International Journal of
Mathematical Modelling & Computations
Vol. 13, No. 02, Spring 2023, 1- 13
DOI:10.30495/ijm2c.2023.1974373.1269

Examining $(3+1)$-Dimensional Extended Sakovich Equation Using Lie Group Methods

Y. AryaNejad[a,*], M. Jafari[a] and A. Khalili[a]

[a] *Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran.*

Abstract. In this paper, we investigate the symmetry group of the $(3+1)$-dimensional Sakovich equation. We obtain the classical and non-classical Lie symmetries for the equation under consideration. Therefore, we respond to the question of classification of the equation symmetries and, as a result, its invariant solutions. Presenting the algebra of symmetries and utilizing Ibragimovs method, we create the optimal system of Lie subalgebras. We obtain the symmetry reductions and invariant solutions of the considered equation using these vector fields.

Received: 01 February 2023, Revised: 15 March 2023, Accepted: 01 June 2023.

Keywords: Lie algebras; Extended Sakovich equation; Invariant solution; Reduction equations.

AMS Subject Classification: 70G65, 34C14, 53C50.

Index to information contained in this paper

1. Introduction

In 2020, Wazwaz extended the Sakovich equation (see [17]) to two new Painlev-integrable models of the same order as the Sakovich equation and of $(2+1)$ and $(3+1)$ dimensions given as (see [19]):

$$u_{xt} + u_{yy} + 2uu_{xy} + 6u^2u_{xx}^2 + u_{xx} + u_{xy} = 0, \tag{1}$$

* Corresponding author. Email: y.aryanejad@pnu.ac.ir

and

$$u_{xt} + u_{yy} + 2uu_{xy} + 6u^2u_{xx}^2 + u_{xx} + u_{xy} + u_{xz} + u_{yz} = 0. \tag{2}$$

Recently, multi-wave and interaction solutions and some Lie symmetry analysis to Equation (1) have been studied [15].
In the present work, we used Lies method to examine and find the answers of Equation (2). Next, an optimal system of subalgebras is presented associated with Lie symmetry algebra. The second-order linear PDE called (2 + 1)-dimensional second-order Sakovich equation was established:

$$u_{xt} + u_{yy} + 2uu_{xy} + 6u^2u_{xx} + 2u_{xx}^2 = 0, \tag{3}$$

which is quadratic in u_{xx}, and satisfies the Painlev test for integrability. The equation of Sakovich doesn't have a scattering expression u_{xxx}, which is not the case in the KdV equation. Nonlinear equations are widely used in evaluating nonlinear wave phenomena. Therefore, such equations have been considered by scientists for years and have made it possible to conduct detailed studies. Korteweg-De Vries (KdV) equation is one of the in-depth equations analyzed. The establishment of a completely integrable model, which describes the true characteristics of the scientific and engineering fields, is in progress, and a wide range of useful findings are being obtained. Several properties of the integrable equations include the presence of a Lax pair, which can be solved by the IST technique, satisfying the Painlev criterion, having infinite symmetry and Hamiltonian and Bi-Hamiltonian formulas, and other criteria [4, 12]. Through the Lie symmetry group process, the problem of symmetry categorization is extensively taken into account for various equations in different spaces [1–4, 6, 16]. On the contrary, Lies approach (symmetry group approach), which is a computational, algorithmic method for obtaining group-invariant solutions, is largely used in resolving differential equations. Through the mentioned process; proper solutions can be achieved through known ones, investigation of the invariant solutions, so also reduction of the ODEs order [5, 10, 13]. Studies in this area are in progress since such equations depict the states and properties of nonlinear phenomena, broaden vision in terms of physical aspects, and then become more practical in engineering and other sciences. So, the search for accurate solutions is important in non-linear equations in several ways, like plasma laser radiation [11, 18].

The paper is presented in several chapters as follows. The infinitesimal generators of the symmetry algebra of Equation (2) are specified along with some other results obtained in Section 2. In Section 3, we make the optimal ideal subalgebras of Eq.(2). Following the third section, we discover the similarity solutions, Lie invariants, and similarity reduction based on the infinitesimal symmetries of Eq.(2). In Section 4, we show reductions for differential equations as well as for definite solutions. In the last section, we obtain the associated non-classical symmetries of Eq.(2).

2. The symmetry computation of Eq.(2)

Normally,

$$\Delta_\beta((x^1, ..., x^m), (u^1, ..., u^n)^{(p)}) = 0, \qquad 1 \leqslant \beta \leqslant t, \tag{4}$$

is a set of PDE of order p. $(u^1, ..., u^n)^{(i)}$ represents the i-order derivative of U regarding x, $0 \leqslant i \leqslant p$. On both X and U, infinitesimal transformations Lie group acts as:

$$\begin{aligned} \tilde{x}^i &= x^i + \varepsilon\xi^i((x^1,...,x^m),(u^1,...,u^n)) + o(\varepsilon^2), \qquad 1 \leqslant i \leqslant m, \\ \tilde{u}^j &= u^j + \varepsilon\phi_j((x^1,...,x^m),(u^1,...,u^n)) + o(\varepsilon^2), \qquad 1 \leqslant j \leqslant n, \end{aligned} \tag{5}$$

in which the infinitesimal transformations for $(x^1,...,x^m)$ and $(u^1,...,u^n)$, are denoted by ξ^i and ϕ_j respectively. A given infinitesimal generator equivalent to the transformations group (5) is

$$V = \sum_{i=1}^{p} \xi^i((x^1,...,x^m),(u^1,...,u^n))\partial_{x^i} + \sum_{j=1}^{q} \phi_j((x^1,...,x^m),(u^1,...,u^n))\partial_{u^j}. \tag{6}$$

