

Some properties of band matrix and its application to the numerical solution one-dimensional Bratu's problem

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Abstract. A Class of new methods based on a septic non-polynomial spline function for the numerical solution one-dimensional Bratu's problem are presented. The local truncation errors and the methods of order 2th, 4th, 6th, 8th, 10th, and 12th, are obtained. The inverse of some band matrixes are obtained which are required in proving the convergence analysis of the presented method. Associated boundary formulas are developed. Convergence analysis of these methods is discussed. Numerical results are given to illustrate the efficiency of methods.

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1. Introduction

Consider the Liouville-Bratu-Gelfand equation [8], [7], [23], [21], [22]

$$\begin{cases} \Delta u(x) + \lambda e^{u(x)} = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

where $\lambda > 0$, and Ω is a bounded domain. We consider the classical Bratu's problem given by the following boundary value problem

$$u''(x) + \lambda e^{u(x)} = 0, \quad u(0) = u(1) = 0, \quad 0 \leq x \leq 1. \quad (2)$$

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The Bratu's problem in one-dimensional planar coordinates (2) has analytical solution in the following form:

$$u(x) = -2 \ln \left[\frac{\cosh\left(\left(x - \frac{1}{2}\right)\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)} \right], \quad (3)$$

where θ is the solution of $\theta = \sqrt{2\lambda} \cosh\left(\frac{\theta}{4}\right)$.

The Bratu's problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ and $\lambda < \lambda_c$ respectively, where the critical value λ_c satisfies the equation $1 = \frac{1}{4}\sqrt{2\lambda_c} \sinh\left(\frac{\theta}{4}\right)$ and it was evaluated in [7],[23], [4], [10]. Also the critical value λ_c is given by $\lambda = 3.513830719$. Bratu's problem is also used in a large variety of applications such as the fuel ignition model of the thermal combustion theory, the model of thermal reaction process, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity about the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology [7], [21], [22],[17], [29], [19], [36], [14], [20].

Many authors obtained analytical and numerical methods for the solution of (2). For example Hikmet Caglar [8] have developed B-spline method, Buckmire and Mounim et al. [7], [22] have used finite difference method, Deeba et al. [9] developed decomposition method, Khuri [18] and Syam et al. [29] have been used Laplace transform decomposition method, Li and S.J. Liao [19] developed homotopy analysis method, Wazwaz [36] have been analyzed Adomian decomposition method, Aregbesola [3] used weighted residual method, Hassan and Erturk [12] applied differential transformation method and J.H. He [14] used He's variational method to solve the Bratu's problem. Rashidinia et al. [24], [25] used non-polynomial spline methods for the solution of fourth and two-order boundary-value problems. Some authors such as Van Daele et al. [35], Ramadan et al. [27], Siraj-ul-Islam et al. [34] and Akram and Siddiqi [2] have been analyzed and developed non-polynomial splines approach to the solution of boundary value problems. Mohsen et al. [23] have been obtained new smoother to enhance multigrid-based methods for Bratu problem. Tirmizi and Twizell [31] have been developed higher-order finite-difference methods for nonlinear second-order two-point boundary-value problems. Usmani and Sakai [33] obtained a connection between quartic spline and Numerov solution of a boundary value problem. Boutayeb and Twizell [5], [30], [6] have been analyzed and developed numerical methods for the solution of sixth-order and eighth-order boundary-value problems. J.H. He [13] has been developed variational method to find the two branch solutions and identify the bifurcation point of (2).

The basic motivation of this paper is discussed convergence analysis of the non-polynomial spline for solutions one-dimensional Bratu problem. Ghazala Akram and Shahid S. Siddiqi [1] used septic spline for interpolation at equally spaced knots along with the end conditions and lead to uniform convergence of $O(h^8)$ and also Ramadan et al. [26] used septic spline for the numerical solution of the sixth-order linear boundary value problems and also discussed convergence analysis of the method. R. Jalilian used non-polynomial quintic spline function to the approximate solutions of the one-dimensional Bratus problem and obtained the method of order $O(h^6)$.

The paper is organized in four sections. We use the consistency relation of non-polynomial septic spline for approximate the solution of (2). section 2 is devoted to the description of the method and development of boundary conditions and also we obtain the methods of order 2th, 4th, 6th, 8th, 10th, and 12th. The new approach for convergence analysis is discussed in section 3. Finally, in section 4, numerical evidences are included to show the practical applicability and superiority of our method and compare with the other

methods.

