left(\frac{b_2y^2}{y^2+b} + 1 - A - d_3\right) \tag{16}$$

Now from(14) and (16), we get

$$\xi(y) = s - \frac{sy}{L} - H - \frac{b_1y}{(y^2+b)\eta}\left(\frac{b_2y^2}{y^2+b} - 1 + A - d_3\right) \tag{17}$$

then at $y = 0$, $\xi(0) = s - H > 0$, and at $y = \delta$, $\xi(\delta) < 0$ if $s < \dfrac{\delta s}{L} + H$.

Then there exist a positive point $y_6 \in (0, \delta)$ such that $\xi(y_6) = 0$ and $\xi'(y_6) < 0$.

Positive value of z_6 can be obtained by using the value of y_6 from equation (16).

(vi). The interior equilibrium point $\bar{E}(\bar{x}, \bar{y}, \bar{z})$ is obtained by the following equations

$$r - \frac{rx}{K} - \frac{a_1xy}{x^2+a} = 0 \tag{18}$$

$$s - \frac{sy}{L} + \frac{a_2x^2}{x^2+a} - \frac{b_1zy}{y^2+b} - H = 0 \tag{19}$$

$$\frac{b_2y^2}{y^2+b} - d_3 + (1-A) - \eta z = 0 \tag{20}$$

Now from equation (18), we obtained

$y = \dfrac{1}{x}\left(-\dfrac{rx^3}{a_1} + \dfrac{Krx^2}{a_1} - \dfrac{arx}{a_1} + \dfrac{Kar}{a_1}\right)$

$y = \dfrac{1}{x}w(x)$, where $w(x) = -\dfrac{rx^3}{a_1} + \dfrac{Krx^2}{a_1} - \dfrac{arx}{a_1} + \dfrac{Kar}{a_1}$.

From equation (20), we obtained $z = \frac{1}{\eta}\left(\frac{b_2y^2}{y^2+b} + 1 - A - d_3\right)$now using the values of y and z in equation(19), we get, following equation

$$\begin{aligned} g(x) = {} & \left(w^2(x) + bx^2\right)^2 \left[(s-H)\eta x(x^2+a) - \frac{s}{L}w(x)\eta(x^2+a) + a_2x^3\eta\right] \\ & -b_1w^3(x)(x^2+a)x^2 - (1-A-d_3)\left(w^2(x)+bx^2\right)x^3\left(x^2+a\right) \end{aligned} \tag{21}$$

From (21), it is easy to see that,

(a). $g(0) = -\frac{s}{La_1^5}K^5a^6\eta < 0$

(b). $g(K) = bK^5(K^2+a)\,[b(s-H)\eta - (1-A-d_3)] + a_2b^2K^7\eta$,
it is positive if
$b(s-H)\eta > 1-A-d_3$ as $s > H$ and $1 - A > d_3$.

(c). A little algebraic manipulation yields that, $g'(x) > 0$ also positive.
Then, there exist a positive root $x = \bar{x}$ of (21) in $(0, K)$. Using this value in equation (18)-(20), we get the values of $\bar{y}$ and $\bar{z}$, as $\bar{y} = \frac{w(\bar{x})}{\bar{x}}$ and
$\bar{z} = \frac{1}{\eta}\left(\frac{b_2\bar{y}^2}{\bar{y}^2+b} + 1 - A - d_3\right)$

5. Local Stability Analysis

In this section we derived the conditions for the local stability of equilibrium points by computing the eigenvalues of the Jacobian matrix, J of the system (1)-(3), where

$$J(E) = \begin{bmatrix} r - \frac{2xr}{K} - \frac{2aa_1xy}{(x^2+a)^2} & \frac{-a_1x^2}{x^2+a} & 0 \\ \frac{2aa_2xy}{(x^2+a)^2} & s - \frac{2ys}{L} + \frac{a_2x^2}{x^2+a} - \frac{2bb_1yz}{(y^2+b)^2} - H & \frac{-b_1y^2}{y^2+b} \\ 0 & \frac{2bb_2yz}{(y^2+b)^2} & \frac{b_2y^2}{y^2+b} - d_3 + 1 - A - 2\eta z \end{bmatrix} \tag{22}$$

The existence of local stability of equilibrium points are give as.
The eigen values of the characteristic equation of jacobian matrix at the equilibrium point $E_0(0,0,0)$ are given as r, $s-H$ and $1-A-d_3$.All the eigen values are positive, hence equilibrium point E_0 is unstable.

