*Journal of*
*Linear and Topological Algebra*
*Vol.* 02, *No.* 04, 2013, 243- 254

# Approximate solution of fourth order differential equation in Neumann problem

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Received 19 August 2013; revised 9 November 2013; accepted 4 December 2013.

**Abstract.** Generalized solution on Neumann problem of the fourth order ordinary differential equation in space $W^2_\alpha(0,b)$ has been discussed , we obtain the condition on B.V.P when the solution is in classical form. Formulation of Quintic Spline Function has been derived and the consistency relations are given.Numerical method,based on Quintic spline approximation has been developed. Spline solution of the given problem has been considered for a certain value of $\alpha$. Error analysis of the spline method is given and it has been tested by an example.



**Keywords:** Fourth order ordinary differential equation, Neumann problem, generalized solution, quintic spline function, error analysis.

**2010 AMS Subject Classification:** 65M06, 65M12.

## 1. Introduction

The most important class of operator equation in the fourth order is:

$$Lu \equiv (x^\alpha u'')'' + Au = f, \tag{1}$$

when $x \in [0,b], 0 \leqslant \alpha \leqslant 4,\ f \in L_2((0,b),\mathcal{H})$, the operator $A$ has a complete system of eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}}$, which form a Riesz basis in $\mathcal{H}$. Degenerate equations encountered in solving many important problems of applied character(the theory of small deformation surfaces of rotation, the membrane theory of shells, the bending of plates of variable thickness with a sharp edge). Particularly, these equations are important in the gas dynamic.

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Tricomi [19]has studied on the second-order equation with non-characteristic degeneration. Keldysh [9] had played a fundamental role in the theory of degenerate equations.He studied the first boundary value problem of the second-order equation with characteristic degeneration.Bicadze in [2] formulated weight problems which has been created by Fichera [5]. Generally speaking the 4th order B.V.P can not be solved analytically only in the special case the analytic solution is available so that the numerical Approximation of the solution are interested . Numerical discussion of fourth-order boundary value problems are given in [20], [10],[11].The Quintic spline methods for the solution of linear fourth-order two boundary value problems are given in [10],[12],[11]. Rashidinia discussed solving fourth-order linear boundary-value problems [12].
In this paper we emphasize on the generalized solution for the Neumann problem in $W^2_\alpha(0,b)$. In section-2 we consider one-dimension form of equation (1) and defined spaces $\dot{W}^2_\alpha(0,b), W^2_\alpha(0,b)$ and there norms also we defined generalized solution of the Neumann problem.In section-3 Formulation of Quintic Spline Function has been derived and the consistency relations are given which it is useful in approximation of the solution of 4th B.V.P.In section-4 Error Analysis of the method is given and finally in section-5 Numerical illustration has been given.

## 2. Neumann problem for one-dimensional case

We consider the one-dimensional of equation (1),that $Au = au$, $a \in \mathbb{C}, a = const$,

$$Lu \equiv (x^\alpha u'')'' + au = f, \quad 0 \leqslant \alpha \leqslant 4, \quad f \in L_2(0,b). \tag{2}$$

To discuss the generalized solution in Neumann problem,we need to define spaces $\dot{W}^2_\alpha(0,b)$and $W^2_\alpha(0,b)$.

### 2.1 *Space $\dot{W}^2_\alpha(0,b)$ and $W^2_\alpha(0,b)$*

Definition 2
The weighted Sobolev space $\dot{W}^2_\alpha(0,b)$ is the completion of $\dot{C}^2[0,b]$ with the norm

$$\|u\|^2_{\dot{W}^2_\alpha(0,b)} = \int_0^b x^\alpha\, |u''(x)|^2 dx, \quad \alpha \geqslant 0, \tag{3}$$

when $\dot{C}^2[0,b]$ be a set of twice continuously differentiable functions and $u(x)$ defined on $[0,b]$ and satisfying the conditions

$$u(0) = u'(0) = u(b) = u'(b) = 0. \tag{4}$$

The elements of $\dot{W}^2_\alpha(0,b)$ are continuously differentiable functions on $[\varepsilon, b]$ for every $0 < \varepsilon < b$, whose first derivatives are absolutely continuous and $u(b) = u'(b) = 0$.