We apply x, y,z and t instead of x^1, x^2, x^3 and x^4 respectively, and for simplicity

$$\begin{gathered} \xi^j = \xi^j(x,y,z,t,u), \qquad j = 1,\cdots,4, \\ \phi = \phi(x,y,z,t,u). \end{gathered}$$

Here, an infinitesimal transformations one parameter Lie group is taken to apply the process for Eq.(2) as:

$$\begin{aligned} \tilde{x} &= x + \varepsilon\xi^1 + \sigma(\varepsilon^2), \\ \tilde{y} &= y + \varepsilon\xi^2 + \sigma(\varepsilon^2), \\ \tilde{z} &= z + \varepsilon\xi^3 + \sigma(\varepsilon^2), \\ \tilde{t} &= t + \varepsilon\xi^4 + \sigma(\varepsilon^2), \\ \tilde{u} &= u + \varepsilon\phi + \sigma(\varepsilon^2). \end{aligned}$$

The equivalent symmetry generator is:

$$V = \xi^1\partial_x + \xi^2\partial_y + \xi^3\partial_z + \xi^4\partial_t + \phi\partial_u. \tag{7}$$

The criterion of invariance has associated with the equations:

$$\begin{gathered} Pr^{(2)}V[u_{xt} + u_{yy} + 2uu_{xy} + 6u^2u_{xx}^2 + u_{xx} + u_{xy} + u_{xz} + u_{yz}] = 0. \text{ whenever} \\ u_{xt} + u_{yy} + 2uu_{xy} + 6u^2u_{xx}^2 + u_{xx} + u_{xy} + u_{xz} + u_{yz} = 0. \end{gathered}$$

Since infinitesimal transformation are based on x,y,z, t and u, adjusting the coefficients at 0, we obtain:

$$\begin{cases} -4\xi^1_{uu} = -\xi^4_x = 0, \\ -4\xi^2_{uu} = -4\xi^4_{uu} = 0, \\ -\xi^3_u = -8\xi^3_{ux} = 0, \\ -4\xi^3_{uu} = 4\phi_{uu} = 0, \\ \vdots \end{cases}$$

The number of generated equations is 48. We express the answer to the above set of equations in the form of the following theorem:

Table 1. Lie bracket of Eq.(2).

[,]	ϑ^1	ϑ^2	ϑ^3	ϑ^4	ϑ^5	ϑ^6	ϑ^7
ϑ^1	0	0	0	0	0	0	$\frac{1}{5}\vartheta^1$
ϑ^2	*	0	0	0	0	$-\vartheta^1$	$\frac{2}{5}\vartheta^1+\frac{3}{5}\vartheta^2$
ϑ^3	*	*	0	0	$-\vartheta^1$	$\frac{5}{3}\vartheta^1$	$-\frac{4}{15}\vartheta^1+\frac{3}{5}\vartheta^3$
ϑ^4	*	*	*	0	$\vartheta^1+\vartheta^2$	$-\frac{2}{3}\vartheta^1+\vartheta^3$	$\frac{2}{3}\vartheta^1+\vartheta^4$
ϑ^5	*	*	*	*	0	0	$-\frac{2}{5}\vartheta^6$
ϑ^6	*	*	*	*	*	0	$-\frac{2}{5}\vartheta^6$
ϑ^7	*	*	*	*	*	*	0

Theorem 2.1 *The Lie point symmetry group of Eq.*(2) *has a Lie algebra made by* (7)*, with coefficients as follows:*

$$\begin{aligned}
\xi^1 =&\frac{1}{15}(3x+6y+10t-4z)C_1\\
&+\frac{1}{15}(-10t-15y+25z)C_3+C_5t+C_7-zC_5,\\
\xi^2 =&\frac{3}{5}C_1y+C_5t+C_6,\\
\xi^3 =&\frac{3}{5}C_1z+C_3t+C_4,\\
\xi^4 =&C_1t+C_2,\\
\phi =&\frac{1}{30}(2-12u)C_1-\frac{1}{6}C_3,
\end{aligned} \tag{8}$$

where $C_i \in \mathbb{R}$, $i=1,...,7$.

Corollary 2.2 *One-parameter Lie group of Eq.*(2) *for every point symmetry contains the infinitesimal generators as:*

$$\begin{aligned}
\vartheta^1 =&\partial_x,\\
\vartheta^2 =&\partial_y,\\
\vartheta^3 =&\partial_z,\\
\vartheta^4 =&\partial_t,\\
\vartheta^5 =&(t-z)\partial_x+t\partial_y,\\
\vartheta^6 =&-(\frac{2}{3}t+y-\frac{5}{3}z)\partial_x+t\partial_z-\frac{1}{6}\partial_u,\\
\vartheta^7 =&(\frac{1}{5}x+\frac{2}{5}y+\frac{2}{3}t-\frac{4}{15}z)\partial_x+\frac{3}{5}y\partial_y+\frac{3}{5}z\partial_z\\
&+t\partial_t-(-\frac{1}{15}+\frac{2}{15}u)\partial_u,
\end{aligned} \tag{9}$$