The eigen value of the characteristic equation at $E_1(K,0,0)$ are $-r$, $s + \frac{a_2K^2}{K^2+a} - H$ and $-d_3 + 1 - A$, so equilibrium point E_1 is unstable in direction $y-z$ but stable in direction x, then it is saddle point.

The eigen value of the characteristic equation at$E_2(0,(1-H)L,0)$ are r, $-s-H+2sH$ and $\frac{b_1L^2(1-H)^2}{b+L^2(1-H)^2} - d_3 + (1-A)$, it is unstable in direction $x-z$ and stable in direction-y, so it is saddle point.

As the characteristic roots for $E_3(0,0,z_3)$ are r, $s-H$ and $-d_3+(1-A)-2\eta z_3$, it is unstable in direction $x-y$ and stable in direction z, then it is saddle point.

At $E_4(x_4, y_4, 0)$, characteristic equation is given as,

$$\left[\lambda^2 + (-A_{11} - A_{22})\lambda + A_{11}A_{22} - A_{21}A_{12}\right](A_{33} - \lambda) = 0 \tag{23}$$

Where
$A_{11} = r - \dfrac{2x_4 r}{K} - \dfrac{2aa_1x_4y_4}{(x_4^2+a)^2}$, $A_{12} = \dfrac{-a_1x_4^2}{x_4^2+a}$, $A_{21} = \dfrac{2aa_2x_4y_4}{(x_4^2+a)^2}$
$A_{22} = s - \dfrac{2y_4 s}{L} + \dfrac{a_2x_4^2}{x_4^2+a} - H$, $A_{33} = \dfrac{b_2^2y_4}{y_4^2+b} - d_3 + (1-A)$
The equilibrium point E_4 is unstable along z direction as $A_{33} > 0$.

The eigen values of the characteristic equation at equilibrium point $E_5(x_5, 0, z_5)$ are $r - \dfrac{2x_5r}{K}$, $s + \dfrac{a_2x_5^2}{x_5^2+a} - H$ and $-d_3 + (1-A) - 2\eta z_5$, it is stable in direction $x - z$ and unstable in direction y, hence it is saddle point.

Characteristic equation about the equilibrium point $E_6(0, y_6, z_6)$ is given as,

$$(\lambda - r)\left(\lambda^2 + A_4\lambda + B_4\right) = 0 \tag{24}$$

Where
$A_4 = H - s + \dfrac{2y_6s}{L} + \dfrac{2bb_1y_6z_6}{(y_6^2+b)^2} + \eta z_6$
$B_4 = \left(H - s + \dfrac{2y_6s}{L} + \dfrac{2bb_1y_6z_6}{(y_6^2+b)^2}\right)\eta z_6 + \dfrac{2bb_1b_2y_6^3z_6}{(y_6^2+b)^3}$
From equation (24), we eigen values $\lambda_1 = r$, $\lambda_{2,3} = \dfrac{-A \pm \sqrt{A^2 - 4B}}{2}$. The equilibrium point E_6 is unstable along x as $\lambda_1 = r > 0$

Behavior of the system at the interior equilibrium point $\bar{E}(\bar{x}, \bar{y}, \bar{z})$ seen by the characteristic equation of this point by jacobian matrix (22) is given as,

$$\lambda^3 + \overline{A}\lambda^2 + \overline{B}\lambda + \overline{C} = 0$$

where $\overline{A}, \overline{B}$ and $\overline{C}$ is given as
$\overline{A} = C_{11} + C_{22} + C_{33}$, $\overline{B} = -C_{11}C_{22} - C_{11}C_{33} - C_{22}C_{33} - C_{32}C_{23} - C_{12}C_{21}$
$\overline{C} = C_{11}C_{22}C_{33} - C_{11}C_{23}C_{32} - C_{12}C_{21}C_{33}$
and value of C_{ij} for all $i, j = 1, 2, 3$ given by following equations
$C_{11} = r - \dfrac{2\bar{x}r}{K} - \dfrac{2aa_1\bar{x}\bar{y}}{(\bar{x}^2+a)^2}$, $C_{12} = \dfrac{-a_1\overline{x}^2}{\overline{x}^2+a}$, $C_{21} = \dfrac{2aa_2\overline{xy}}{(\overline{x}^2+a)^2}$
$C_{22} = s - \dfrac{2\bar{y}s}{L} + \dfrac{a_2\overline{x}^2}{\overline{x}^2+a} - \dfrac{2bb_1\overline{yz}}{(\overline{y}^2+b)^2} - H$, $C_{23} = \dfrac{-b_1^2\overline{y}}{\overline{y}^2+b}$, $C_{32} = \dfrac{2bb_1\overline{yz}}{(\overline{y}^2+b)^2}$,
$C_{33} = \dfrac{b_2^2\overline{y}}{\overline{y}^2+b} - d_3 + (1-A) - 2\eta\overline{z}$
Then by Routh-Hurwitz criteria equilibrium $\overline{E}(\overline{x}, \overline{y}, \overline{z})$, is locally asymptotically stable if, $\overline{A} > 0, \overline{C} > 0$ and $\overline{AB} > \overline{C}$ otherwise it is not stable.