2. Description of the method and development of boundary conditions

Let us consider a mesh with nodal points x_i on $[a, b]$ such that:

$$\Delta : a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b,$$

where $h = \frac{b-a}{n}$ for $i = 1(1)n$. For each segment $[x_i, x_{i+1}]$, $i = 0, 1, 2, \dots, n - 1$ by using non-polynomial spline [26] we have

$$\alpha_1(M_{i-3} + M_{i+3}) + \alpha_2(M_{i-2} + M_{i+2}) + \alpha_3(M_{i-1} + M_{i+1}) + \alpha_4 M_i - [(u_{i+3} + u_{i-3}) + (\beta_1)(u_{i+2} + u_{i-2}) + (\beta_2)(u_{i+1} + u_{i-1}) + (\beta_3)u_i] \frac{1}{h^2}, i = 3, \dots, n - 3, \tag{4}$$

where

$$\begin{aligned} \alpha_1 &= \frac{(120\theta - 20\theta^3 + \theta^5 - 120\text{Sin}[\theta])}{20\theta^2(-6\theta + \theta^3 + 6\text{Sin}[\theta])}, \\ \alpha_2 &= \frac{-2(240\theta + 20\theta^3 - 13\theta^5 + \theta(120 - 20\theta^2 + \theta^4)\text{Cos}[\theta] - 360\text{Sin}[\theta])}{20\theta^2(-6\theta + \theta^3 + 6\text{Sin}[\theta])}, \\ \alpha_3 &= \frac{(840\theta + 100\theta^3 + 67\theta^5 + (960\theta + 80\theta^3 - 52\theta^5)\text{Cos}[\theta] - 1800\text{Sin}[\theta])}{20\theta^2(-6\theta + \theta^3 + 6\text{Sin}[\theta])}, \\ \alpha_4 &= -\frac{4(240\theta + 20\theta^3 - 13\theta^5 + 3\theta(120 + 20\theta^2 + 11\theta^4)\text{Cos}[\theta] - 600\text{Sin}[\theta])}{20\theta^2(-6\theta + \theta^3 + 6\text{Sin}[\theta])}, \\ \beta_1 &= -\frac{40\theta^2(-\theta(12 + \theta^2) + \theta(-6 + \theta^2)\text{Cos}[\theta] + 18\text{Sin}[\theta])}{20\theta^2(-6\theta + \theta^3 + 6\text{Sin}[\theta])}, \\ \beta_2 &= -\frac{20\theta^2(42\theta + 5\theta^3 + 4\theta(12 + \theta^2)\text{Cos}[\theta] - 90\text{Sin}[\theta])}{20\theta^2(-6\theta + \theta^3 + 6\text{Sin}[\theta])}, \\ \beta_3 &= \frac{80\theta^2(12\theta + \theta^3 + 3\theta(6 + \theta^2)\text{Cos}[\theta] - 30\text{Sin}[\theta])}{20\theta^2(-6\theta + \theta^3 + 6\text{Sin}[\theta])}, \end{aligned}$$

By expanding (4) in Taylor series about x_i , we obtain the following local truncation error:

$$\begin{aligned} t_i &= -(2 + 2\beta_1 + 2\beta_2 + \beta_3) u_i + h^2 u_i^{(2)} (-9 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 - 4\beta_1 - \beta_2) \\ &\quad + \frac{1}{12} h^4 u_i^{(4)} (-81 + 108\alpha_1 + 48\alpha_2 + 12\alpha_3 - 16\beta_1 - \beta_2) \\ &\quad + \frac{1}{360} h^6 u_i^{(6)} (-729 + 2430\alpha_1 + 480\alpha_2 + 30\alpha_3 - 64\beta_1 - \beta_2) \\ &\quad + \frac{1}{20160} h^8 u_i^{(8)} (-6561 + 40824\alpha_1 + 3584\alpha_2 + 56\alpha_3 - 256\beta_1 - \beta_2) \\ &\quad + \frac{h^{10}}{1814400} u_i^{(10)} (-59049 + 590490\alpha_1 + 23040\alpha_2 + 90\alpha_3 - 1024\beta_1 - \beta_2) \end{aligned}$$

$$\begin{aligned}
& + \frac{h^{12}}{239500800} u_i^{(12)} (-531441 + 7794468\alpha_1 + 135168\alpha_2 + 132\alpha_3 - 4096\beta_1) \\
& + \frac{h^{14} u_i^{(14)}}{43589145600} (-4782969 + 96722262\alpha_1 + 745472\alpha_2 + 182\alpha_3 - 16384\beta_1 - \beta_2) \\
& + O(h^{15}). \tag{5}
\end{aligned}$$

By using the above truncation error to eliminate the coefficients of various powers h we can obtain classes of the methods. For different choices of parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2$ and β_3 , we get the class of methods such as:

(I) *Second-order method.* For $\alpha_1 = 50, \alpha_2 = 10, \alpha_3 = \alpha_4 = 0, \beta_1 = 24, \beta_2 = 15$ and $\beta_3 = -80$, we have

$$t_i = 450h^4 u_i^{(4)} + O(h^5).$$

(II) *Fourth-order method.* For $\alpha_1 = 50, \alpha_2 = 10, \alpha_3 = -450, \alpha_4 = 900, \beta_1 = 24, \beta_2 = 15$, and $\beta_3 = -80$, we have

$$t_i = 307h^6 u_i^{(6)} + O(h^7).$$

(III) *Sixth-order method.* For $\alpha_1 = \frac{1}{42}, \alpha_2 = \frac{20}{7}, \alpha_3 = \frac{397}{14}, \alpha_4 = \frac{1208}{21}, \beta_1 = 24, \beta_2 = 15$, and $\beta_3 = -80$, we have

$$t_i = \frac{h^8 u_i^{(8)}}{252} + O(h^9).$$

(IV) *Eighth-order method.* For $\alpha_1 = \frac{337}{4927}, \alpha_2 = \frac{39712}{44343}, \alpha_3 = \frac{16285}{44343}, \alpha_4 = 0, \beta_1 = \frac{-33536}{14781}, \beta_2 = \frac{18755}{14781}$, and $\beta_3 = 0$, we have

$$t_i = \frac{65629h^{10} u_i^{(10)}}{27936090} + O(h^{11}).$$

(V) *Tenth-order method.* For $\alpha_1 = \frac{2867}{62559}, \alpha_2 = \frac{34180}{20853}, \alpha_3 = \frac{158339}{20853}, \alpha_4 = \frac{525032}{62559}, \beta_1 = \frac{6272}{993}, \beta_2 = -\frac{7265}{993}$, and $\beta_3 = 0$, we have

$$t_i = \frac{3319h^{12} u_i^{(12)}}{35390520} + O(h^{13}).$$

(VI) *Twelve-order method.* For $\alpha_1 = \frac{1857}{49483}, \alpha_2 = \frac{110322}{49483}, \alpha_3 = \frac{989739}{49483}, \alpha_4 = \frac{2175924}{49483}, \beta_1 = \frac{112266}{7069}, \beta_2 = \frac{112995}{7069}$, and $\beta_3 = -\frac{464660}{7069}$, we have

$$t_i = \frac{114669h^{14} u_i^{(14)}}{19812993200} + O(h^{15}).$$

we assume that

$$M_i = -\lambda e^{u_i}, \tag{6}$$

where $M_i = S_i''(x_i)$, u_i is the approximation of the exact value $u(x_i)$ and $S_i(x)$ is non-polynomial septic spline function [26]. By substituting (6) in the spline relation (4), we obtain the nonlinear equations in the following form.

$$\begin{aligned} &\alpha_1(\lambda e^{u_{i-3}} + \lambda e^{u_{i+3}}) + \alpha_2(\lambda e^{u_{i-2}} + \lambda e^{u_{i+2}}) + \alpha_3(\lambda e^{u_{i-1}} + \lambda e^{u_{i+1}}) + \alpha_4 \lambda e^{u_i} \\ &+ \frac{1}{h^2}((u_{i+3} + u_{i-3}) + (\beta_1)(u_{i+2} + u_{i-2}) + (\beta_2)(u_{i+1} + u_{i-1}) + (\beta_3)u_i) = 0, \\ &i = 3, \dots, n - 3. \end{aligned} \tag{7}$$

To obtain unique solution for the nonlinear system (7) we need four more equations. Following [15] we define the following identities:

$$\begin{aligned} &\sum_{k=0}^4 \gamma_k u_k + h^2 \sum_{k=1}^{12} \eta_k u_k'' + t_1 h^{14} u_0^{(14)} = 0, \quad i = 1, \\ &\sum_{k=0}^5 \mu_k u_k + h^2 \sum_{k=1}^{12} \sigma_k u_k'' + t_2 h^{14} u_0^{(14)} = 0, \quad i = 2, \\ &\sum_{k=0}^5 \mu_k u_{n-k} + h^2 \sum_{k=1}^{12} \sigma_k u_{n-k}'' + t_{n-2} h^{14} u_0^{(14)} = 0, \quad i = n - 2, \\ &\sum_{k=0}^4 \gamma_k u_{n-k} + h^2 \sum_{k=1}^{12} \eta_k u_{n-k}'' + t_{n-1} h^{14} u_0^{(14)} = 0, \quad i = n - 1, \end{aligned} \tag{8}$$