We present the Lie algebra for Eq.(2) with Table (1). The data in the ij^{th} row and column of Table (1) are marked with $[X^i, \vartheta^j] = \vartheta^i\vartheta^j - \vartheta^j\vartheta^i$ where $i, j = 1, ..., 7$. For instance, the flow of ϑ^7 in Corollary 2.2 is

$$\Phi_\epsilon = (ye^{\frac{3}{5}\epsilon}+\frac{5}{6}te^{\epsilon}-\frac{2}{3}e^{\frac{3}{5}\epsilon}+e^{\frac{1}{5}\epsilon}(-y-\frac{5}{6}t+\frac{2}{3}z+x), ye^{\frac{3}{5}\epsilon}, ze^{\frac{3}{5}\epsilon}, te^{\epsilon}, \frac{1}{6}+e^{-\frac{2}{5}\epsilon}(-\frac{1}{6}+u)).$$

Table 2. Adjoint representation of $\mathbf{g}$

Ad	ϑ^1	ϑ^2	ϑ^3	ϑ^4	ϑ^5	ϑ^6	ϑ^7
ϑ^1	$\vartheta^1+\frac{1}{5}s\vartheta^7$	ϑ^2	ϑ^3	ϑ^4	ϑ^5	ϑ^6	ϑ^7
ϑ^2	$\vartheta^1-s\vartheta^6+\frac{2}{5}s\vartheta^7$	$\vartheta^2+\frac{3}{5}s\vartheta^7$	ϑ^3	ϑ^4	ϑ^5	ϑ^6	ϑ^7
ϑ^3	$\vartheta^1-s\vartheta^5+\frac{5}{3}s\vartheta^6-\frac{4}{15}s\vartheta^7$	ϑ^2	$\vartheta^3+\frac{3}{5}s\vartheta^7$	ϑ^4	ϑ^5	ϑ^6	ϑ^7
ϑ^4	$\vartheta^1+s\vartheta^5-\frac{2}{3}s\vartheta^6+\frac{2}{3}s\vartheta^7$	$\vartheta^2+s\vartheta^5$	$\vartheta^3+s\vartheta^6$	$\vartheta^4+s\vartheta^7$	ϑ^5	ϑ^6	ϑ^7
ϑ^5	$\vartheta^1+s\vartheta^3-s\vartheta^4$	$\vartheta^2-s\vartheta^4$	ϑ^3	ϑ^4	$\vartheta^5-\frac{2}{5}s\vartheta^7$	ϑ^6	ϑ^7
ϑ^6	$\vartheta^1+s\vartheta^2-\frac{5}{3}s\vartheta^3+(\frac{5}{6}s^2+\frac{2}{3}s)\vartheta^4$	ϑ^2	$\vartheta^3-s\vartheta^4$	ϑ^4	ϑ^5	$\vartheta^6-\frac{2}{5}s\vartheta^7$	ϑ^7
ϑ^7	$e^{\frac{-1}{5}s}\vartheta^1+(e^{\frac{-3}{5}s}-e^{\frac{-1}{5}s})\vartheta^2$ $+(\frac{-2}{3}e^{\frac{-3}{5}s}+\frac{2}{3}e^{\frac{-1}{5}s})\vartheta^3$ $+(\frac{5}{6}e^{-s}\frac{-5}{6}e^{\frac{-1}{5}s})\vartheta^4$	$e^{\frac{-3}{5}s}\vartheta^2$	$e^{\frac{-3}{5}s}\vartheta^3$	$e^{-s}\vartheta^4$	$e^{\frac{2}{5}s}\vartheta^5$	$e^{\frac{2}{5}s}\vartheta^6$	ϑ^7

3. Classification of one-dimensional subalgebras

The one-parameter optimal system of Eq.(2) can be determined utilizing the symmetry group. Such subgroups must be obtained by presenting various solutions. Therefore, invariant solutions should be searched not connected to a transformation in the whole symmetry group. An optimal set of subalgebras is yielded. The issue of categorizing 1D subalgebras would be similar to the question of categorizing the adjoint representation orbits. One representative is considered from each group of equivalent subalgebras, to solve an optimum set of subalgebras problems [13, 14]. The adjoint representation of each ϑ^t, $t=1,...,7$ is defined as:

$$\mathrm{Ad}(\exp(s.\vartheta^t).\vartheta^r)=\vartheta^r-s.[\vartheta^t,\vartheta^r]+\frac{s^2}{2}.[\vartheta^t,[\vartheta^t,\vartheta^r]]-\cdots, \tag{10}$$

where s represents the parameter and $[\vartheta^t,\vartheta^r]$ is presented in Table (1) for $t,r=1,\cdots,7$ ([13], page 199). Let $\mathfrak{g}$ is the Lie algebra generated by (9). The adjoint action for g is given according to Table (2). Considering the Ibragimovs method, an optimal system of one-dimensional subalgebras is presented in the form of the following theorem.

