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Estimates for the Generalized Fourier-Bessel Transform in the Space $L^2_{\alpha,n}$

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Abstract Some estimates are proved for the generalized Fourier-Bessel transform in the space $L^2_{\alpha,n}$ on certain classes of functions characterized by the generalized continuity modulus.

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1. Introduction

In [5], Abilov et al. proved two useful estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator. This result has been generalized in [8] for the Fourier-Bessel transform in the space $L^2(\mathbb{R}^+, x^{2\alpha+1}dx)$, $\alpha > -1/2$.
In this paper, we consider a second-order singular differential operator $\mathcal{B}$ on the half line which generalizes the Bessel operator $\mathcal{B}_\alpha$, we prove some estimates in certain classes of functions characterized by a generalized continuity modulus and connected with the generalized Fourier-Bessel transform associated to $\mathcal{B}$ in $L^2_{\alpha,n}$ analogs of the statements proved in [5]. For this purpose, we use a generalized translation operator.
In section 2, we give some definitions and preliminaries concerning the generalized Fourier-Bessel transform. Some estimates are proved in section 3.

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2. Preliminaries on the Generalized Fourier-Bessel Transform

Integral transforms and their inverses the Bessel transform are widely used to solve various problems in calculus, mechanics, mathematical physics, and computational mathematics (see [3,4]).
In this section, we develop some results from harmonic analysis related to the singular differential operator $\mathcal{B}$. Further details can be found in [1] and [6]. In all what follows assume where $\alpha > -1/2$ and n a non-negative integer.
Consider the second-order singular differential operator on the half line defined by

$$\mathcal{B}f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx} - \frac{4n(\alpha+n)}{x^2}f(x),$$

For $n = 0$, we obtain the classical Bessel operator

$$\mathcal{B}_\alpha f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx}.$$

Let M be the map defined by

$$Mf(x) = x^{2n} f(x), \quad n = 0, 1, ...$$

Let $L^p_{\alpha,n}$, $1 \leqslant p < \infty$, be the class of measurable functions f on $[0, \infty[$ for which

$$\|f\|_{p,\alpha,n} = \|M^{-1}f\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{1/p}.$$

If $p = 2$, then we have $L^2_{\alpha,n} = L^2([0, \infty[, x^{2\alpha+1})$.
For $\alpha > \frac{-1}{2}$, we introduce the normalized spherical Bessel function j_α defined by

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1) J_\alpha(x)}{x^\alpha}, \tag{1}$$

where $J_\alpha(x)$is the Bessel function of the first kind and $\Gamma(x)$ is the gamma-function (see [7]). The function $y = j_\alpha(x)$ satisfies the differential equation

$$\mathcal{B}_\alpha y + y = 0$$

with the condition initial $y(0) = 0$ and $y'(0) = 0$. The function $j_\alpha(x)$ is infinitely differentiable, even and moreover entire analytic.
In the terms of $j_\alpha(x)$, we have (see[2])

$$1 - j_\alpha(x) = O(1), \quad x \geqslant 1. \tag{2}$$

$$1 - j_\alpha(x) = O(x^2), \quad 0 \leqslant x \leqslant 1. \tag{3}$$

$$\sqrt{hx} J_\alpha(hx) = O(1), \quad hx \geqslant 0. \tag{4}$$

For $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$, put

$$\varphi_\lambda(x) = x^{2n} j_{\alpha+2n}(\lambda x). \tag{5}$$

where $j_{\alpha+2n}$ is the Bessel kernel of index $\alpha + 2n$ given by (1).
From [1,6] recall the following properties.

PROPOSITION 2.1 *(**a**) φ_λ satisfies the differential equation*

$$\mathcal{B}\varphi_\lambda = -\lambda^2 \varphi_\lambda.$$

*(**b**) For all $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$*

$$|\varphi_\lambda(x)| \leqslant x^{2n} e^{|Im\lambda||x|}.$$

The generalized Fourier-Bessel transform that we call it the integral transform defined by

$$\mathcal{F}_\mathcal{B} f(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda(x) x^{2\alpha+1} dx, \lambda \geqslant 0, f \in L^1_{\alpha,n}.$$

Let $f \in L^1_{\alpha,n}$ such that $\mathcal{F}_\mathcal{B}(f) \in L^1_{\alpha+2n} = L^1([0, \infty[, x^{2\alpha+4n+1}dx)$. Then the inverse generalized Fourier-Bessel transform is given by the formula

$$f(x) = \int_0^{+\infty} \mathcal{F}_\mathcal{B} f(\lambda)\varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n}\lambda^{2\alpha+4n+1}d\lambda, \quad a_{\alpha+2n} = \frac{1}{4^{\alpha+2n}(\Gamma(\alpha+2n+1))^2}.$$