by using Taylor 's expansion we obtain the unknown coefficients in (8) as follows:

$$\begin{aligned} &(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4) = (65, -104, 14, 24, 1), \quad t_1 = t_{n-1} = \left(\frac{1210210269217}{326918592000}\right), \\ &(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8, \eta_9, \eta_{10}, \eta_{11}, \eta_{12}) = \left(\frac{-2248215317}{19958400}, \frac{4539179}{17600}, \right. \\ &\left. \frac{-6055918291}{6652800}, \frac{9918918899}{4989600}, \frac{-892246279}{285120}, \frac{18019157507}{4989600}, \frac{-30650022317}{9979200}, \right. \\ &\left. \frac{2380569353}{1247400}, \frac{-16851712321}{19958400}, \frac{503717713}{1995840}, \frac{-130447723}{2851200}, \frac{18976637}{4989600}\right), \\ &(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5,) = (26, 14, -80, 15, 24, 1), \quad t_2 = t_{n-2} = \left(\frac{521085679991}{373621248000}\right), \\ &(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}) = \left(\frac{-7712745923}{159667200}, \right. \end{aligned}$$

$$\frac{259885933}{5702400}, \frac{-19222747601}{53222400}, \frac{8351791919}{11404800}, \frac{-3712133107}{3193344}, \frac{95883047}{71280},$$

$$\frac{-2614370381}{2280960}, \frac{28466192407}{39916800}, \frac{-7204033313}{22809600}, \frac{16761737}{177408}, \frac{-43435489}{2534400},$$

$$\frac{113795873}{79833600}.$$

3. Convergence analysis

In this section, we investigate the convergence analysis of the sixth-order method and also in the same way we can prove the convergence analysis for any of the other methods. The equations (7) along with boundary condition (8) yields nonlinear system of equations, and may be written in a matrix form as

$$A_0 U^{(1)} + \lambda h^2 B \mathbf{f}^{(1)}(U^{(1)}) = R^{(1)}, \quad (9)$$

$$(\text{where } \mathbf{f}^{(1)}(U^{(1)}) = e^{U^{(1)}} = (e^{u_1^{(1)}}, \dots, e^{u_{n-1}^{(1)}})^t,$$

the matrices A_0 and B are an $(n-1) \times (n-1)$ -dimensional which have the following forms

$$A_0 = -P_{n-1}^3(1, 2, 1) + 30P_{n-1}^2(1, 2, 1) - 120P_{n-1}(1, 2, 1), \quad (10)$$

where

$$P_{n-1}(x, z, y) = \begin{pmatrix} z & -y & & & \\ -x & z & -y & & \\ & \ddots & \ddots & \ddots & \\ & & -x & z & -y \\ & & & -x & z \end{pmatrix}, \quad (11)$$

$$B =$$

$$\begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \eta_8 & \eta_9 & \eta_{10} & \eta_{11} & \eta_{12} \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & & & & & & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & & \\ & & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 & \alpha_2 & & \\ \sigma_{12} & \sigma_{11} & \sigma_{10} & \sigma_9 & \sigma_8 & \sigma_7 & \sigma_6 & \sigma_5 & \sigma_4 & \sigma_3 & \sigma_2 & \sigma_1 \\ \eta_{12} & \eta_{11} & \eta_{10} & \eta_9 & \eta_8 & \eta_7 & \eta_6 & \eta_5 & \eta_4 & \eta_3 & \eta_2 & \eta_1 \end{bmatrix}. \tag{12}$$

$$R^{(1)} = \begin{pmatrix} -65u_0, \\ -26u_0, \\ -u_0 - \lambda\alpha_1, \\ 0 \\ \vdots \\ 0 \\ -u_n - \lambda\alpha_1, \\ -26u_0, \\ -65u_0, \end{pmatrix}. \tag{13}$$

We assume that

$$A_0\bar{U}^{(1)} + \lambda h^2 B\mathbf{f}^{(1)}(\bar{U}^{(1)}) = R^{(1)} + t^{(1)}, \tag{14}$$

where the vector $\bar{U}^{(1)} = u(x_i), (i = 1, 2, \dots, n - 1)$, is the exact solution and $t^{(1)} = [t_1, t_2, \dots, t_{n-1}]^T$, is the vector of local truncation error.

