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Topological number for locally convex topological spaces with continuous semi-norms

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Abstract. In this paper we introduce the concept of topological number for locally convex topological spaces and prove some of its properties. It gives some criterions to study locally convex topological spaces in a discrete approach.

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1. Introduction

In functional analysis, the notion of a topological vector space plays a central role. A lot of deep and interesting results are only valid for certain subclasses of topological vector spaces. Certainly, one of the most important better-behaved types of topological vector spaces are the locally convex ones. Locally convex spaces are encountered repeatedly when discussing weak topologies on a Banach space, sets of operators on Hilbert spaces or the theory of distributions [2, 5, 6]. On the other hand, it is well-known that a topological vector space X is locally convex if and only if it can be generated by a collection of semi-norms, in the sense that, there exists a family $\mathcal{P}$ of continuous semi-norms on X such that

$$\{\{x \in X : \ p(x) < \epsilon\} : \ p \in \mathcal{P}, \ \epsilon > 0\}$$

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is a base for the neighborhood system of 0. A typical semi-norm is generated by a convex, balanced and absorbing subset of X. In other words, if $C \subset X$ is a convex, balanced and absorbing set then the map $p_C : X \longrightarrow \mathbb{R}^+$ defined by $p_C(x) := \inf\{t : t \geqslant 0, x \in tC\}$ is a semi-norm on X which is called the Minkowski functional. The map p_C, as a semi-norm, induces a topology on X which is denoted by τ_C. If $C = B(0,1) = \{x \in X : \| x \| \leqslant 1\}$ then τ_C coincides with norm topology on X. One can see [3] for more information about locally convex spaces.

In this paper, we first present some elementary properties of τ_C; then we study τ_C by assigning a cardinal number $N = N(\tau_C)$ to this topology which is called *topological number*. This cardinal number has some beautiful properties and, some how, measures that how far is the topology τ_C from the norm topology.

Finally, we will extend this idea for locally convex topologies, generated by a family of continuous semi-norms.

In the remaining of the paper, a $C.B.A$ set means a convex, balanced and absorbing set.

2. Some elementary properties of τ_C

Let $X = \mathbb{R}^2$ and $C := \mathbb{R} \times [-1,1]$ and $x_n := ((-1)^n, 0)$, then clearly $p_C(x_n) \to 0$ and so $x_n \to (0,0)$ in τ_C, but $\{x_n\}$ does not converge in norm, so $\tau_C \neq \tau_{\|.\|}$. Therefore, the first question which naturally arise is, "when does τ_C coincides with the norm topology $\tau_{\|.\|}$ on X? At first we try to answer this question. It is easily seen that if $A \subseteq B$ then $\tau_B \subseteq \tau_A$. We have the following theorem as well.

Theorem 2.1 $\tau_A \subseteq \tau_B$ if and only if $p_A : (X, \tau_B) \to \mathbb{R}^+$ is continuous.

Proof. Let $\tau_A \subseteq \tau_B$ and $x_n \to x$ in τ_B, then $x_n \to x$ in τ_A and so we have $p_A(x_n - x) \to 0$. But $|p_A(x_n) - p_A(x)| \leqslant p_A(x_n - x)$, therefore $p_A(x_n) \to p_A(x)$ and so $p_A : (X, \tau_B) \longrightarrow \mathbb{R}^+$ is continuous.

Now let $p_A : (X, \tau_B) \longrightarrow \mathbb{R}^+$ be continuous and $x_n \to x$ in τ_B, then $x_n - x \to 0$ in τ_B which implies $p_A(x_n - x) \to 0$, so $x_n \to x$ in τ_A and $\tau_A \subseteq \tau_B$. ■

Corollary 2.2 $\tau_A = \tau_B$ if and only if $p_A : (X, \tau_B) \to \mathbb{R}^+$ and $p_B : (X, \tau_A) \to \mathbb{R}^+$ are continuous.

Let C be a $C.B.A$ subset of X and define the relation $\sim_C$ on X as follows:

$$\text{"}x \sim_C y \iff p_C(x - y) = 0.\text{"}$$

Then " $\sim_C$ " is clearly an equivalence relation on X. For $x_0 \in X$, the equivalence class of x_0 is denoted by $[x_0]_\sim$.

***Example* 2.3** Let $X = \mathbb{R}^2$ and $C = \mathbb{R} \times [-1,1]$, then $[0]_\sim = \{(x,y) : y = 0\}$ which is the x-axes.

Lemma 2.4 Let C be a $C.B.A$ subset of X and $x_0 \in X$. If $x_1, x_2 \in [x_0]_\sim$ then the line passing through x_1 and x_2 is completely contained in $[x_0]_\sim$.