PROPOSITION 2.2 *[1,6] (**c**) For every $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ we have the Plancherel formula*

$$\int_0^{+\infty} |f(x)|^2 x^{2\alpha+1} dx = \int_0^{+\infty} |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

*(**d**) The generalized Fourier-Bessel transform $\mathcal{F}_\mathcal{B}$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2([0, +\infty[, \mu_{\alpha+2n})$.*

Define the generalized translation operator T^h, $h \geqslant 0$ by the relation

$$T^h f(x) = (xh)^{2n} \tau^h_{\alpha+2n}(M^{-1}f)(x), x \geqslant 0,$$

where $\tau^h_{\alpha+2n}$ is the Bessel translation operators of order $\alpha + 2n$ defined by

$$\tau^h_\alpha f(x) = c_\alpha \int_0^\pi f(\sqrt{x^2 + h^2 - 2xh\cos t}) \sin^{2\alpha} t dt,$$

where

$$c_\alpha = \left(\int_0^\pi \sin^{2\alpha} t dt\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(\pi)\Gamma(\alpha+\frac{1}{2})}.$$

For $f \in L^2_{\alpha,n}$, we have

$$\mathcal{F}_\mathcal{B}(T^h f)(\lambda) = \varphi_\lambda(h)\mathcal{F}_\mathcal{B}(f)(\lambda), \tag{6}$$

$$\mathcal{F}_\mathcal{B}(\mathcal{B}f)(\lambda) = -\lambda^2\mathcal{F}_\mathcal{B}(f)(\lambda), \tag{7}$$

(see [1,6] for details).
The generalized modulus of continuity of function $f \in L^2_{\alpha,n}$ is defined as

$$w(f,\delta)_{2,\alpha,n} = \sup_{0<h\leqslant\delta} \|T^h f(x) - h^{2n} f(x)\|_{2,\alpha,n}, \delta > 0.$$

3. Main Result

The goal of this work is to prove several some estimates for the integral

$$J_N^2(f) = \int_N^{+\infty} |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

in certain classes of functions in $L^2_{\alpha,n}$.

LEMMA 3.1 *For $f \in L^2_{\alpha,n}$, we have*

$$\|T^h f(x) - h^{2n} f(x)\|^2_{2,\alpha,n} = h^{4n}\int_0^{+\infty} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

Proof By using the formula (6), we conclude that

$$\mathcal{F}_\mathcal{B}(T^h f - h^{2n} f)(\lambda) = h^{2n}(j_{\alpha+2n}(\lambda h) - 1)\mathcal{F}_\mathcal{B} f(\lambda). \tag{8}$$

Now by formula (8) and Plancherel equality, we have the result. ■

THEOREM 3.2 *Given $f \in L^2_{\alpha,n}$. Then there exist a constant $C > 0$ such that, for all $N > 0$,*

$$J_N(f) = O(N^{2n}\omega(f, CN^{-1})_{2,\alpha,n}).$$

Proof Firstly, we have

$$J_N^2(f) \leqslant \int_N^{+\infty} |j| d\mu + \int_N^{+\infty} |1-j| d\mu, \tag{9}$$

with $j = j_p(\lambda h)$, $p = \alpha + 2n$ and $d\mu = |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$. The parameter $h > 0$ will be chosen in an instant.
In view of formulas (1) and (4), there exist a constant $C_1 > 0$ such that

$$|j| \leqslant C_1(\lambda h)^{-p-\frac{1}{2}}.$$

Then

$$\int_N^{+\infty} |j| d\mu \leqslant C_1 (hN)^{-p-\frac{1}{2}} J_N^2(f).$$

Choose a constant C_2 such that the number $C_3 = 1 - C_1 C_2^{-p-\frac{1}{2}}$ is positif.
Setting $h = C_2/N$ in the inequality (9), we have

$$C_3 J_N^2(f) \leqslant \int_N^{+\infty} |1-j| d\mu. \tag{10}$$

By Hölder inequality the second term in (10) satisfies

$$\begin{aligned}\int_N^{+\infty} |1-j| d\mu &= \int_N^{+\infty} |1-j|.1.d\mu \\ &\leqslant \left(\int_N^{+\infty} |1-j|^2 d\mu\right)^{1/2} \left(\int_N^{+\infty} d\mu\right)^{1/2} \\ &\leqslant \left(\int_N^{+\infty} |1-j|^2 d\mu\right)^{1/2} J_N(f).\end{aligned}$$