By using (9) and (14) we get

$$AE^{(1)} = [A_0 + \lambda h^2 BF_k(U^{(1)})]E^{(1)} = t^{(1)}, \tag{15}$$

where

$$E^{(1)} = \bar{U}^{(1)} - U^{(1)},$$

$$\mathbf{f}^{(1)}(\bar{U}^{(1)}) - \mathbf{f}^{(1)}(U^{(1)}) = F_k(U^{(1)})E^{(1)}, \tag{16}$$

and $F_k(U^{(1)}) = \text{diag}\{\frac{\partial e^{u_i^{(1)}}}{\partial u_i^{(1)}}\}, (i = 1, 2, \dots, n - 1)$, is a diagonal matrix of order $n - 1$.

To prove the existence of A^{-1} , since $A = A_0 + h^4BF_k(U^{(1)})$, we have to show $A_0 = -P_{n-1}^3(1, 2, 1) + 30P_{n-1}^2(1, 2, 1) - 120P_{n-1}(1, 2, 1)$, is nonsingular.

by using Henrici [11] we have

$$\|(P_{n-1}(1, 2, 1))^{-1}\| \leq \frac{(b - a)^2}{8h^2}. \tag{17}$$

It is clear that the matrix A_0 is nonsingular and also $\|A_0^{-1}\| < \omega$ where ω is a positive number ($\|\cdot\|$ is the L_∞ norm).

Several results can now be derived directly from the Eq. (11). We shall now consider the inversion of the product of matrices $P_{n-1}(-1, z, -1)$ and $P_{n-1}(1, z', 1)$. We first note that the matrices $P_{n-1}(-1, z, -1)$ and $P_{n-1}(1, z', 1)$ are commuted. By considering $F(z) = P_{n-1}(-1, z, -1)$ and $G(z') = P_{n-1}(1, z', 1)$ then we have the following lemmas [16].

Lemma 1. If we consider the matrices $F(z)$ and $G(z')$ then:

$$(F^n(z)G^n(z'))^{-1} = \frac{1}{(z+z')^n} \sum_{i=0}^n \binom{n}{i} F^{-(n-i)}(z)G^{-i}(z'), z \neq -z'. \quad (18)$$

Proof:

$$[(F^n(z)G^n(z'))] \left[\frac{1}{(z+z')^n} \sum_{i=0}^n \binom{n}{i} F^{-(n-i)}(z)G^{-i}(z') \right] = \frac{1}{(z+z')^n}$$

$$\left[\sum_{i=0}^n \binom{n}{i} (F^n(z)G^n(z')) F^{-(n-i)}(z)G^{-i}(z') \right] = \frac{1}{(z+z')^n}$$

$$\left[\sum_{i=0}^n \binom{n}{i} (G^{n-i}(z')F^i(z)) \right] = \frac{1}{(z+z')^n} [(z+z')^n I_n] = I_n, z \neq -z'.$$

And similarly

$$\frac{1}{(z+z')^n} \left[\sum_{i=0}^n \binom{n}{i} F^{-(n-i)}(z)G^{-i}(z') \right] [(F^n(z)G^n(z'))] = \frac{1}{(z+z')^n}$$

$$\left[\sum_{i=0}^n \binom{n}{i} F^{(i)}(z)G^{(n-i)}(z') \right] = \frac{1}{(z+z')^n} [(z+z')^n I_n] = I_n, z \neq -z'.$$

This completes the proof of the Lemma 1.

Lemma 2. By consider the matrices $F(z)$ and $G(z')$ then we have:

$$(F^n(z)F^n(z'))^{-1} = \frac{1}{(z-z')^n} \sum_{i=0}^n (-1)^i \binom{n}{i} F^{-(n-i)}(z')F^{-i}(z), z \neq z'.$$

Proof:

$$[(F^n(z)F^n(z'))] \left[\frac{1}{(z-z')^n} \sum_{i=0}^n (-1)^i \binom{n}{i} F^{-(n-i)}(z')F^{-i}(z) \right] =$$

$$\frac{1}{(z - z')^n} \left[\sum_{i=0}^n (-1)^i \binom{n}{i} (F^n(z)F^n(z'))F^{-(n-i)}(z')F^{-i}(z) \right] =$$

$$\frac{1}{(z - z')^n} \left[\sum_{i=0}^n (-1)^i \binom{n}{i} (F^i(z')F^{n-i}(z)) \right] =$$

$$\frac{1}{(z - z')^n} [(z - z')^n I_n] = I_n, z \neq z'.$$

And also we get

$$\left[\frac{1}{(z - z')^n} \sum_{i=0}^n (-1)^i \binom{n}{i} F^{-(n-i)}(z')F^{-i}(z) \right] [(F^n(z)F^n(z'))] =$$

$$\frac{1}{(z - z')^n} \left[\sum_{i=0}^n (-1)^i \binom{n}{i} F^{-(n-i)}(z')F^{-i}(z)F^n(z)F^n(z') \right] =$$

$$\frac{1}{(z - z')^n} \left[\sum_{i=0}^n (-1)^i \binom{n}{i} F^{n-i}(z)(F^i(z')) \right] = \frac{1}{(z - z')^n} [(z - z')^n I_n] = I_n, z \neq z'.$$

This completes the proof of the Lemma 2.