Proposition 3 For every $u \in \dot{W}^2_\alpha(0,b)$ close to $x = 0$, we have the following estimates

$$\text{(i)} \qquad |u(x)|^2 \leqslant C_1 x^{3-\alpha} \|u\|^2_{\dot{W}^2_\alpha(0,b)}, \text{ for } \alpha \neq 1, 3,$$

$$\text{(ii)} \qquad |u'(x)|^2 \leqslant C_2 x^{1-\alpha} \|u\|^2_{\dot{W}^2_\alpha(0,b)}, \text{ for } \alpha \neq 1. \tag{5}$$

Following ( [7]) in the above solutions when $\alpha = 1$and $\alpha = 3$ the factors $x^{3-\alpha}$ in (i) must be replaced by $x^2|\ln x|$ and $|\ln x|$ respectively.when $\alpha = 1$ the factor $x^{1-\alpha}$ in (ii) must be replaced by $|\ln x|$.

It follows from relations (5) that, for $\alpha < 1$ (weak degeneracy), the boundary conditions $u(0) = u'(0) = 0$ are “retained”, while for $1 \leqslant \alpha < 3$ (strong degeneracy), only the condition $u(0) = 0$ is ”retained”, for $\alpha \geq 3$, both $u(0)$ and $u'(0)$ in general may be infinite. For example, if $u(x) = x^{\beta}\varphi(x)$ when $\varphi(x) \in C^2[0, b]$, $\varphi(b) = \varphi'(b) = 0$ and $\varphi(0) \neq 0$, then for $\alpha > 3$ and $\frac{(3-\alpha)}{2} < \beta < 0$, the function $u(x)$ belongs to $\dot{W}^2_{\alpha}(0, b)$, but $u(0)$ and $u'(0)$ do not exist.

Proposition 4 For every $0 \leqslant \alpha \leqslant 4$, we have a continuous embedding

$$\dot{W}^2_{\alpha}(0, b) \hookrightarrow L_2(0, b). \tag{6}$$

It is compact for $0 \leqslant \alpha < 4$, and it verify that the sequence $u_n(x) = n^{-1/2}x^{-1/2}|\ln x|^{-1/2-1/n}$ is bounded in $\dot{W}^2_4(0, b)$, but it does not contain a subsequence convergent in $L_2(0, b)$.(proof in [6])

Remark 5 The embedding (6) for $\alpha > 4$ is fail.

When $\alpha > 4$, we use the function $u(x) = x^{-1/2}\varphi(x)$ that $\varphi(x) \in C^2[0, b]$, $\varphi(b) = \varphi'(b) = 0$ and $\varphi(0) \neq 0$ and $u \in \dot{W}^2_{\alpha}(0, b)$ but $u \notin L_2(0, b)$.

Corollary 6 If the function $u$ has a bounded piecewise-continuous derivative of the second order in $[\varepsilon, b]$ for arbitrary $0 < \varepsilon < b$, $\|u\|_{\dot{W}^2_{\alpha}(0,b)} < \infty$, $u(b) = u'(b) = 0$ and near to $x = 0$ hold the inequalities (5), then $u \in \dot{W}^2_{\alpha}(0, b)$.

Definition 7 The weighted Sobolev space $W^2_{\alpha}(0, b)$ is the completion of $C^2[0, b]$ with the norm

$$\|u\|^2_{W^2_{\alpha}(0,b)} = \int_0^b (x^{\alpha}\,|u''(x)|^2 + |u(x)|^2)\,dx, \quad \alpha \geqslant 0, \tag{7}$$

with the corresponding scalar product

$$\{u, v\}_{\alpha} = (x^a u'', v'') + (u, v).$$

Proposition 8 For every $u \in W^2_{\alpha}(0, b)$ we have

$$\text{(i)} \qquad |u(x)|^2 \leqslant (c_1 + c_2x^{3-\alpha})\|u\|^2_{W^2_{\alpha}(0,b)}, \text{ for } \alpha \neq 1, 3,$$

$$\text{(ii)} \qquad |u'(x)|^2 \leqslant (c_3 + c_4x^{1-\alpha})\|u\|^2_{W^2_{\alpha}(0,b)}, \text{ for } \alpha \neq 1. \tag{8}$$