Proof. Let $x_1, x_2 \in [x_0]_\sim$, then $p_C(x_1 - x_0) = p_C(x_2 - x_0) = 0$. Now for each $t \in \mathbb{R}$ put

$x_t := (1-t)x_1 + tx_2$, then

$$\begin{aligned} p_C(x_t - x_0) &= p_C((1-t)x_1 + tx_2 - (1-t)x_0 - tx_0) \\ &= p_C((1-t)(x_1 - x_0) + t(x_2 - x_0)) \\ &\leqslant |\ 1-t\ |\ p_C(x_1 - x_0) + |t| p_C(x_2 - x_0) \\ &= 0. \end{aligned}$$

Therefore $x_t \in [x_0]_\sim$, and this completes the proof. ■

Lemma 2.5 Let C be a $C.B.A$ subset of X. If C is unbounded then there exists $x_0 \in X$ such that $x_0 \neq 0$ *and* $p_C(x_0) = 0$.

Proof. Since C is an unbounded and $C.B.A$ subset of X, then there is a line passing through the origin such that $\mathbb{L} \subseteq C$. Then clearly, $(1/\lambda)\mathbb{L} = \mathbb{L} \subseteq C$ and so $\mathbb{L} \subseteq \lambda C$ for all $\lambda > 0$. So, $p_C(x) = 0$ for all $x \in \mathbb{L}$. ■

Corollary 2.6 If C is an unbounded subset of X, then τ_C is not Hausdorff and consequently $\tau_C \neq \tau_{\|.\|}$.

Theorem 2.7 If A is a bounded and $C.B.A$ subset of X and $0 \in Int(A)$, then $\tau_A = \tau_{\|.\|}$.

Proof. Since A is bounded then $A \subseteq B(0, M)$, for some $M > 0$, therefore $\tau_{B(0,M)} = \tau_{B(0,1)} = \tau_{\|.\|} \subseteq \tau_A$. On the other hand, since $0 \in Int(A)$, there exists $\varepsilon > 0$ such that $B(0, \varepsilon) \subseteq A$ so $\tau_A \subseteq \tau_{B(0,\varepsilon)} = \tau_{B(0,1)} = \tau_{\|.\|}$, and it completes the proof. ■

Theorem 2.8 Let X be a normed space and $\dim X = n < \infty$. Then a convex and balanced set $C \subseteq X$ is absorbing if and only if $0 \in Int(C)$.

Proof. If $0 \in Int(C)$ then C is clearly absorbing. Now let C be absorbing, $\mathfrak{B} = \{e_1, e_2, ..., e_n\}$ be a basis for X and $\varepsilon > 0$ be such that $\pm\varepsilon e_j \in C$ *for* $j = 1, 2, 3, ..., n$. Put $V := co(\pm\varepsilon e_1, ..., \pm\varepsilon e_n)$, then $0 \in V$ and so $0 \in Int(C)$. ■

Corollary 2.9 If $\dim X < +\infty$ and C is a $C.B.A$ subset of X then $\tau_C = \tau_{\|.\|}$ if and only if C is bounded.

The following example shows that Theorem 2.8 does not hold if $\dim X$ is not finite.

***Example* 2.10**

Let X be a normed space with $\dim X = \aleph_0$ and let $\mathfrak{B} = \{e_n\}_{n=1}^{\infty}$ be a normalized basis for X (i.e., $\|e_n\| = 1$ for $n = 1, 2, 3, ...$).

Put $C := co(\{\pm\frac{1}{2^n} e_n : n = 1, 2, 3, ...\})$, then

(a) $0 \notin Int(C)$.
(b) C is absorbing.

To show (a), let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$, then $\| \frac{1}{2^n} e_n \| = \frac{1}{2^n} < \varepsilon$ so $\frac{1}{2^n} e_n \in B(0, \varepsilon)$; since $\frac{1}{2^n} e_n$ is a vertex of C then $B(0, \varepsilon) \cap C^c \neq \varnothing$ and so $0 \notin Int(C)$.

To show (b), let $x \in X$ and $x = \sum_{j=1}^{k} \lambda_j e_j$, then $x = \sum_{j=1}^{k} \mu_j sgn\lambda_j . \frac{1}{2^j} e_j$ for some $\mu_j \geqslant 0$. Let $\varepsilon > 0$ be such that $\varepsilon \sum_{j=1}^{k} \mu_j = \sum_{j=1}^{k} \varepsilon\mu_j = 1$. Then we have $\varepsilon x = \sum_{j=1}^{k} \varepsilon\mu_j sgn\lambda_j . \frac{1}{2^j} e_j \in C$. Since C is convex and balanced then $\alpha x \in C$ for all α with $|\alpha| \leqslant \varepsilon$, so C is absorbing.