Lemma 3. By consider the matrices $F(z)$ and $G(z')$ then we have:

$$(G^n(z)G^n(z'))^{-1} = \frac{1}{(z - z')^n} \sum_{i=0}^n (-1)^i \binom{n}{i} G^{-(n-i)}(z')G^{-i}(z), z \neq z'.$$

Proof:

$$[(G^n(z)G^n(z'))] \left[\frac{1}{(z - z')^n} \sum_{i=0}^n (-1)^i \binom{n}{i} G^{-(n-i)}(z')G^{-i}(z) \right] =$$

$$\frac{1}{(z - z')^n} \left[\sum_{i=0}^n (-1)^i \binom{n}{i} (G^n(z)G^n(z'))G^{-(n-i)}(z')G^{-i}(z) \right] =$$

$$\frac{1}{(z - z')^n} \left[\sum_{i=0}^n (-1)^i \binom{n}{i} (G^i(z')G^{n-i}(z)) \right] =$$

$$\frac{1}{(z - z')^n} [(z - z')^n I_n] = I_n, z \neq z'.$$

And

$$\begin{aligned} & \left[\frac{1}{(z-z')^n} \sum_{i=0}^n (-1)^i \binom{n}{i} G^{-(n-i)}(z') G^{-i}(z) [(G^n(z) G^n(z'))] \right] = \\ & \frac{1}{(z-z')^n} \left[\sum_{i=0}^n (-1)^i \binom{n}{i} G^{-(n-i)}(z') G^{-i}(z) G^n(z) G^n(z') \right] = \\ & \frac{1}{(z-z')^n} \left[\sum_{i=0}^n (-1)^i \binom{n}{i} G^{n-i}(z) (G^i(z')) \right] = \frac{1}{(z-z')^n} [(z-z')^n I_n] = I_n, z \neq z'. \end{aligned}$$

This completes the proof of the Lemma 3.

Theorem 1. If $Y < \frac{1}{\lambda h^2 \|B\| \|A_0^{-1}\|}$, then the matrix A given by (15) is monotone

($Y = \max |\frac{\partial e_i^{u_i(1)}}{\partial u_i^{(1)}}|, i = 1, 2, \dots, n-1$).

Proof: From (15) we have

$$A = A_0 + \lambda h^2 B F_k(U^{(1)}),$$

hence $AA_0^{-1} = I + \lambda h^2 B F_k(U^{(1)}) A_0^{-1}$, so that

$$\begin{aligned} A_0 A^{-1} &= (I + \lambda h^2 B F_k(U^{(1)}) A_0^{-1})^{-1} = \\ &= I - (\lambda h^2 B F_k(U^{(1)}) A_0^{-1}) + (\lambda h^2 B F_k(U^{(1)}) A_0^{-1})^2 - (\lambda h^2 B F_k(U^{(1)}) A_0^{-1})^3 + \dots \\ &= [I - (\lambda h^2 B F_k(U^{(1)}) A_0^{-1})] [I + (\lambda h^2 B F_k(U^{(1)}) A_0^{-1})^2 + (\lambda h^2 B F_k(U^{(1)}) A_0^{-1})^4 + \dots]. \end{aligned}$$

Also if $\rho(\lambda h^2 B F_k(U^{(1)}) A_0^{-1}) < 1$ then, the two infinite series convergence. Let $\|F_k(U^{(1)})\| \leq Y = \max |\frac{\partial e_i^{u_i(1)}}{\partial u_i^{(1)}}|, i = 1, 2, \dots, n-1$, then

$$A^{-1} =$$

$$[A_0^{-1} - A_0^{-1} \lambda h^2 B F_k(U^{(1)}) A_0^{-1}] [I + (\lambda h^2 B F_k(U^{(1)}) A_0^{-1})^2 + (\lambda h^2 B F_k(U^{(1)}) A_0^{-1})^4 + \dots],$$

where the infinite series is nonnegative. Hence to show that A is monotone, it sufficient to show that $[A_0^{-1} - A_0^{-1} \lambda h^2 B F_k(U^{(1)}) A_0^{-1}] > 0$. Here we have

$$A_0^{-1} > A_0^{-1} \lambda h^2 B F_k(U^{(1)}) A_0^{-1} \Rightarrow I > A_0^{-1} \lambda h^2 B F_k(U^{(1)}) \Rightarrow$$

$$\|\lambda h^2 A_0^{-1} B F_k(U^{(1)})\| \leq \lambda h^2 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\| < 1. \quad (19)$$