6. Global Stability for the Positive Equilibrium Point

The global stability of system is given by the following theorem.

THEOREM 6.1 *The positive equilibrium point $\bar{E}(\bar{x},\bar{y},\bar{z})$ is globally asymptotically stable, if the following inequality holds.*

(i). $(aa_2\bar{x} - a_1x_{max}\bar{x})^2 < \dfrac{2sa}{L}(\bar{x}^2+a)\left(\frac{ra}{K}(\bar{x}^2+a)+a_1a\bar{y}\right)$

(ii). $\left(b_2b(\bar{y}+y_{max}) - b_1\bar{y}^2 - b_1b\bar{y}\right)^2 < \dfrac{2\eta b^2 s}{L}(\bar{y}^2+b)^2$

Proof **:** *Consider the positive definite function, $V(x,y,z)$ as*

$$V(x,y,z) = x - \bar{x} - \bar{x}log\frac{x}{\bar{x}} + y - \bar{y} - \bar{y}log\frac{y}{\bar{y}} + z - \bar{z} - \bar{z}log\frac{z}{\bar{z}} \tag{25}$$

Where function $V(x,y,z)$ is continuous in R^3_+. Now the derivative of V with respect to time along the solution of the system is given as.

$$\frac{dV}{dt} = \frac{(x-\bar{x})}{x}\frac{dx}{dt} + \frac{(y-\bar{y})}{y}\frac{dy}{dt} + \frac{(z-\bar{z})}{z}\frac{dz}{dt} \tag{26}$$

then from the system of equations (1)-(3), we obtained

$$\frac{dV}{dt} = \frac{dV_1}{dt} + \frac{dV_2}{dt} + \frac{dV_3}{dt}$$

Where

$$\frac{dV_1}{dt} = (x-\bar{x})\left[-\frac{r}{K}(x-\bar{x}) - \frac{a_1xy}{x^2+a} + \frac{a_1\bar{x}\bar{y}}{\bar{x}^2+a}\right]$$

$$\frac{dV_2}{dt} = (y-\bar{y})\left[-\frac{s}{L}(y-\bar{y}) + \frac{a_2x^2}{x^2+a} - \frac{b_1zy}{y^2+b} - \frac{a_2\bar{x}^2}{\bar{x}^2+a} + \frac{b_1\bar{y}^2\bar{z}}{\bar{y}^2+b}\right] \text{ and}$$

$$\frac{dV_3}{dt} = (z-\bar{z})\left[-\eta(z-\bar{z}) + \frac{b_2y^2}{y^2+b} - \frac{b_2\bar{y}^2}{\bar{y}^2+b}\right]$$

Then by numerical simulation we obtained following,

$$\frac{dV}{dt} = -a_{11}(x-\bar{x})^2 - a_{22}(y-\bar{y})^2 - a_{33}(z-\bar{z})^2 + a_{12}(x-\bar{x})(y-\bar{y}) + a_{23}(y-\bar{y})(z-\bar{z})$$

Where

$$a_{11} = \frac{r}{K} + \frac{a_1a\bar{y} - a_1x\bar{x}\bar{y}}{(x^2+a)(\bar{x}^2+a)},\quad a_{22} = \frac{s}{L} + \frac{b_1z(y^2\bar{y}+y\bar{y}+b)}{(\bar{y}^2+b)(y^2+b)},\quad a_{33} = \eta,$$

$$a_{12} = \frac{a_2a(x+\bar{x}) - aa_1\bar{x}^2 - aa_1x}{(x^2+a)(\bar{x}^2+a)} \text{ and } a_{23} = \frac{b_2b(y+\bar{y}) - b_1\bar{y}^3 - bb_1\bar{y}}{(\bar{y}^2+b)(y^2+b)}$$