3. Topological number for topologies induced by a C.B.A set

Definition 3.1 Let C be a $C.B.A$ subset of X and τ_C be the topology generated by P_C. Define

$$N(\tau_C) := \min\{\text{card}(\mathfrak{A}) : \mathfrak{A} \subseteq C, co(\mathfrak{A}) \textit{ is a C.B.A set and } \tau_{co(\mathfrak{A})} = \tau_C\}.$$

$N(\tau_C)$ is called the *topological number* of τ_C.

***Example* 3.2** Let $X = \mathbb{R}^2$ and $C = \overline{B}(0,1)$ and put $\mathfrak{A} := \{\pm i, \pm j\}$. Then if $D = co(\mathfrak{A})$, we will have $\tau_D = \tau_C = \tau_{\|.\|}$. It is clear that card$(\mathfrak{A})$ is minimum among all of the sets with the same property, so $N(\tau_C) = 4$. Similarly, for $X = \mathbb{R}^3$ and $C = \overline{B}(0,1)$, if we put $\mathfrak{A} := \{\pm i, \pm j, \pm k\}$ then $\tau_{co(\mathfrak{A})} = \tau_C = \tau_{\|.\|}$ and $N(\tau_C) = N(\tau_{\|.\|}) = 6$.

The following theorem generalizes the previous example.

Theorem 3.3 If X is a normed space with $\dim X = n$ then $N(\tau_{\|.\|}) = 2n$.

Proof. Put $C := \overline{B}(0,1)$ and $\mathfrak{A} := \{\pm e_1, \pm e_2, ..., \pm e_n\}$ where $\mathfrak{B} = \{e_1, e_2, ..., e_n\}$ is a basis for X. If $D = co(\mathfrak{A})$ then $\tau_D = \tau_C = \tau_{\|.\|}$ and card$(\mathfrak{A}) = 2n$, therefore $N(\tau_{\|.\|}) \leqslant 2n$. Let $N(\tau_{\|.\|}) = 2k < 2n$ then there exists $\mathfrak{A} := \{\pm f_1, \pm f_2, ..., \pm f_k\}$ such that $co(A)$ is a C.B.A set and $\tau_{co(\mathfrak{A})} = \tau_{\|.\|}$. Since $k < n$ then we may find $f \neq 0$ such that $f \notin$ span$(\{f_1, f_2, ..., f_k\})$, therefore $\lambda f \notin$ span$(\{f_1, f_2, ..., f_k\})$ for all $\lambda > 0$ and so $\lambda f \notin co(\mathfrak{A})$ for all $\lambda > 0$ which is a contradiction to the fact that $co(\mathfrak{A})$ is absorbing, so $N(\tau_{\|.\|}) = 2n$. ■

Theorem 3.4 Let $\dim X = n$ and C be a $C.B.A$ subset of X. Then $\tau_C \neq \tau_{\|.\|}$ if and only if $N(\tau_C) \geqslant \aleph_0$.

Proof. Let $N(\tau_C) \geqslant \aleph_0$. If $\tau_C = \tau_{\|.\|}$ then $N(\tau_C) = 2n < \aleph_0$, so $\tau_C \neq \tau_{\|.\|}$. Now let $\tau_C \neq \tau_{\|.\|}$ and $N(\tau_C) < +\infty$; then we may find $\mathfrak{A} \subseteq C$ such that card$(\mathfrak{A}) < +\infty$, $\tau_C = \tau_{co(\mathfrak{A})}$ and $co(\mathfrak{A})$ is balanced and absorbing. Since card$(\mathfrak{A}) < +\infty$ then $co(\mathfrak{A})$ is bounded and so $\tau_{co(\mathfrak{A})} = \tau_{\|.\|}$ which is a contradiction, thus $N(\tau_C) \geqslant \aleph_0$. ■

***Example* 3.5** Let $X = \mathbb{R}^2$ and $C := \mathbb{R} \times [-1, 1]$. Clearly, if $C = co(\mathfrak{A})$ then card$(\mathfrak{A}) \geqslant \aleph_0$. Of course here we may choose $\mathfrak{A}$ such that card$(\mathfrak{A}) = \aleph_0$ and so $N(\tau_C) = \aleph_0$.

Theorem 3.6 Let $\dim X = \alpha$ where α is an infinite cardinal number and $\tau_C = \tau_{\|.\|}$, then $N(\tau_C) = \alpha$.