Then

$$Y < \frac{1}{\lambda h^2 \|B\| \|A_0^{-1}\|}.$$

Theorem 2. Let $u(x)$ be the exact solution of the boundary value problem (2) and assume $u_i, i = 1, 2, \dots, n - 1$, be the numerical solution obtained by solving the system (9). Then we have

$$\|E\| \equiv O(h^6),$$

provided $Y < \frac{17325}{279548636\lambda h^2\omega}$, where

$$\alpha_1 = \frac{1}{42}, \alpha_2 = \frac{20}{7}, \alpha_3 = \frac{397}{14}, \alpha_4 = \frac{1208}{21}, \beta_1 = 24, \beta_2 = 15, \beta_3 = -80,$$

Proof: We can write the error equation (15) in the following form

$$E^{(1)} = (A_0 + \lambda h^2 B F_k(U^{(1)}))^{-1} t^{(1)} = (I + \lambda h^2 A_0^{-1} B F_k(U^{(1)}))^{-1} A_0^{-1} t^{(1)},$$

$$\|E^{(1)}\| \leq \|(I + \lambda h^2 A_0^{-1} B F_k(U^{(1)}))^{-1}\| \|A_0^{-1}\| \|t^{(1)}\|,$$

it follows that

$$\|E^{(1)}\| \leq \frac{\|A_0^{-1}\| \|t^{(1)}\|}{1 - \lambda h^2 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\|}, \tag{20}$$

provided $\lambda h^2 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\| < 1$. Following [16] we have

$$\|t^{(1)}\| \leq \frac{h^8 M_8}{252}, \tag{21}$$

where $M_8 = \max|u^{(8)}(\xi)|, a \leq \xi \leq b$.

From inequalities (20), (21), $\|A_0^{-1}\| < \omega$, $\|F_k(U^{(1)})\| \leq Y$ ($Y = \max|\frac{\partial e_i^{u_i^{(1)}}}{\partial u_i^{(1)}}|, i = 1, 2, \dots, n - 1$.) and $\|B\| \leq \frac{279548636}{17325}$ we obtain

$$\|E\| \leq \frac{17325\omega h^8 M_8}{252(1 - 279548636\lambda h^2\omega Y)} \equiv O(h^6), \tag{22}$$

provided

$$Y < \frac{17325}{279548636\lambda h^2\omega}. \tag{23}$$

Corollary

(i) For $\alpha_1 = 50, \alpha_2 = 10, \alpha_3 = \alpha_4 = 0, \beta_1 = 24, \beta_2 = 15$ and $\beta_3 = -80$, we have

$$\|E\| \equiv O(h^2).$$

(ii) For $\alpha_1 = 50, \alpha_2 = 10, \alpha_3 = -450, \alpha_4 = 900, \beta_1 = 24, \beta_2 = 15$, and $\beta_3 = -80$, we get

$$\|E\| \equiv O(h^4).$$

(iii) For $\alpha_1 = \frac{1}{42}, \alpha_2 = \frac{20}{7}, \alpha_3 = \frac{397}{14}, \alpha_4 = \frac{1208}{21}, \beta_1 = 24, \beta_2 = 15$, and $\beta_3 = -80$, we obtain

$$\|E\| \equiv O(h^6).$$

(iv) For $\alpha_1 = \frac{337}{4927}, \alpha_2 = \frac{39712}{44343}, \alpha_3 = \frac{16285}{44343}, \alpha_4 = 0, \beta_1 = \frac{-33536}{14781}, \beta_2 = \frac{18755}{14781}$, and $\beta_3 = 0$, we have

$$\|E\| \equiv O(h^8).$$

(v) For $\alpha_1 = \frac{2867}{62559}, \alpha_2 = \frac{34180}{20853}, \alpha_3 = \frac{158339}{20853}, \alpha_4 = \frac{525032}{62559}, \beta_1 = \frac{6272}{993}, \beta_2 = -\frac{7265}{993}$, and $\beta_3 = 0$, we get

$$\|E\| \equiv O(h^{10}).$$

(vi) For $\alpha_1 = \frac{1857}{49483}, \alpha_2 = \frac{110322}{49483}, \alpha_3 = \frac{989739}{49483}, \alpha_4 = \frac{2175924}{49483}, \beta_1 = \frac{112266}{7069}, \beta_2 = \frac{112995}{7069}$, and $\beta_3 = -\frac{464660}{7069}$, we have

$$\|E\| \equiv O(h^{12}).$$

4. Numerical Illustrations

In order to test the viability of the proposed method based on non-polynomial spline and to demonstrate its convergence computationally, we consider the boundary-value problem (2). This problem has been solved using our methods with different values of n and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3$. The maximum absolute errors in solutions are tabulated in Tables 2-5. The maximum absolute errors in solutions of this problem are compared with method in [15] for $n = 16, 32, 64, 128$. The tables show that our results are more accurate. All calculations were implemented using Mathematica6.0 with Working Precision 50.