Therefor the sufficient condition for $\dfrac{dV}{dt} \leqslant 0$, given by the following inequalities

(a). $a_{ii} > 0$ *for all* $i = 1,2,3.$
(b). $a_{12}^2 < 2a_{11}a_{22}.$
(c). $a_{23}^2 < 2a_{22}a_{33}.$

then by using above inequalities, we obtained the following conditions,

(i). $(aa_2\bar{x} - a_1x_{max}\bar{x})^2 < \dfrac{2sa}{L}(\bar{x}^2+a)\left(\dfrac{ra}{K}(\bar{x}^2+a)+a_1a\bar{y}\right)$

(ii). $\left(b_2b(\bar{y}+y_{max}) - b_1\bar{y}^2 - b_1b\bar{y}\right)^2 < \dfrac{2\eta b^2 s}{L}(\bar{y}^2+b)^2$

Hence the theorem.

7. Permanence of System

In this section we described the permanence of the solution of the system.

THEOREM 7.1 *Assume that* $r\left(K^2+a\right) > a_1K\delta$, $s\left(\delta^2+b\right) > H\left(\delta^2+b\right)+b\delta\varepsilon$ *and* $(1-A) > d_3$, *then the solution of system is permanent.*

Proof : *From first equation of the system, we have*

$$\frac{dx}{dt} \geqslant rx\left(1-\frac{x}{K}\right)-\frac{a_1K\delta}{K^2+a}x$$

Using comparison principle, we set

$$\liminf x(t) \geqslant \frac{K}{r}\left(r-\frac{a_1K\delta}{K^2+a}\right)=k(say)>0. \tag{27}$$

From second equation of the system, we have

$$\frac{dy}{dt} \geqslant sy\left(1-\frac{y}{L}\right)-Hy-\frac{b_1\delta\varepsilon y}{\delta^2+b} \tag{28}$$

$$=\left(s-H-\frac{b_1\delta\varepsilon}{\delta^2+b}-\frac{sy}{L}\right)y \tag{29}$$

Using comparison principle, we set

$$\liminf y(t) \geqslant \frac{L}{s}\left(s-H-\frac{b_1\delta\varepsilon}{\delta^2+b}\right)=\delta_0(say)>0 \tag{30}$$

From third equation of the system, we have

$$\frac{dz}{dt} \geqslant \left((1-A)-d_3-\eta z\right)z \tag{31}$$

Using comparison principle, we get

$$\liminf z(t) \geqslant \frac{((1-A)-d_3)}{\eta}=\epsilon_0(say)>0 \tag{32}$$

hence the theorem.

8. Numerical Simulation

In this section, stability of the non-linear system 1-3, in the positive octant, is investigated numerically by using the following set of parameters.
$r=0.51$, $K=100$, $a_1=0.98$, $a=40$, $s=0.5$, $L=60$, $a_2=0.80$, $b_1=0.58$, $H=0.02$, $b_2=0.4$, $b=40$, $A=0.50$, $\eta=0.04$, $d_3=0.01$.
The eigenvalues of the system are $\lambda_{1,2}=-0.159146\pm0.51166i$ and $\lambda_3=-0.509632$ at interior equilibrium point $\bar{E}(5.7151, 6.2350, 17.1803)$. The real part of eigen values are negative. So, the interior equilibrium $\bar{E}$ is locally asymptotically stable in the octant region. Figure (1a) and figure (1b) shows that the equilibrium points $\bar{E}$ is locally and globally asymptotically stable, for $a=40$, $b=40$ and $A=0.50$. From figure (2a) - (2b), we observed that at $b=25$ and $A=0.50$ densities of the

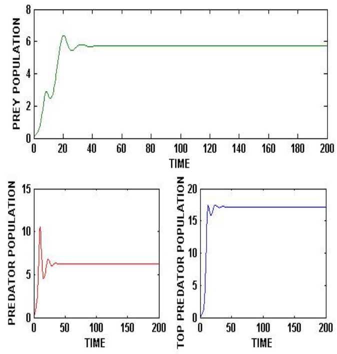

Figure 1. Time series.