Following ([18]) in the above solutions when $\alpha = 1$and $\alpha = 3$ the factors $x^{3-\alpha}$ in (i) must be replaced by $x^2|\ln x|$ and $|\ln x|$ respectively.when $\alpha = 1$ the factor $x^{1-\alpha}$ in (ii) must be replaced by $|\ln x|$.

Proposition 9 The following embedding for arbitrary $0 \leqslant \alpha \leqslant 4$ is continuous and compact.

$$W^2_{\alpha}(0, b) \subset L_2(0, b). \tag{9}$$

### 2.2 *Generalized Solution Of The Neumann Problem*

In this subsection we define that generalized solution and when it is classical get some conditions.

Definition 10 The function $u \in W_\alpha^2(0,b)$ is called a generalized solution of the Neumann problem for the equation (2) if for every $v \in W_\alpha^2(0,b)$ we have the equality

$$(x^\alpha u'', v'') + a(u,v) = (f,v). \tag{10}$$

If the generalized solution $u \in W_\alpha^2(0,b)$ for the equation (2) is classical,then we get the following conditions ([14])

$$u''(b) = u'''(b) = 0, \quad (x^\alpha u''(x))|_{x=0} = (x^\alpha u''(x))'|_{x=0} = 0. \tag{11}$$

Indeed, integrating the equality (10) by parts,we obtain that for every $v \in W_\alpha^2(0,b)$ the equality

$$((x^\alpha u'')'', v) + a(u,v) + (x^\alpha u''(x)\bar{v}'(x) - (x^\alpha u''(x))'\bar{v}(x))|_{x=0}^{x=b} = (f,v),$$

is valid.
In the following section we discuss the numerical approximation the solution of the Neumann problem by using Quintic spline function.

## 3. Quintic Spline Function

We consider a uniform mesh $\Delta$, with nodal points $x_i$ on $[a,b]$ such that

$$\begin{aligned} &\Delta : a = x_0 < x_1 < ... < x_N = b, \\ &x_i = a + ih, \quad i = 1, ..., N, \end{aligned} \tag{12}$$

We denote a function value,$u(x_i)$ by$u_i$. Quintic spline function $S_i(x)$ interpolating to a function $u(x)$ on $[a,b]$ and it is defined as follows:
**1.**In each subinterval$[x_{i-1}, x_i]$,$S_i(x)$ is a polynomial of at most degree five;
**2.**The first,second,third and fourth derivatives of $S_i(x)$are continuing on $[x_{i-1}, x_i] \subseteq [a,b]$ which are denoting the following:

$$\begin{aligned} &(\text{i})S_i'(x_i^-) = S_i'(x_i^+), && (\text{ii})S_i''(x_i^-) = S_i''(x_i^+), \\ &(\text{iii})S_i'''(x_i^-) = S_i'''(x_i^+), && (\text{ii})S_i^{(4)}(x_i^-) = S_i^{(4)}(x_i^+), \end{aligned} \tag{13}$$

More ever we denote:

$$\begin{aligned} &(\text{i})S_i(x_{i-1}) = u_{i-1}, && (\text{ii})S_i(x_i) = u_i, \\ &(\text{iii})S_i''(x_{i-1}) = M_{i-1}, && (\text{iv})S_i''(x_i) = M_i, \\ &(\text{v})S_i^{(4)}(x_{i-1}) = F_{i-1}, && (\text{vi})S_i^{(4)}(x_i) = F_i, \end{aligned} \tag{14}$$