Proof. Let $\tau_C = \tau_{\|.\|}$ and let $\mathfrak{B} = \{e_j\}_{j \in J}$ be a normalized basis for X with card$(J) = \alpha$. Put $\mathfrak{A} := \{\pm e_j\}_{j \in J}$ and $D := co(\mathfrak{A})$. We show that $\tau_D = \tau_{\|.\|}$. It is clear that $0 \in Int(D)$. Moreover D is Bounded, Since if $x \in D = co(\mathfrak{A})$ then $x = \sum_{j=1}^{m} \lambda_j f_j$ where $f_j = \pm e_{k_j}$, $\lambda_j \geqslant 0$ and $\sum_{j=1}^{m} \lambda_j = 1$, therefore

$$\|x\| = \|\sum_{j=1}^{m} \lambda_j f_j\| \leqslant \sum_{j=1}^{m} |\lambda_j| \|f_j\| = \sum_{j=1}^{m} \lambda_j = 1.$$

Consequently $D \subseteq B(0,1)$ and so D is Bounded. Hence by Theorem 2.7 $\tau_D = \tau_{\|.\|} = \tau_C$, so $\tau_C = \tau_{co(\mathfrak{A})}$ and card$(\mathfrak{A}) = \alpha$. Therefore $N(\tau_C) \leqslant \alpha$.

Now, let in contrast $N(\tau_C) = \beta < \alpha$, then there exists $\mathfrak{A} = \{f_j\}_{j\in\xi}$ such that $\text{card}(\xi) = \beta$ and $\tau_C = \tau_{co(\xi)}$. Since $\beta < \alpha = \dim X$ then we can find $0 \neq f \in X$ such that $f \notin \text{span}(\mathfrak{A})$ so $\lambda f \notin \text{span}(\mathfrak{A}) \supseteq co(\mathfrak{A})$ for all $\lambda > 0$ which is a contradiction to the fact that $co(\mathfrak{A})$ is absorbing; therefore $N(\tau_C) = \alpha$. ■

Corollary 3.7 If $\dim X = \alpha$, C is a $C.B.A$ subset of X and $\tau_C = \tau_{\|\cdot\|}$ then $N(\tau_C) = 2\alpha$.

Lemma 3.8 Let $C \subseteq X$, $D \subseteq Y$ be $C.B.A$ sets and $A \subseteq C$, $B \subseteq D$ be such that $co(A)$ and $co(B)$ are $C.B.A$ sets. If $\tau_C = \tau_{co(A)}$ and $\tau_D = \tau_{co(B)}$ then $\tau_{C\times D} = \tau_{co(A\times B)}$.

Proof. Since $A \times B \subseteq C \times D$ then $co(A \times B) \subseteq C \times D$, therefore $\tau_{C\times D} \subseteq \tau_{co(A\times B)}$. Now, let $(x_n, y_n) \to (x, y)$ in $\tau_{C\times D}$. Since

$$\begin{aligned} p_{C\times D}((x_n, y_n) - (x, y)) &= \inf\{t \geqslant 0 : t[(x_n, y_n) - (x, y)] \in C \times D\} \\ &= \inf\{t \geqslant 0 : (t(x_n - x), t(y_n - y)) \in C \times D\} \\ &= \inf\{t \geqslant 0 : t(x_n - x) \in C, t(y_n - y) \in D\} \\ &\geqslant \inf\{t \geqslant 0 : t(x_n - x) \in C\} \\ &\geqslant 0 \end{aligned}$$

then

$$\inf\{t \geqslant 0 : t(x_n - x) \in C\} \to 0.$$

Similarly,

$$\inf\{t \geqslant 0 : t(y_n - y) \in D\} \to 0.$$

Hence $x_n \to x$ in τ_C and $y_n \to y$ in τ_D. But $\tau_C = \tau_{co(A)}$ and $\tau_D = \tau_{co(B)}$, so $x_n \to x$ in $\tau_{co(A)}$ and $y_n \to y$ in $\tau_{co(B)}$. Therefore, $p_{co(A)}(x_n - x) \to 0$ and $p_{co(B)}(y_n - y) \to 0$ which implies

$$\inf\{t \geqslant 0 : t(x_n - x) \in co(A)\} \to 0$$

and

$$\inf\{t \geqslant 0 : t(y_n - y) \in co(B)\} \to 0.$$

So, for a given $\varepsilon > 0$, there exists a natural number N such that $\inf\{t \geqslant 0 : t(x_n - x) \in co(A)\} < \frac{\varepsilon}{2}$ and $\inf\{t \geqslant 0 : t(y_n - y) \in co(B)\} < \frac{\varepsilon}{2}$ for all $n \geqslant N$. Now, for such natural number $n \geqslant N$, we may choose $0 \leqslant t_1, t_2 < \varepsilon$ such that:

$$t_1(x_n - x) \in co(A),$$

and

$$t_2(y_n - y) \in co(B).$$

Put $t := \min\{t_1, t_2\}$; then $t < \varepsilon$, $t(x_n - x) \in co(A)$ and $t(y_n - y) \in co(B)$, because $co(A)$ *and* $co(B)$ are balanced. Therefore

$$\inf\{t \geqslant 0 : t(x_n - x) \in co(A), t(y_n - y) \in co(B)\} < \varepsilon,$$

hence

$$\inf\{t \geqslant 0 : t[(x_n, y_n) - (x, y)] \in co(A) \times co(B)\} < \varepsilon.$$

So $p_{co(A)\times co(B)}((x_n, y_n) - (x, y)) \to 0$ and therefore $(x_n, y_n) \to (x, y)$ in $\tau_{co(A)\times co(B)}$. Thus

$$\tau_{co(A\times B)} = \tau_{co(A)\times co(B)} \subseteq \tau_{C\times D} \subseteq \tau_{co(A\times B)}.$$

Therefore $\tau_{co(A\times B)} = \tau_{C\times D}$. ■

Theorem 3.9 Let $C \subseteq X$, $D \subseteq Y$ be $C.B.A$ sets. Then

$$N(\tau_{C\times D}) = N(\tau_C) + N(\tau_D).$$

Proof. Let $N(\tau_C) = \alpha$ and $N(\tau_D) = \beta$. Then there exist $A \subseteq C$ and $B \subseteq D$ such that $A = \{e_i\}_{i\in\Gamma}$, $\text{card}(\Gamma) = \alpha$ and $\tau_C = \tau_{co(A)}$ and $B = \{f_j\}_{j\in\Lambda}$, $\text{card}(\Lambda) = \beta$ and $\tau_D = \tau_{co(B)}$. Put $\Omega := \{(e_i, 0), (0, f_j)\}_{i\in\Gamma, j\in\Lambda}$, then $\text{card}(\Omega) = \alpha + \beta$. First we show that $\tau_{C\times D} = \tau_{co(\Omega)}$. Since each $(e_i, 0)$ and $(0, e_j)$ belongs to $C \times D$, then $co(\Omega) \subseteq C \times D$ and therefore $\tau_{C\times D} \subseteq \tau_{co(\Omega)}$. On the other hand if we put $\Delta := \{(e_i, f_j)\}_{i\in\Gamma, j\in\Lambda}$, then by Lemma 3.8 $\tau_{co(\Delta)} = \tau_{C\times D}$. But $co(\Delta) \subseteq 2co(\Omega)$, therefore $\tau_{co(\Omega)} = \tau_{2co(\Omega)} \subseteq \tau_{co(\Delta)} = \tau_{C\times D}$. Hence $\tau_{C\times D} \subseteq \tau_{co(\Omega)}$. It implies that $N(\tau_{C\times D}) \leqslant \alpha + \beta$.

The prove of the equality is obvious if $\alpha, \beta < \infty$, so, without loss of generality, we assume that β is an infinite cardinal number.

Case1: If $\alpha = \beta$, then $\alpha + \beta = \alpha \leqslant N(\tau_{C\times D}) \leqslant \alpha + \beta$ and hence $N(\tau_{C\times D}) = \alpha + \beta$.

Case2: If $\alpha < \beta$, then $\alpha + \beta = \beta \leqslant N(\tau_{C\times D}) \leqslant \alpha + \beta$ and this proves the equality. ■

4. Linear topology and continuous semi-norms

If ρ is a semi-norm then we define C_ρ and W_ρ as follows:

$$C_\rho := \{x \in X : \ \rho(x) \leqslant 1\}$$

and

$$W_\rho := \{x \in X : \ \rho(x) = 0\}.$$

It is clear that C_ρ is a C.B.A set and W_ρ is a linear subspace of X. Then we may define the topological number for the topology induced by the semi-norm ρ to be the topological number, corresponding to the C.B.A set C_ρ. In other words $N(\tau_\rho) := N(\tau_{C_\rho})$. Now we are going to generalize the concept of topological number for more general topologies. To do this, we need the concept of linear topology.

Definition 4.1 Let W be a linear subspace of X. A sequence $\{x_n\}_{n\geqslant 1}$ is said to tend to an element x in the linear topology corresponding to W if and only if $dist(x_n - x, W) \to 0$. The topology corresponding to the previous definition is called the linear topology corresponding to the linear space W and is denoted by τ_W.