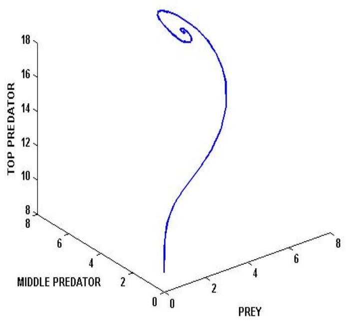

Figure 2. Global stablity for positive equilibrium point $\bar{E}$ at $a = 40, b = 40$ and $A = 0.5$.

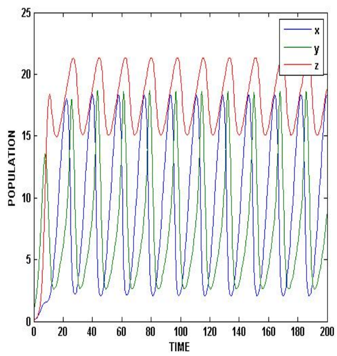

Figure 3. The time series of system at $b = 25$.

populations not settle down to their equilibrium values. Now from figure (3a)-(3b), shows the bifurcation diagram of the system. In figure (3a) and figure (3b), the system bifurcates when the value of $a > 70$ and $0.15 < b < 0.30$ respectively. In figure (4a) and (4b), when the value of alternative resource $0.15 < A < 0.3$ system is unstable for middle and top predator. In figure (5a) and (5b), we see that alternative resource has positive effect on middle predator. As the value of $'A'$ increases density of middle predator population increases and decreases the density of top predator. Now in figure (6a)-(6c), here we observed that when the values of

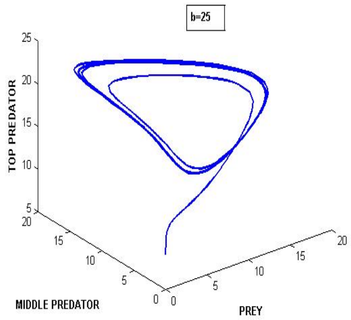

Figure 4. Bifurcation diagram of the system.

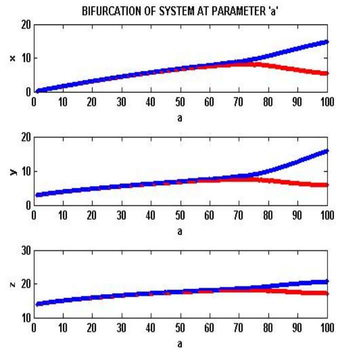

Figure 5. At $a = 40$ and $A = 0.50$.

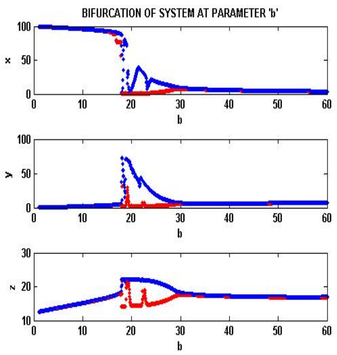

Figure 6. At $a = 40$ and $A = 0.50$.

$b = 25$ and alternative resource $A = 0.50$ equilibrium points become periodic, but equilibrium point again stable when value of alternative resource A increase up to 0.75.

9. Conclusion

In this paper, a food chain model with Holling type III functional response in presence of alternative resource is proposed. Here conditions of preliminary properties like positivity, boundedness, local and global stabilities of model is studied. The analytical results have been verified through numerical simulations.

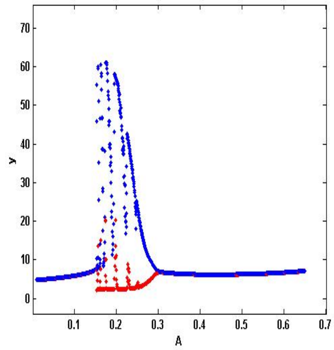

Figure 7. At $a = 40$ and $b = 40$.

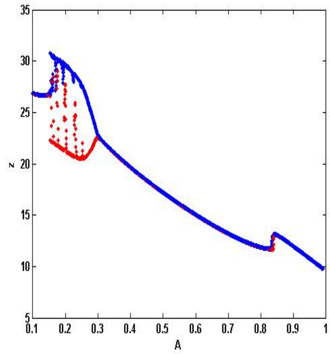

Figure 8. At $a = 40$ and $b = 40$.