The spline function $S_i(x)$ for $x \in [x_{i-1}, x_i]$ is defined by:

$$\begin{aligned} S_i(x) &= (\frac{x-x_{i-1}}{h})u_i + (\frac{x_i - x}{h})u_{i-1} \\ &+ \frac{h^2}{3!}[(\frac{x-x_{i-1}}{h})^3 M_i - (\frac{x-x_{i-1}}{h})M_i + (\frac{x_i-x}{h})^3 M_{i-1} - (\frac{x_i-x}{h})M_{i-1}] \\ &+ \frac{h^4}{5!}[((\frac{x-x_{i-1}}{h})^5 - \frac{10}{3}(\frac{x-x_{i-1}}{h})^3 + \frac{7}{3}(\frac{x-x_{i-1}}{h}))F_i + (\frac{x-x_{i-1}}{h})F_{i-1}]. \end{aligned} \quad (15)$$

By the continuity of the first and third derivatives and eliminating$M_i$'s we obtain the following useful relation.

$$\begin{aligned} &u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2} \\ &- \frac{1}{120}h^4(F_{i-2} + 26F_{i-1} + 66F_i + 26F_{i+1} + F_{i+2}) + \tau_{F_i} = 0, \quad i = 3, ..., N-3, \end{aligned} \quad (16)$$

and from equation (2) we obtain:

$$u^{(4)}(x) = x^{-\alpha}(f(x) - a.u(x) - 2.\alpha.x^{(\alpha-1)}.u'''(x) - \alpha.(\alpha-1).x^{(\alpha-2)}.u''(x)), \quad (17)$$

by substituting Equation (17) into (16) we obtain the main relation of the method.To obtain the unique solution we need to associated four more equations with (16) .Since we have $u''(x)$ and $u'''(x)$ in (17) we have to approximate them.We use Taylor series and method of undetermined coefficients that:

$$\begin{aligned} &u_1 - 2u_2 + u_3 - h^2M_0 - 2h^3T_0 - \frac{h^4}{120}(321F_1 - 72F_2 + F_3) - \tau_{F_1} = 0, \\ \Rightarrow &-2u_1 + 5u_2 - 4u_3 + u_4 + h^2M_0 + h^3T_0 \\ &- \frac{h^4}{120}(-\frac{379}{3}F_1 + \frac{581}{3}F_2 - \frac{55}{3}F_3 + F_4) - \tau_{F_2} = 0, \\ \Rightarrow &u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2} \\ &- \frac{h^4}{120}(F_{i-2} + 26F_{i-1} + 66F_i + 26F_{i+1} + F_{i+2}) - \tau_{F_i} = 0, \quad i = 3(1)N-3 \\ \Rightarrow &-2u_{N-1} + 5u_{N-2} - 4u_{N-3} + u_{N-4} + h^2M_N + h^3T_N \\ &- \frac{h^4}{120}(-\frac{379}{3}F_{N-1} + \frac{581}{3}F_{N-2} - \frac{55}{3}F_{N-3} + F_{N-4}) - \tau_{F_{N-2}} = 0, \\ \Rightarrow &u_{N-1} - 2u_{N-2} + u_{N-3} - h^2M_N - 2h^3T_N \\ &- \frac{h^4}{120}(321F_{N-1} - 72F_{N-2} + F_{N-3}) - \tau_{F_{N-1}} = 0. \end{aligned} \quad (18)$$

Now for approximation $u''(x)$ by same manner so that the following system :

$$u_1 - 2u_2 + u_3 - \frac{h^2}{20}(M_1 + 9M_2 + M_3) - \tau_{M_1} = 0,$$
$$-2u_1 + 5u_2 - 4u_3 + u_4 - \frac{h^2}{20}(-4M_1 - 101M_2 + 46M_3 + M_4) - \tau_{M_2} = 0,$$
$$u_{i-2} + 2u_{i-1} - 6u_i + 2u_{i+1} + u_{i+2} - \frac{h^2}{20}(M_{i-2} + 26M_{i-1} + 66M_i + 26M_{i+1} + M_{i+2}) - \tau_{M_i} = 0,$$
$$-2u_{N-1} + 5u_{N-2} - 4u_{N-3} + u_{N-4} - \frac{h^2}{20}(-4M_{N-1} - 101M_{N-2} + 46M_{N-3} + M_{N-4})$$
$$-\tau_{M_{N-2}} = 0,$$
$$u_{N-1} - 2u_{N-2} + u_{N-3} - \frac{h^2}{20}(M_{N-1} + 9M_{N-2} + M_{N-3}) - \tau_{M_{N-1}} = 0. \tag{19}$$