Theorem 3.1 *Considering the Ibragimovs method, the 1D optimal system of subalgebras for Eq.*(2) *are as follows:*

$$\begin{array}{ll}
1:\ \vartheta^1, & 8:\vartheta^2\pm\vartheta^3,\\
2:\ \vartheta^2, & 9:\vartheta^2\pm\vartheta^6,\\
3:\ \vartheta^3, & 10:\vartheta^3\pm\vartheta^5\\
4:\ \vartheta^4, & 11:\vartheta^4\pm\vartheta^5,\\
5:\ \vartheta^5, & 12:\vartheta^4\pm\vartheta^6,\\
6:\ \vartheta^6, & 13:\vartheta^5\pm\vartheta^6,\\
7:\ \vartheta^7, & 14:\vartheta^4\pm\vartheta^5\pm\vartheta^6,
\end{array}$$

Proof According to Table 1, we find that the center is empty. Therefore, it is necessary to specify the subalgebras:

$$\langle\vartheta^1,\vartheta^2,\vartheta^3,\vartheta^4,\vartheta^5,\vartheta^6,\vartheta^7\rangle.$$

Considering basis $\{\vartheta^1, \cdots, \vartheta^7\}$ and a vector field $X = \sum_{i=1}^{7} a_i \vartheta^i$, for $t = 1, \cdots, 7$, the map:

$$\begin{cases} \mathrm{Ad}(\exp(s\vartheta^t).X) : \mathfrak{g} \to \mathfrak{g} \\ X \mapsto \mathrm{Ad}(\exp(s\vartheta^t).X) \end{cases}$$

is a linear function. The associated matrix of functions $\mathrm{Ad}(\exp(s_i\vartheta^i).X)$, $1 \leqslant i \leqslant 7$ are reported as:

$$\mathrm{Ad}(\exp(s_i\vartheta^i).X) = [a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7] M^i_{7\times 7} \begin{bmatrix} \vartheta^1 \\ \vartheta^2 \\ \vartheta^3 \\ \vartheta^4 \\ \vartheta^5 \\ \vartheta^6 \\ \vartheta^7 \end{bmatrix},$$

where

$$M^1 = \begin{bmatrix} 1&0&0&0&0&0&\frac{-1}{5}s_1 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}, \quad M^2 = \begin{bmatrix} 1&0&0&0&0&s_2&\frac{-2}{5}s_2 \\ 0&1&0&0&0&0&\frac{-3}{5}s_2 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix},$$

$$M^3 = \begin{bmatrix} 1&0&0&0&s_3&\frac{-5}{3}s_3&\frac{4}{15}s_3 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&\frac{-3}{5}s_3 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}, \quad M^4 = \begin{bmatrix} 1&0&0&0&-s_4&\frac{2}{3}s_4&\frac{-2}{3}s_4 \\ 0&1&0&0&-s_4&0&0 \\ 0&0&1&0&0&s_4&0 \\ 0&0&0&1&0&0&-s_4 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix},$$

$$M^5 = \begin{bmatrix} 1&0&-s_5&s_5&0&0&0 \\ 0&1&0&s_5&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&\frac{2}{5}s_5 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}, M^6 = \begin{bmatrix} 1&-s_6&\frac{5}{3}s_6&\frac{5}{6}s_6^2-\frac{2}{3}s_6&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&s_6&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&\frac{2}{5}s_6 \\ 0&0&0&0&0&0&1 \end{bmatrix},$$

and for instance, for matrix M^7 we have:

$$\mathrm{Ad}(\exp(s_7\vartheta^7).X) = [a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7]\begin{bmatrix} e^{\frac{-1}{5}s_7} & \begin{matrix} e^{\frac{-3}{5}s_7} \\ -e^{\frac{-1}{5}s_7}\end{matrix} & \begin{matrix}\frac{-2}{3}e^{\frac{-3}{5}s_7} \\ +\frac{2}{3}e^{\frac{-1}{5}s_7}\end{matrix} & \begin{matrix}-\frac{5}{6}e^{-s_7} \\ -\frac{5}{6}e^{\frac{-1}{5}s_7}\end{matrix} & 0 & 0 & 0 \\ 0 & e^{\frac{-3}{5}s_7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{-3}{5}s_7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-s_7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{2}{5}s_7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{2}{5}s_7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix}\vartheta^1 \\ \vartheta^2 \\ \vartheta^3 \\ \vartheta^4 \\ \vartheta^5 \\ \vartheta^6 \\ \vartheta^7\end{bmatrix}.$$

From $\mathrm{Ad}(\exp(s_1\vartheta^1)) \circ \mathrm{Ad}(\exp(s_2\vartheta^2)) \circ \cdots \circ \mathrm{Ad}(\exp(s_7\vartheta^7))$, we can shorten X as follows:

For $a_7 \neq 0$, the coefficients a_1, a_2, a_3, a_4, a_5 and a_6 can be disappeared by setting, $s_1 = \dfrac{-5a_1}{a_7}$, $s_2 = \dfrac{-5a_2}{3a_7}$, $s_3 = \dfrac{-5a_3}{3a_7}$, $s_4 = \dfrac{-a_4}{a_7}$, $s_5 = \dfrac{5a_5}{2a_7}$, $s_6 = \dfrac{5a_6}{2a_7}$ respectively. When required, by scaling X, we assume $a_7 = 1$. Thus, x turns into the case (7).