The following theorem shows that any topology induced by a continuous semi-norm is indeed a linear topology.

Theorem 4.2 Let ρ be a continuous semi-norm and $W_\rho := \rho^{-1}(\{0\})$. Then the topology induced by ρ is equivalent to the linear topology corresponding to W_ρ.

Proof. Let $\{x_n\}_{n\geqslant 1}$ be a sequence in X such that $x_n \to 0$ in W_ρ, then $\text{dist}(x_n, W_\rho) \to 0$ or $\inf\{\|x_n - w\| :\ w \in W_\rho\} \to 0$. So we may choose a sequence $\{w_n\}_{n\geqslant 1}$ in W_ρ such that $\|x_n - w_n\| \to 0$. Since ρ is continuous and $\rho(x_n) = \rho(x_n) - \rho(w_n) \leqslant \rho(x_n - w_n)$ then $\rho(x_n) \to 0$ and so $x_n \to 0$ in τ_ρ. To prove the converse, let $\rho(x_n) \to 0$, then

$$\inf\{t :\ t \geqslant 0 \ and\ x_n \in tC_\rho\} \to 0.$$

Choose a sequence $\{t_n\}_{n\geqslant 1}$ of positive numbers such that $t_n \searrow 0$ and $x_n \in t_n C_\rho$ for $n \in \mathbb{N}$. It is easily seen that $W_\rho = \bigcap_{n=1}^{\infty} t_n C_\rho$. So $\text{dist}(x_n, W_\rho) \to 0$, thus $x_n \to 0$ in W_ρ, which gives the result. ■

Note: If ρ is not continuous then the previous theorem may fail. To see this, let X be a normed space with $\dim X = \aleph_0$ and $\mathfrak{B} = \{e_n\}_{n\geqslant 1}$ be a basis for X. Put $C := co(\{\pm\frac{1}{2^n} e_n :\ n \in \mathbb{N}\})$, then C is a C.B.A set and $0 \notin Int(C)$. Let $\{\varepsilon_n\}_{n\geqslant 1}$ be a sequence of positive numbers such that $\varepsilon_n \searrow 0$. Then for $n \in \mathbb{N}$, we may find $x_n \in X$ such that $x_n \in B(0, \varepsilon_n) \setminus C$, therefore $x_n \to 0$ and $p_C(x_n) \geqslant 1$. Thus we have found a sequence $\{x_n\}_{n\geqslant 1}$ in X such that $\text{dist}(x_n, W_{p_C}) \to 0$ but $p_C(x_n)$ does not tend to 0.

Corollary 4.3 Let X be an inner product space and ρ be a continuous semi-norm on X and $W_\rho := \rho^{-1}(\{0\})$. Then for any sequence $\{x_n\}_{n\geqslant 1}$ in X we have

$$\rho(x_n) \to 0 \ if\ and\ only\ if\ \|proj_{W_\rho{}^\perp}^{x_n}\| \to 0.$$

Corollary 4.4 For any continuous semi-norm ρ, if $\{x_n\}_{n\geqslant 1}$ is a sequence in X such that $x_n \to x$ in norm then $x_n \to x$ in τ_ρ; in other words $\tau_\rho \subseteq \tau_{\|.\|}$.

Lemma 4.5 ([3]) Let X be a Banach space and V, W be closed linear subspaces of X such that $V + W$ is closed. Then there exists a constant $c \geqslant 0$ such that

$$\text{dist}(x, V \cap W) \leqslant c(\text{dist}(x, V) + \text{dist}(x, W))$$

for all $x \in X$.

Corollary 4.6 Let $\dim X < \infty$ and W_1 , W_2 be linear subspaces of X. If $\{x_n\}_{n\geqslant 1}$ is a sequence in X such that $\text{dist}(x_n, W_1) \to 0$ and $\text{dist}(x_n, W_2) \to 0$ then $\text{dist}(x_n, W_1 \cap W_2) \to 0$.

Corollary 4.7 Let $\dim X < \infty$ and W_i $(i = 1, 2, ..., k)$ be linear subspaces of X. If $\{x_n\}_{n\geqslant 1}$ is a sequence in X such that $\text{dist}(x_n, W_i) \to 0$ $(i = 1, 2, ..., k)$, then $\text{dist}(x_n, \cap_{i=1}^{k} W_i) \to 0$.

Let $\mathfrak{A} := \{\rho_i\}_{i\in I}$ be a family of semi-norms on X. The topology induced by the family $\mathfrak{A}$ is denoted by $\tau_{\mathfrak{A}}$.