And the approximation of $u'''(x)$:

$$u_1 - 2u_2 + u_3 - h^2M_0 - \frac{43h^3}{120}T_0 - \frac{h^3}{60}(73T_1 + \frac{49}{2}T_2 + T_3) - \tau_{T_1} = 0,$$
$$-2u_1 + 5u_2 - 4u_3 + u_4 + h^2M_0 + \frac{59}{180}h^3T_0 - \frac{h^3}{60}(\frac{-159}{2}T_1 + 14T_2 + \frac{145}{6}T_3 + T_4) - \tau_{T_2} = 0,$$
$$-u_{i-2} + 2u_{i-1} - 2u_{i+1} + u_{i+2}$$
$$-\frac{h^3}{60}(T_{i-2} + 26T_{i-1} + 66T_i + 26T_{i+1} + T_{i+2}) - \tau_{T_i} = 0, \quad i = 3(1)N-3$$
$$-2u_{N-1} + 5u_{N-2} - 4u_{N-3} + u_{N-4} + h^2M_N + \frac{59h^3}{180}T_N - \frac{h^3}{60}(-\frac{159}{2}T_{N-1} + 14T_{N-2}$$
$$+\frac{145}{6}T_{N-3} + T_{N-4}) - \tau_{T_{N-2}} = 0,$$
$$u_{N-1} - 2u_{N-2} + u_3 - h^2M_N - \frac{43h^3}{120}T_N - \frac{h^3}{60}(73T_{N-1} + \frac{49}{2}T_{N-2} + T_{N-3}) - \tau_{T_{N-1}} = 0. \tag{20}$$

The above system of (17), (18),(19),(20) can be dented in the Matrix form as follows:

$$F = x^{-\alpha}(G - aU - 2\alpha x^{\alpha-1}T - \alpha(\alpha-1)x^{\alpha-2}M), \tag{21}$$

$$A_1U - h^2W_1M_0^* - h^3V_1T_0^* - \frac{h^4}{120}C_1F - R_F = 0, \tag{22}$$

$$A_2U - \frac{h^2}{20}C_2M - R_M = 0, \tag{23}$$

$$A_3U - h^2W_3M_0^* - h^3V_3T_0^* - \frac{h^3}{60}C_3T - R_T = 0, \tag{24}$$

When:

$$A_1 = \begin{pmatrix} 1 & -2 & 1 & & & & & & 0 \\ -2 & 5 & -4 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ & & \cdot & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & -4 & 5 & -2 \\ 0 & & & & & & 1 & -2 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & -2 & 1 & & & & & & 0 \\ -2 & 5 & -4 & 1 & & & & & \\ 1 & 2 & -6 & 2 & 1 & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & 1 & 2 & -6 & 2 & 1 \\ & & & & & 1 & -4 & 5 & -2 \\ 0 & & & & & & 1 & -2 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & -2 & 1 & & & & & & 0 \\ -2 & 5 & -4 & 1 & & & & & \\ -1 & 2 & 0 & -2 & 1 & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & -1 & 2 & 0 & -2 & 1 \\ & & & & & 1 & -4 & 5 & -2 \\ 0 & & & & & & 1 & -2 & 1 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 321 & -72 & 1 & & & & & & 0 \\ -\frac{379}{3} & \frac{581}{3} & -\frac{55}{3} & 1 & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & & \cdot & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 1 & -\frac{55}{3} & \frac{581}{3} & -\frac{379}{3} \\ 0 & & & & & & 1 & -72 & 321 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 1 & 9 & 1 & & & & & & 0 \\ -4 & -101 & 46 & 1 & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 1 & 46 & -101 & -4 \\ 0 & & & & & & 1 & 9 & 1 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} 73 & \frac{49}{2} & 1 & & & & & & 0 \\ -\frac{159}{2} & 14 & \frac{145}{6} & 1 & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 1 & \frac{145}{6} & 14 & -\frac{159}{2} \\ 0 & & & & & & 1 & \frac{49}{2} & 73 \end{pmatrix}$$