Let $a_7 = 0$. Consider a vector

$$(a_1, a_2, a_3, a_4, a_5, a_6, 0). \tag{11}$$

For $a_6 \neq 0$, the coefficients a_1, a_3 disappeared by adjusting, $s_2 = \dfrac{a_1}{a_6}$, $s_4 = \dfrac{-a_3}{a_6}$ respectively. Thus, (11) is reduced to

$$(0, a_2, 0, a_4, a_5, a_6, 0). \tag{12}$$

Let $a_7 = 0$ and $a_6 = a_5 \neq 0$, for vector (12), the coefficient a_2 can be disappeared by setting $\mathbf{s}_4 = \dfrac{-a_2}{a_5}$. Thus, by scaling X, we suppose $a_4 = 1$ and $a_6 = a_5 = \pm 1$. Thus, X gives rise to the case (14). For $a_4 = 0$, by scaling X, $a_5 = 1$ and $a_6 = \pm 1$ are supposed. Thus, X gives rise to the case (13).

Let $a_7 = a_5 = 0$ and $a_6 \neq 0$. Thus, (12) is reduced to

$$(0, a_2, 0, a_4, 0, a_6, 0). \tag{13}$$

Let $a_7 = a_5 = 0$ and $a_6 = a_4 \neq 0$, for vector (13), the coefficient a_2 is vanished by $s_5 = \dfrac{a_2}{a_4}$ adjusting a. Thus, we assume $a_4 = 1$, $a_6 = \pm 1$, by scaling X. Hence, X turns into the case (12).

Let $a_7 = a_5 = a_4 = 0$ and $a_6 \neq 0$, in vector (13). Thus, by scaling X, we assume $a_2 = 1$ and $a_6 = \pm 1$. Thus, X gives rise to the case (9). For $a_2 = 0$, we assume $a_6 = 1$, by scaling X. Therefore, X gives rise to the case (6).

Let $a_7 = a_6 = 0$. Thus, (11) is reduced to

$$(a_1, a_2, a_3, a_4, a_5, 0, 0). \tag{14}$$

For $a_7 = a_6 = 0$ and $a_5 \neq 0$, for vector (14), the coefficients a_1 , a_2 are vanished by setting, $s_3 = \dfrac{a_1}{a_5}$,$s_4 = \dfrac{-a_2}{a_5}$ respectively. Therefore, (11) is reduced to

$$(0, 0, a_3, a_4, a_5, 0, 0). \tag{15}$$

Let $a_7 = a_6 = 0$ and $a_5 = a_4 \neq 0$, for vector (15), the coefficient a_3 can be disappeared by setting $s_6 = \dfrac{a_3}{a_4}$. Thus, by scaling X, we suppose $a_4 = 1$, $a_5 = \pm 1$. Thus, X turns into the case (11).

Let $a_7 = a_6 = a_4 = 0$ and $a_5 \neq 0$, in vector (15). Thus, by scaling X, we assume $a_3 = 1$ and $a_5 = \pm 1$. Thus, X gives rise to the case (10). For $a_3 = 0$, by scaling X, we assume $a_5 = 1$. Thus, X gives rise to the case (5).

Let $a_7 = a_6 = a_5 = 0$. Thus, (14) is reduced to

$$(a_1, a_2, a_3, a_4, 0, 0, 0). \tag{16}$$

Let $a_7 = a_6 = a_5 = 0$ and $a_4 \neq 0$, for vector (16), the coefficients a_1, a_2, a_3 can be disappeared by setting $s_5 = \dfrac{a_1}{a_4}$, $s_5 = \dfrac{a_2}{a_4}$, $s_6 = \dfrac{a_3}{a_4}$ respectively. Thus, we assume $a_4 = 1$, by scaling X. Thus, X gives rise to the case (4).

Let $a_7 = a_6 = a_5 = a_4 = 0$. Thus, (14) is reduced to

$$(a_1, a_2, a_3, 0, 0, 0, 0). \tag{17}$$

Let $a_7 = a_6 = a_5 = a_4 = 0$ and $a_3 \neq 0$, for vector (17), the coefficient a_1 is vanished by setting $s_5 = \dfrac{-a_1}{a_3}$. Thus, by scaling X, we assume $a_2 = 1$, and $a_3 = \pm 1$. Thus, X gives rise to the case (8). For $a_2 = 0$, we suppose $a_3 = 1$, by scaling X. Hence, X gives rise to the case (3).

Let $a_7 = a_6 = a_5 = a_4 = a_3 = 0$. Thus, (17) is reduced to

$$(a_1, a_2, 0, 0, 0, 0, 0). \tag{18}$$

Let $a_7 = a_6 = a_5 = a_4 = a_3 = 0$ and $a_2 \neq 0$, for vector (18), the coefficient a_1 is vanished by adjusting $s_6 = \dfrac{-a_1}{a_2}$. Thus, we assume $a_2 = 1$ by scaling X. Therefore, X gives rise to the case (2). For $a_2 = 0$, we suppose $a_1 = 1$, by scaling X. Thus, X gives rise to the case (1).
We have obtained all cases, and the proof is complete. ■

4. Equivalent solutions of Equation (2)

First, symmetry reduction of Eq.(2) is classified, taking into account the subalgebras of Theorem 3.1. It is essential to look for a new form of Equation (2) in specific coordinates. In these new coordinates, reduction occurs. Independent variables p, q and r must be found for the infinitesimal generator to create these coordinates.

Hence, the equation is expressed in novel coordinates through the chain rule reducing the system. Table 3.1 shows the similarity variables p_i, q_i, r_i and h_i for 1D subalgebras in Theorem 3.1. Using each similarity variable, the reduced PDE of Eq.(2) is found (Table 4).