Theorem 4.8 Let $\dim X < \infty$ and $\mathfrak{A} = \{\rho_i\}_{i\in I}$ be a family of semi-norms on X and $W_{\rho_i} := {\rho_i}^{-1}(\{0\})$. If there exist a finite number of subspaces $W_{\rho_{i_1}}, ..., W_{\rho_{i_m}}$ such that $\cap_{k=1}^{m} W_{\rho_{i_k}} = \{0\}$, then $\tau_{\mathfrak{A}} = \tau_{\|.\|}$.

Proof. Since $\dim X < \infty$ then each ρ_i is continuous and so $\tau_{\mathfrak{A}} \subseteq \tau_{\|.\|}$. Now, let $x_n \to 0$ in $\tau_{\mathfrak{A}}$, then $\rho_i(x_n) \to 0$ for $i \in I$, therefore $\text{dist}(x_n, W_{\rho_i}) \to 0$ for $i \in I$. Then by the previous corollary $\text{dist}(x_n, \cap_{k=1}^{m} W_{\rho_{i_k}}) \to 0$ and so $\text{dist}(x_n, 0) \to 0$ or $\|x_n\| \to 0$. Thus $x_n \to 0$ in norm and hence $\tau_{\|.\|} \subseteq \tau_{\mathfrak{A}}$ which gives the result. ■

5. Topological number for locally convex spaces with continuous semi-norms

Definition 5.1 Let $\mathfrak{A} = \{\rho_i\}_{i\in I}$ be a family of semi-norms on X. The topological number of the induced topology by the family $\mathfrak{A}$ is defined as follows:

$$N(\tau_{\mathfrak{A}}) := \min\{N(\tau_{C_{\rho_1}\cap...\cap C_{\rho_k}}) : \ k \in \mathbb{N}, \ \rho_1, ..., \rho_k \in \mathfrak{A}\}.$$

The following theorem and its corollary generalizes Theorem 3.3.

Theorem 5.2 Let X be a normed space with $\dim X = n$ and let $\mathfrak{A} = \{\rho_i\}_{i\in I}$ be a family of semi-norms. Then $N(\tau_{\mathfrak{A}}) = 2n$ if and only if there exist a finite number of semi-norms $\rho_1, ..., \rho_k \in \mathfrak{A}$ such that $C_{\rho_1} \cap ... \cap C_{\rho_k}$ is bounded.

Proof. Let $C_{\rho_1} \cap ... \cap C_{\rho_k}$ be bounded for some $\rho_1, ..., \rho_k \in \mathfrak{A}$. Combining Corollary 2.9 and Theorem 3.3 we will have $N(\tau_{C_{\rho_1}\cap...\cap C_{\rho_k}}) = 2n$, so $N(\tau_{\mathfrak{A}}) \leqslant 2n$. On the other hand, since for each C.B.A set C, $N(\tau_C) \geqslant 2n$ then $N(\tau_{\mathfrak{A}}) = 2n$. If for all finite number of semi-norms $\rho_1, ..., \rho_k \in \mathfrak{A}$, $C_{\rho_1} \cap ... \cap C_{\rho_k}$ is unbounded, then by Corollary 2.9 and Theorem 3.4, $N(\tau_{C_{\rho_1}\cap...\cap C_{\rho_k}}) \geqslant \aleph_0$ for all finite numbers of semi-norms $\rho_1, ..., \rho_k \in \mathfrak{A}$. Therefore $N(\tau_{\mathfrak{A}}) \geqslant \aleph_0$ which gives the result. ■

Corollary 5.3 If $\dim X = n$ and $N(\tau_{\mathfrak{A}}) = 2n$ then $\tau_{\mathfrak{A}} = \tau_{\|.\|}$.

Proof. If $N(\tau_{\mathfrak{A}}) = 2n$ then there exist finite number of semi-norms $\rho_1, ..., \rho_k \in \mathfrak{A}$ such that $C_{\rho_1} \cap ... \cap C_{\rho_k}$ is bounded, but $\cap_{i=1}^{k} W_{\rho_i} \subseteq W_{\rho_j} \subseteq C_{\rho_j}$ for $j = 1, 2, ..., k$, therefore $\cap_{i=1}^{k} W_{\rho_i} \subseteq \cap_{i=1}^{k} C_{\rho_i}$. Since $\cap_{i=1}^{k} C_{\rho_i}$ is bounded and $\cap_{i=1}^{k} W_{\rho_i}$ is a linear subspace of X then $\cap_{i=1}^{k} W_{\rho_i} = \{0\}$, so by Theorem 4.8 $\tau_\rho = \tau_{\|.\|}$. ■

Definition 5.4 Let ρ and φ be two semi-norms on X and Y respectively. The map $\rho \times \varphi : X \times Y \to \mathbb{R}^+$ is defined as follows:

$$(\rho \times \varphi)(x, y) := \max\{\rho(x), \varphi(y)\}.$$

It is easily seen that $\rho \times \varphi$ is a semi-norm on $X \times Y$.