$$
\begin{aligned}
&G = (f_1, ..., f_{N-2}, f_{N-1})^T,\\
&U = (u_1, u_2, ..., u_{N-2}, u_{N-1})^T,\\
&F = (u_1^{(4)}, u_2^{(4)}, ..., u_{N-2}^{(4)}, u_{N-1}^{(4)})^T = (F_1, F_2, ..., F_{N-2}, F_{N-1})^T,\\
&M = (u_1^{(2)}, u_2^{(2)}, ..., u_{N-2}^{(2)}, u_{N-1}^{(2)})^T = (M_1, M_2, ., ., ., M_{N-2}, M_{N-1})^T,\\
&T = (u_1^{(3)}, u_2^{(3)}, ..., u_{N-2}^{(3)}, u_{N-1}^{(3)})^T = (T_1, T_2, ., ., ., ., T_{N-2}, T_{N-1})^T,\\
&M_0^* = (u_0'', u_0'', 0, ..., 0, u_N'', u_N'')^T = (M_0, M_0, ., ., ., M_N, M_N)^T,\\
&T_0^* = (u_0''', u_0''', 0, ..., 0, u_N''', u_N''')^T = (T_0, T_0, ., ., ., ., T_N, T_N)^T,\\
&W_1 = (1, -1, 0, ..., 0, -1, 1)^T,\\
&W_3 = (1, -1, 0, ..., 0, -1, 1)^T,\\
&V_1 = (2, -1, 0, ..., 0, -1, 2)^T,\\
&V_3 = (\frac{43}{120}, -\frac{59}{180}, 0, ..., 0, -\frac{59}{180}, \frac{43}{120})^T,\\
&R_T = (\tau_{T_1}, \tau_{T_2}, ..., \tau_{T_{N-2}}, \tau_{T_{N-1}})^T,\\
&R_M = (\tau_{M_1}, \tau_{M_2}, ..., \tau_{M_{N-2}}, \tau_{M_{N-1}})^T,\\
&R_F = (\tau_{F_1}, \tau_{F_2}, ..., \tau_{F_{N-2}}, \tau_{F_{N-1}})^T,
\end{aligned}
$$

By substituting Equations (21),(23),(24) into (22) the main method has the following Matrix form:

$$
\mathbf{A}U - \mathbf{B} - \tau_{Error} = 0, \tag{25}
$$

Where:

$$
\begin{aligned}
&\mathbf{A} = A_1 + \frac{h^2\alpha C_1}{6x^2C_2}(\alpha-1)A_2 + \frac{h\alpha C_1}{xC_3}A_3 + \frac{h^4 aC_1}{120x^\alpha},\\
&\mathbf{B} = \frac{h^4 C_1}{120x^\alpha}G + (\frac{h^3\alpha C_1}{xC_3}W_3 + h^2W_1)M_0^* + (\frac{h^4\alpha C_1}{xC_3}V_3 + h^3V_1)T_0^*,\\
&\tau_{Error} = (\frac{h\alpha C_1}{xC_3})R_T + \frac{h^2\alpha C_1}{6x^2C_2}(\alpha-1)R_M + R_F.
\end{aligned} \tag{26}
$$