For example, we calculated the invariants related to subalgebra $H_5 := \vartheta^2 + \vartheta^3$, by solving the characteristic equation as follows:

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dz}{1} = \frac{dt}{0} = \frac{du}{0}.$$

Thus, the new variables are:

$$p = x, \qquad q = y - z, \qquad r = t, \qquad h = u,$$

where $h(p,q,r)$ meets a decreased PDE with three variables as:

$$h_{pr} + h_{qq} - 2hh_{pq} + 6h^2 h_{pp} + 2h_{pp}^2 + h_{pp} - h_{qq} = 0. \tag{19}$$

Subalgebra $\vartheta^2 + \vartheta^3$, and the decreased equation (19) are presented in Tables 3 and 4, by the case (5). Equivalent solution of Eq. (19) becomes:

$$h(p,q,r) = u(x,y,z,t) = -\frac{4C_2^3 \cosh\theta - C_4 \cosh\theta - 12C_2^3}{6C_2 \cosh\theta},$$

where $\theta = \frac{1}{6}\frac{1}{C_2}(6C_1C_2 + 6C_2^2 x - 6tC_2^2 - 16tC_2^6 + tC_4^2 + 6C_4C_2 z - 6C_4C_2 y))^2$. Using a similar argument, for the vector ϑ^2, Equation (2) is decreased as:

$$h_{pr} + 6h^2 h_{pp} + 2h_{pp}^2 + h_{pp} + h_{pq} = 0, \tag{20}$$

where the independent variables are as $p = x, q = z, r = t$ and the dependent function is as $u = h(p,q,r)$. The equivalent solution of Eq. (20) becomes:

$$h(p,q,r) = u(x,y,z,t) = -\frac{2C_2^2}{3}\frac{(\cosh\theta - 3)}{\cosh\theta},$$

where $\theta = \left(C_1 + C_2 x + C_3 t - \frac{8}{3} z C_2^5 - zC_3 - zC_2\right)^2$. Also, for the vector ϑ^3, Equation (2) decreased as:

$$h_{pr} + h_{qq} + 2hh_{pq} + 6h^2 h_{pp} + 2h_{pp}^2 + h_{pp} + h_{pq} = 0, \tag{21}$$

where the independent variables are as $p = x, q = y, r = t$ and the dependent function is as $u = h(p,q,r)$. The equivalent solution of Eq. (21) is derived as:

$$h(p,q,r) = u(x,y,z,t) = -\frac{4C_2^3 \cosh\theta + C_3 \cosh\theta - 12C_2^3}{6C_2 \cosh\theta},$$

where $\theta = (\frac{1}{6}\frac{1}{C_2}(6C_1C_2 + 6C_2^2 x + 6yC_3C_2 - 16tC_2^6 - 6tC_3C_2 - 6tC_2^2 - 5tC_3^2))^2$. For case (4), in Tables 3 and 4, the corresponding solution is derived as:

$$h(p,q,r) = u(x,y,z,t) = -\frac{4C_2^3 \cosh\theta + C_3 \cosh\theta - 12C_2^3}{6C_2 \cosh\theta},$$

Table 3. Similarity solution.

i	H_i	p_i	q_i	r_i	w_i	u_i
1	ϑ^1	y	z	t	u	$h(p,q,r)$
2	ϑ^2	x	z	t	u	$h(p,q,r)$
3	ϑ^3	x	y	t	u	$h(p,q,r)$
4	ϑ^4	x	y	z	u	$h(p,q,r)$
5	$\vartheta^2+\vartheta^3$	x	$y-z$	t	u	$h(p,q,r)$
6	$\vartheta^4+\vartheta^5$	y	$-\frac{1}{2}t^2+z$	$-\frac{5}{9}t^3+\frac{1}{3}t^2+yt-\frac{5}{3}zt+x$	$u+\frac{1}{6}t$	$h(p,q,r)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4. Reduced equations.

1	$h_p+h_{rq}=0,$
2	$h_{pr}+6h^2h_{pp}+2h_{pp}^2+h_{pp}+h_{pq}=0,$
3	$h_{pr}+h_{qq}+2hh_{pq}+6h^2h_{pp}+2h_{pp}^2+h_{pp}+h_{pq}=0,$
4	$h_{qq}+2hh_{pq}+6h^2h_{pp}+2h_{pp}^2+h_{pp}+h_{pq}+h_{pr}+h_{qr}=0,$
5	$h_{pr}-2hh_{pq}+6h^2h_{pp}+2h_{pp}^2+h_{pp}=0,$
6	$6h_q-10h_r+6h_p+12hh_{rp}+36h_r-h^2+12h_r^2+6h_r+6h_{rp}+6h_{rq}+6h_{pq}=0.$
$\vdots$	$\vdots$

where $\theta=(\frac{1}{6}\frac{1}{C_3+C_2}(6C_1C_3+6C_1C_2+6xC_3C_2+6C_2^2x+6C_3^2y+6yC_3C_2-16zC_2^6-6zC_3C_2-6zC_2^2-5zC_3^2))^2$.