Conclusion

The approximate solutions of the one-dimensional Bratu's problem by using non-polynomial spline, shows that our methods are better in the sense of accuracy and applicability. These have been verified by the maximum absolute errors $\max|e_i|$ given

Table 1. Maximum absolute errors sixth-order method in [15]

n	$\lambda = 1$	$\lambda = 2$	$\lambda = 3.51$
8	5.64×10^{-9}	4.53×10^{-8}	3.51×10^{-5}
16	4.66×10^{-11}	1.76×10^{-9}	1.45×10^{-7}
32	8.33×10^{-13}	2.13×10^{-11}	1.02×10^{-9}
64	9.21×10^{-15}	2.87×10^{-13}	1.48×10^{-11}
128	-	2.47×10^{-14}	1.58×10^{-13}

Table 2. Observed maximum absolute errors sixth-order method

n	$\lambda = 1$	$\lambda = 2$	$\lambda = 3.51$
16	1.25×10^{-11}	4.69×10^{-10}	3.01×10^{-5}
32	1.87×10^{-13}	5.02×10^{-12}	3.75×10^{-9}
64	2.89×10^{-15}	7.64×10^{-14}	3.32×10^{-11}
128	4.52×10^{-17}	1.19×10^{-15}	5.21×10^{-13}
256	7.05×10^{-19}	1.85×10^{-17}	8.71×10^{-15}

Table 3. Observed maximum absolute errors twelve-order method

n	$\lambda = 1$	$\lambda = 2$	$\lambda = 3.51$
16	4.87×10^{-13}	1.13×10^{-10}	2.99×10^{-5}
32	2.02×10^{-17}	1.93×10^{-14}	1.57×10^{-9}
64	2.08×10^{-21}	5.25×10^{-19}	1.63×10^{-14}
128	1.31×10^{-25}	6.49×10^{-24}	1.86×10^{-15}
256	7.82×10^{-30}	2.67×10^{-27}	-

Table 4. Observed maximum absolute errors for $\lambda = 1$

n	Second-order	Fourth-order	Eight-order	Tenth-order
16	2.27×10^{-3}	2.64×10^{-5}	5.04×10^{-11}	4.63×10^{-13}
32	6.12×10^{-4}	1.71×10^{-6}	1.51×10^{-13}	1.74×10^{-17}
64	1.55×10^{-4}	1.07×10^{-7}	5.96×10^{-16}	3.38×10^{-21}
128	3.90×10^{-5}	6.74×10^{-9}	2.30×10^{-18}	5.42×10^{-24}
256	9.76×10^{-6}	4.21×10^{-10}	8.98×10^{-21}	4.96×10^{-27}
512	2.44×10^{-6}	2.63×10^{-11}	3.50×10^{-23}	4.85×10^{-30}
1024	6.11×10^{-7}	1.65×10^{-12}	1.37×10^{-25}	4.73×10^{-33}

in tables. Some properties of band matrices are obtained, which are required in proving the convergence analysis of the finite difference and spline methods.

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Table 5. Observed maximum absolute errors for $\lambda = 2$

n	Second-order	Fourth-order	Eight-order	Tenth-order
16	1.39×10^{-2}	3.68×10^{-4}	3.26×10^{-9}	1.13×10^{-10}
32	3.81×10^{-3}	2.28×10^{-5}	7.07×10^{-12}	1.93×10^{-14}
64	9.72×10^{-4}	1.42×10^{-6}	2.96×10^{-14}	5.25×10^{-19}
128	2.44×10^{-4}	8.86×10^{-8}	1.14×10^{-16}	6.49×10^{-24}
256	6.11×10^{-5}	5.53×10^{-9}	4.43×10^{-19}	2.67×10^{-28}
512	1.52×10^{-5}	3.45×10^{-10}	1.72×10^{-21}	3.30×10^{-32}
1024	3.82×10^{-6}	2.16×10^{-11}	6.75×10^{-24}	2.43×10^{-36}

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