## 4. Error Analysis

The Local Truncation error of the given method can be obtained as:

$$\tau_M = \begin{cases} \tau_{M_1} = -\frac{1}{240}h^6.u^{(6)}(\varepsilon_1) & i = 1, \\ \tau_{M_2} = \frac{1}{240}h^7.u^{(7)}(\varepsilon_2) & i = 2, \\ \tau_{M_i} = \frac{1}{120}h^6.u^{(6)}(\varepsilon_i) & i = 3, ..., N-3, \\ \tau_{M_{N-2}} = \frac{1}{240}h^7.u^{(7)}(\varepsilon_{N-2}) & i = N-2, \\ \tau_{M_{N-1}} = -\frac{1}{240}h^6.u^{(6)}(\varepsilon_{N-1}) & i = N-1, \end{cases} \tag{27}$$

$$\tau_T = \begin{cases} \tau_{T_1} = \frac{17}{360}h^6.u^{(6)}(\varepsilon_1) & i = 1, \\ \tau_{T_2} = \frac{139}{180}h^6.u^{(6)}(\varepsilon_2) & i = 2, \\ \tau_{T_i} = -\frac{1}{120}h^7.u^{(7)}(\varepsilon_i) & i = 3, ..., N-3, \\ \tau_{T_{N-2}} = \frac{139}{180}h^6.u^{(6)}(\varepsilon_{N-2}) & i = N-2, \\ \tau_{T_{N-1}} = \frac{17}{360}h^6.u^{(6)}(\varepsilon_{N-1}) & i = N-1, \end{cases} \tag{28}$$

$$\tau_F = \begin{cases} \tau_{F_1} = \frac{61}{90}h^6.u^{(6)}(\varepsilon_1) & i = 1, \\ \tau_{F_2} = \frac{133}{72}h^6.u^{(6)}(\varepsilon_2) & i = 2, \\ \tau_{F_i} = -\frac{1}{12}h^6.u^{(6)}(\varepsilon_i) & i = 3, ..., N-3, \\ \tau_{F_{N-2}} = \frac{133}{72}h^6.u^{(6)}(\varepsilon_{N-2}) & i = N-2, \\ \tau_{F_{N-1}} = \frac{61}{90}h^6.u^{(6)}(\varepsilon_{N-1}) & i = N-1, \end{cases} \tag{29}$$

Where:

$$\begin{aligned} & x_0 < \varepsilon_1 < x_1, \\ & x_1 < \varepsilon_2 < x_2, \\ & x_{i-1} < \varepsilon_i < x_i, \\ & x_{N-2} < \varepsilon_{N-1} < x_{N-1}, \\ & x_{N-1} < \varepsilon_N < x_N. \end{aligned} \tag{30}$$

## 5. Numerical Illustration

In order to test the utility of the proposed method we have solved the following example. The exact solution is known and the maximum absolute error in the solution is tabulated.

**Example :**
Consider the linear boundary value problem :

$$\begin{aligned} & (u''(x))'' - u(x) = x^3 - x^2, \\ & u''_0 = 0, \qquad u''_\pi = 0, \\ & u'''_0 = 0, \qquad u'''_\pi = 0. \end{aligned}$$

With the Exact solution

$$u(x) = -x^3 + x^2 + \frac{(-2 - 6e^{\pi} + 6e^{-\pi} + 2e^{-\pi}e^{\pi} + 3e^{-\pi}\pi + 6\pi + 3\pi e^{\pi})\cos(x)}{(1+e^{\pi})(1+e^{-\pi})}$$
$$- \frac{(-2e^{\pi} + 3\pi e^{\pi} + 2e^{-\pi} - 3e^{-\pi}\pi - 6 + 6e^{-\pi}e^{\pi})\sin(x)}{(1+e^{\pi})(1+e^{-\pi})} + \frac{(-8+3\pi)e^{-x}}{1+e^{-\pi}} + \frac{(4+3\pi)e^{x}}{1+e^{\pi}}, \tag{31}$$

This problem has been solved using the Quintic spline with different values of $N$ the maximum absolute errors in the solution are computed and tabulated in the table.

**Table 1.** The maximum absolute errors in the solution of Example

| N | step lengths | Absolu Error in solution |
|---|---|---|
| 50 | $\frac{\pi}{50}$ | 3.01(-2) |
| 75 | $\frac{\pi}{75}$ | 1.66(-3) |
| 100 | $\frac{\pi}{100}$ | 7.8(-4) |

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