5. Non-classical symmetries

Here, using the method of non-classical symmetries, we try to get closer to the solutions to the equation. Indeed, the method of non-classical symmetries is applied to get other solutions for a system of PDEs and ODEs. For years, this method has been applied in many types of research and plays an influential role in solving partial differential equations. Now, we apply a variant of this method used by Cai Guoliang et al. [7].
Using the notation

$$\xi^j=\xi^j(x,y,z,t,u),\qquad j=1,\cdots,4,$$
$$\phi=\phi(x,y,z,t,u),$$

one relates to the infinitesimal generator V given by

$$V=\xi^1\partial_t+\xi^2\partial_x+\xi^3\partial_y+\xi^4\partial_z+\phi\partial_u, \tag{22}$$

the following first order PDE system

$$\xi^1u_t+\xi^2u_x+\xi^3u_y+\xi^4u_z-\phi, \tag{23}$$

representing the features of the vector field V. Equation (23) is known as the invariant surface condition in our context. To obtain the nonclassical symmetries, we have:

$$\begin{aligned}Pr^{(2)}V[u_{xt}+u_{yy}+2uu_{xy}+6u^2u_{xx}^2+u_{xx}+u_{xy}+u_{xz}+u_{yz}]=0.\ \text{ whenever}\\ u_{xt}+u_{yy}+2uu_{xy}+6u^2u_{xx}^2+u_{xx}+u_{xy}+u_{xz}+u_{yz}=0,\\ \xi^1u_t+\xi^2u_x+\xi^3u_y+\xi^4u_z-\phi=0.\end{aligned}$$

To obtain a system of the determining equations with respect to $\xi^1, \xi^2, \xi^3, \xi^4$ and ϕ, the governing equation of (24) should be solved. Solving the determining equations of (24) results in the non-classical symmetries of (2). During the solving of the non-classical symmetries of (2), there are two cases needed to discuss:

$$\xi^1 = 1, \quad \xi^1 = 0. \tag{24}$$

For case $\xi^1 = 1$, we can get the following determining nonlinear PDE system for the symmetries of (2):

$$\begin{aligned}
&4\xi^4_u\xi^4_x = 0, 4\varphi_{uu} = 0, -8\xi^3_x = 0, -4\xi^3_{xx} = 0, -8\xi^4_x = 0, -4\xi^4_{xx} = 0,\\
&4\xi^3_u\xi^3_x + 4\varphi_{uu} = 0, -4\xi^2_{xx} + 8\varphi_{xu} = 0, 4\xi^4_u\xi^3_x + 4\xi^3_u\xi^4_x + 4\varphi_{uu} = 0,\\
&\cdots.
\end{aligned}$$

The number of generated equations is 25. The general solution of this nonlinear PDE system is the following result.

Theorem 5.1 *In case $\xi^1 = 1$, for equation (2), we have the following infinitesimals:*

$$\begin{aligned}
\xi^2 =& \frac{1}{5}\frac{5C_3C_1t + C_1x - 5C_1zC_3 + 4C_1z + 5C_4C_1 + 30z + 5C_3C_2}{C_1t + C_2},\\
\xi^3 =& \frac{1}{5}\frac{5C_1tF_1(t) + 3C_1y + 5F_1(t)C_2}{C_1t + C_2},\\
\xi^4 =& \frac{1}{5}\frac{2C_1t + 3C_1z + 30t + 5C_5}{C_1t + C_2},\\
\phi =& -\frac{1}{5}\frac{(2uC_1 + 5)}{C_1t + C_2},
\end{aligned} \tag{25}$$

where $C_i \in R$, $i = 1, ..., 5$ and

$$5\left(\frac{d}{dt}F_1(t)\right)C_1t - 5C_1C_3 + 2C_1 + 5C_1F_1(t) + 5\left(\frac{d}{dt}F_1(t)\right)C_2 - 20 = 0.$$

From (25) we get the following infinitesimal generators:

Corollary 5.2 *In case $\xi^1 = 1$, for Eq.(2) in addition to infinitesimal generators in Corollary 2.2 we get the additional nonclassical operators:*

$$\begin{aligned}
\vartheta^1 =& \partial_t + \frac{1}{5}\frac{(x + 34z - 30y)}{t}\partial_x + \frac{1}{5}\frac{(18t + 3y)}{t}\partial_y\\
&+ \frac{1}{5}\frac{(32t + 3z)}{t}\partial_z - \frac{1}{5}\frac{(2u + 5)}{t}\partial_u,\\
\vartheta^2 =& \partial_t + \frac{1}{5}(30z - 30y)\partial_x + 4t\partial_y + 6t\partial_z - \partial_u.
\end{aligned}$$

For case $\xi^1 = 0$, set $\xi^4 = 1$. We can get the following determining nonlinear PDE system for the symmetries of (2):

$$
\begin{aligned}
&4\varphi_{uu}=0,-8\xi^3_x=0,-\xi^3_x=0,-4\xi^3_{xx}=0,-6\xi^2_x+2\varphi_u=0,-4\xi^2_{xx}+8\varphi_{xu}=0,\\
&-8\xi^3_z-2\xi^3_y+\xi^2_x-\xi^3_x-2u\xi^3_x+\xi^3\xi^3_x-\xi^3=0,\xi^2_x-\xi^2_u\varphi+\xi^3\xi^3_y=0,\cdots
\end{aligned}
$$

The number of generated equations is 19. The general solution of this nonlinear PDE system is the following result.