From Lemma 3.1, we conclude that

$$\int_N^{+\infty} |1-j|^2 d\mu \leqslant h^{-4n} \|T^h f(x) - h^{2n} f(x)\|_{2,\alpha,n}^2.$$

Therefore

$$\int_N^{+\infty} |1-j| d\mu \leqslant h^{-2n} \|T^h f(x) - h^{2n} f(x)\|_{2,\alpha,n} J_N(f).$$

For $h = C_2/N$, we obtain

$$C_3 J_N^2(f) \leqslant C_2^{-2n} N^{2n} w(f, C_2/N)_{2,\alpha,n} J_N(f).$$

Consequently

$$C_2^{2n} C_3 J_N(f) \leqslant N^{2n} w(f, C_2/N)_{2,\alpha,n}.$$

for all $N > 0$. The theorem is proved with $C = C_2$. ∎

Theorem 3.3 *Let $f \in L^2_{\alpha,n}$. Then, for all $N > 0$,*

$$\omega(f, N^{-1})_{2,\alpha,n} = O\left(N^{-4(n+1)} \left(\sum_{l=0}^{N-1} (l+1)^3 J_l^2(f)\right)^{\frac{1}{2}}\right).$$

Proof From Lemma 3.1, we have

$$\|T^h f(x) - h^{2n} f(x)\|_{2,\alpha,n}^2 = h^{4n} \int_0^{+\infty} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

This integral is divided into two

$$\int_0^{+\infty} = \int_0^{N} + \int_N^{+\infty} = I_1 + I_2,$$

where $N = [h^{-1}]$. We estimate them separately.
From formula (2), we have the estimate

$$I_2 \leqslant C_4 \int_N^{+\infty} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = C_4 J_N^2(f).$$

Now we estimate I_1. From formula (3), we have

$$I_1 \leqslant C_5 h^4 \int_0^{N} \lambda^4 |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = C_5 h^4 \sum_{l=0}^{N-1} \int_l^{l+1} \lambda^4 |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$
$$= C_5 h^4 \sum_{l=0}^{N-1} a_l \left(J_l^2(f) - J_{l+1}^2(f)\right),$$

with $a_l = (l+1)^4$.
For all integers $m \geqslant 1$, the Abel transformation shows

$$\sum_{l=0}^{m} a_l \left(J_l^2(f) - J_{l+1}^2(f)\right) = a_0 J_0^2(f) + \sum_{l=1}^{m} (a_l - a_{l-1}) J_l^2(f) - a_m J_{m+1}^2(f)$$
$$\leqslant a_0 J_0^2(f) + \sum_{l=1}^{m} (a_l - a_{l-1}) J_l^2(f),$$

because $a_m J_{m+1}^2(f) \geqslant 0$.
Hence

$$I_1 \leqslant C_5 h^4 \left(J_0^2(f) + \sum_{l=1}^{N-1} \left((l+1)^4 - l^4\right) J_l^2(f) - N^4 J_N^2(f) \right).$$

Moreover by the finite increments theorem, we have $(l+1)^4 - l^4 \leqslant 4(l+1)^3$. Then

$$I_1 \leqslant C_5 N^{-4} \left(J_0^2(f) + 4 \sum_{l=1}^{N-1} (l+1)^3 J_l^2(f) - N^4 J_N^2(f) \right),$$

since $N \leqslant \frac{1}{h}$. Combining the estimates for I_1 and I_2 gives

$$\|T^h f(x) - h^{2n} f(x)\|_{2,\alpha,n}^2 = O\left(N^{-4-4n} \sum_{l=0}^{N-1} (l+1)^3 J_l^2(f) \right),$$

which implies

$$\omega(f,N^{-1})_{2,\alpha,n}=O\left(N^{-4(n+1)}\left(\sum_{l=0}^{N-1}(l+1)^3J_l^2(f)\right)^{\frac{1}{2}}\right),$$

and this ends the proof. ■

4. Conclusions

In this work we have succeeded to generalize a result in [8] for the generalized Fourier-Bessel transform in the space $L^2_{\alpha,n}$. We proved that the modulus of smoothness $\omega(f,\delta)_{2,\alpha,n},\delta>0$, possesses the following properties

$$J_N(f)=O(N^{2n}\omega(f,CN^{-1})_{2,\alpha,n}),$$

$$\omega(f,N^{-1})_{2,\alpha,n}=O\left(N^{-4(n+1)}\left(\sum_{l=0}^{N-1}(l+1)^3J_l^2(f)\right)^{\frac{1}{2}}\right),$$

for all $N>0$.

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