The following Lemma can be easily proved.

Lemma 5.5 Let ρ and φ be two semi-norms on X and Y respectively, then we have:

(i) $C_{\rho\times\varphi} = C_\rho \times C_\varphi$.
(ii) $W_{\rho\times\varphi} = W_\rho \times W_\varphi$.

The following theorem generalizes Theorem 3.9

Theorem 5.6 Let $\mathfrak{A} = \{\rho_i\}_{i\in I}$ and $\mathfrak{B} = \{\varphi_j\}_{j\in J}$ be two families of semi-norms on X and Y respectively. Let $\mathfrak{A} \times \mathfrak{B} := \{\rho_i \times \varphi_j\}_{(i,j)\in I\times J}$ be the family of semi-norms defined on $X \times Y$. Then

$$N(\tau_{\mathfrak{A}\times\mathfrak{B}}) = N(\tau_{\mathfrak{A}}) + N(\tau_{\mathfrak{B}}).$$

Proof. By the definition of $N(\tau_{\mathfrak{A}})$ and $N(\tau_{\mathfrak{B}})$ we may choose the semi-norms $\rho_1, ..., \rho_k$ on X and $\varphi_1, ..., \varphi_k$ on Y such that $N(\tau_{\mathfrak{A}}) = N(\tau_{C_{\rho_1}\cap ...\cap C_{\rho_k}})$ and $N(\tau_{\mathfrak{B}}) = N(\tau_{C_{\varphi_1}\cap ...\cap C_{\varphi_m}})$, then we have

$$\begin{aligned} C_{\rho_1\times\varphi_1} \cap ... \cap C_{\rho_k\times\varphi_m} &= (C_{\rho_1} \times C_{\varphi_1}) \cap ... \cap (C_{\rho_k} \times C_{\varphi_m}) \\ &= (C_{\rho_1} \cap ... \cap C_{\rho_k}) \times (C_{\varphi_1} \cap ... \cap C_{\varphi_m}). \end{aligned}$$

Therefore,

$$\begin{aligned} N(\tau_{\mathfrak{A}\times\mathfrak{B}}) \leqslant N(\tau_{C_{\rho_1\times\varphi_1}\cap ...\cap C_{\rho_k\times\varphi_m}}) &= N(\tau_{(C_{\rho_1}\cap ...\cap C_{\rho_k})\times(C_{\varphi_1}\cap ...\cap C_{\varphi_m})}) \\ &= N(\tau_{C_{\rho_1}\cap ...\cap C_{\rho_k}}) + N(\tau_{C_{\varphi_1}\cap ...\cap C_{\varphi_m}}) \qquad (1) \\ &= N(\tau_{\mathfrak{A}}) + N(\tau_{\mathfrak{B}}). \end{aligned}$$

Note that, the second equality holds by Theorem 3.9.

To show the equality we consider two cases:

Case1: If $N(\tau_{\mathfrak{A}})$ and $N(\tau_{\mathfrak{B}})$ are both finite then, by (1), $N(\tau_{\mathfrak{A}\times\mathfrak{B}})$ is finite as well, therefore

$$N(\tau_{\mathfrak{A}\times\mathfrak{B}}) = 2\dim(X \times Y) = 2\dim(X) + 2\dim(Y) = N(\tau_{\mathfrak{A}}) + N(\tau_{\mathfrak{B}}).$$

Case2: Let $N(\tau_{\mathfrak{B}})$ be an infinite cardinal number and $N(\tau_{\mathfrak{A}}) \leqslant N(\tau_{\mathfrak{B}})$, then

$$N(\tau_{\mathfrak{A}}) + N(\tau_{\mathfrak{B}}) = N(\tau_{\mathfrak{B}}) \leqslant N(\tau_{\mathfrak{A}\times\mathfrak{B}}).$$

This proves the equality. ■

Concluding remarks

This paper was an attempt to look at the locally convex spaces in a different way. We assigned a cardinal number to any locally convex space, namely, topological number, and proved some of its properties. In section 3, we first defined the topological number for locally convex spaces generated by a semi-norm. We generalized this concept for locally convex spaces generated by a family of semi-norms. The topological number resembles logarithm, in the sense of Theorems 3.9 and 5.6.

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