Theorem 5.3 *In case $\xi^1=0$, for equation (2) we have the following infinitesimals:*

$$
\begin{aligned}
\xi^2=&-\frac{-C_2z-6y+6t+tC_2-C_4}{-6t+C_1}\\
\xi^3=&-\frac{10t+tC_2-C_3}{-6t+C_1}\\
\phi=&\frac{1}{-6t+C_1},
\end{aligned}
\tag{26}
$$

where $C_i\in R$, $i=1,...,4$.

From (26), we get the following infinitesimal generators:

Corollary 5.4 *In case $\xi^1=0$, for Eq.(2) ,we get the additional non-classical operators:*

$$
\begin{aligned}
\vartheta^1=&\quad -\frac{-6y+6t}{-6t+1}\partial_x-\frac{10t}{-6t+1}\partial_y+\partial_z+\frac{1}{-6t+1}\partial_u,\\
\vartheta^2=&\quad \frac{1}{6}\frac{-z-6y+7t}{t}\partial_x+\frac{11}{6}\partial_y+\partial_z-\frac{1}{6}\frac{1}{t}\partial_u,\\
\vartheta^3=&\quad \frac{1}{6}\frac{-6y+6t}{t}\partial_x+\frac{1}{6}\frac{10t-1}{t}\partial_y+\partial_z-\frac{1}{6}\frac{1}{t}\partial_u,\\
\vartheta^4=&\quad \frac{1}{6}\frac{-1-6y+6t}{t}\partial_x+\frac{5}{3}\partial_y+\partial_z-\frac{1}{6}\frac{1}{t}\partial_u.
\end{aligned}
$$

References

[1] S. C. Anco, M. L. Gandarias and E. Recio, Line-solitons, line-shocks, and conservation laws of a universal KP-like equation in 2+1 dimensions, J. Math. Anal. Appl., **504** (2021) 125319.

[2] Y. AryaNejad, Exact solutions of diffusion equation on sphere, Comput. Methods Differ. Equ., **10 (3)** (2022) 789–798.

[3] Y. AryaNejad, Symmetry Analysis of Wave Equation on Conformally Flat Spaces, J. Geom. Phys., **161** (2021) 104029.

[4] Y. AryaNejad, R. Mirzavand, Conservation laws and invariant solutions of time-dependent Calogero-Bogoyavlenskii-Schiff equation, J. Linear. Topological. Algebra., **11 (4)** (2022) 243–252.

[5] G. W. Bluman and S. C. Anco, Symmetry and Integration Methods for Differential Equations, Springer, (2002).

[6] M. S. Bruzon, M. l. Gandarias, C. Muriel, J. Ramierez, G. S. Saez and F. R.Romero, The Calogero-Bogoyavlenskii-Schiff equation in (z+1) dimensions, Theor. Math. phys., **137 (1)** (2003) 1367–1377.

[7] G. Cai, Y. Wang and F. Zhang, Nonclassical symmetries and group invariant solutions of Burgers-Fisher equations, World Journal of Modelling and Simulation, **3 (4)** (2007) 305–309.

[8] E. Dastranj and H. Sahebi Fard, Exact solutions and numerical simulation for Bakstein-Howison model, Computational Methods for Differential Equations, **10 (2)** (2022) 461–474.

[9] M. L. Gandarias, M. Torrisi and A. Valenti, Symmetry classification, and optimal systems of a non-linear wave equation Internet, J. Non-linear Mech., **39** (2004) 389–398.

[10] N. H. Ibragimov, CRC Handbook of Lie Group Analysis of Differential Equations, New Trends in Theoretical Developments and Computational Methods, CRC Press, Boca Raton, Vol. 3, (1996).

[11] M. Jafari, A. Zaeim and A. Tanhaeivash, Symmetry group analysis and conservation laws of the potential modified KdV equation using the scaling method, Int. J. Geometric Methods in Modern, **19 (7)** (2022) 2250098.

[12] A. P. Mrquez, T. M. Garrido, E. Recio and R. de la Rosa, Lie symmetries and exact solutions for a fourth-order nonlinear diffusion equation, Math. Meth. Appl. Sci., **45 (17)** (2022) 10614–10627.

[13] P. J. Olver, Applications of Lie Groups to Differential Equations, Springer, New York, (1986).

[14] L. V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York, (1982).

[15] Y. S. Ozkan and E. Yasar, Multiwave and interaction solutions and Lie symmetry analysis to a new $(2+1)$-dimensional Sakovich equation, Alexandria Engineering Journal, **59 (6)** (2020) 5285–5293.

[16] S. Rashidi and S. R. Hejazi, Self-adjointness, conservation laws and invariant solutions of the Buckmaster equation, Computational Methods for Differential Equations, **8 (1)** (2020) 85–98.

[17] S. Sakovich, A new Painlev-integrable equation possessing KdV-type solitons, Nonlinear Phenomena in Complex Systems, **22 (3)** (2019) 299–304.

[18] A. M. Wazwaz, New Painleve-integrable $(2+1)$ and $(3+1)$- dimensional KdV and mKdV equations, Rom. J. Phys., **65** (2020) 108.

[19] A. M. Wazwaz, Two new Painlev-integrabl extended Sakovich equations with $(2+1)$ and $(3+1)$ dimensions, International Journal of Numerical Methods for Heat and Fluid Flow, **30 (3)** (2020) 1379–1387.