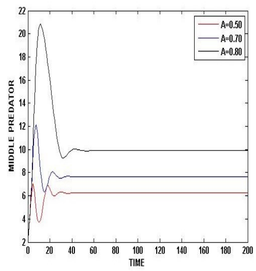

Figure 9. Middle predator population at alternative resource.

Bifurcation analysis is presented with the variation of alternative resource, half saturation constants of holling type III functional response for prey-middle predator and middle- top predator. From bifurcation analysis, we observed that stability of system become extinct for $0.15 < A < 0.30$ when $a = 40$, $b = 40$. System again loss their stability at $a > 70$ and $20 < b < 30$ respectively when $A = 0.50$. Here it is seen that at suitable value of alternative resource $A = 0.75$ at critical value of $b = 25 \in (20, 30)$ system is stable. From the figures (5a) and (5b), it is observed that the equilibrium level of middle and top predator can increase and decrease respectively with increase the value of alternative resource.

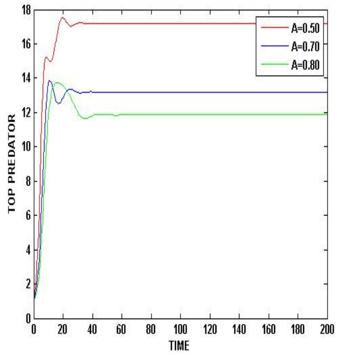

Figure 10. Top predator population at alternative resource.

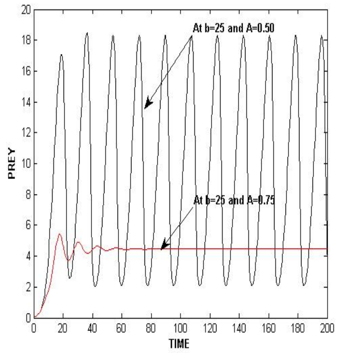

Figure 11. When the values of $b = 25$ and alternative resource $A = 0.50$ equilibrium points become periodic, but equilibrium point again stable when value of alternative resource A increase up to 0.75.

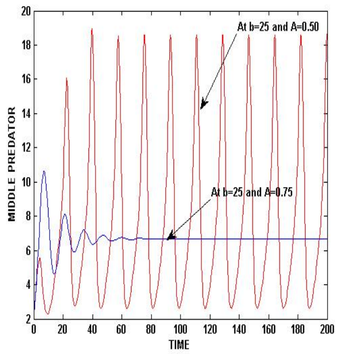

Figure 12.

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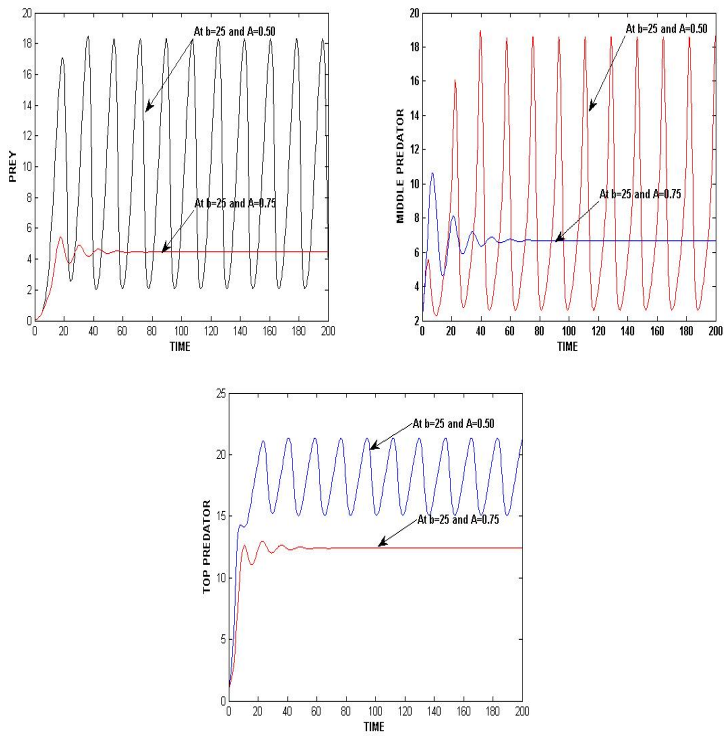

Figure 13. When the values of $b = 25$ and alternative resource $A = 0.50$ equilibrium points become periodic, but equilibrium point again stable when value of alternative resource A increase up to 0.75.

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